A generating function

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July 14, 2022

abstract

In this note we study the function $f(z)=\frac{1}{1-2z-z^2-z^4} \quad , z \in \mathbb{C}$

Introduction

Recall that

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\frac{1}{15}-...\right) \tag{1}$$

In this note we study the function

$$f(z)=\frac{1}{1-2z-z^2-z^4} \quad , z \in \mathbb{C} \tag{2}$$

Results

Entry 1.

$$f(z)=\frac{1}{1-2z-z^2-z^4}=\sum_{n=0}^{\infty} c_n z^n \tag{3}$$

where

$$c_n = 2c_{n-1}+c_{n-2}+c_{n-4} \tag{4}$$

$$\{c_n : n \geq 0\} = \{1,2,5,12,30,74,183,452,1117,...\} \tag{5}$$

Entry 2.

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = u \tag{6}$$

$$u^4+u^2+2u-1=0 \tag{7}$$

$$u = -\frac{1}{2}\sqrt{\frac{-2-11\left(91+6\sqrt{267}\right)^{-1/3}+\left(91+6\sqrt{267}\right)^{1/3}}{3}} + \frac{1}{2}\sqrt{\frac{-4+11\left(91+6\sqrt{267}\right)^{-1/3}-\left(91+6\sqrt{267}\right)^{1/3}}{3}+4\sqrt{\frac{3}{-2-11\left(91+6\sqrt{267}\right)^{-1/3}+\left(91+6\sqrt{267}\right)^{1/3}}}} \tag{8}$$

Entry 3.

$$\pi = 8\sum_{n=0}^{\infty}(-1)^n u^{2n+1}\sum_{k=0}^{[n/2]}\frac{1}{2n-4k+1}\binom{2n-2k}{2k} \tag{9}$$

Entry 4.

$$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{u}{1-u^4}\right)^{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k}u^{4k} = 8\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{2+u}\right)^{2n+1}\sum_{k=0}^{n}\binom{2n+1}{2k}u^{4k} \tag{10}$$

Entry 5.

$$\pi = 4\sum_{n=0}^{\infty}\frac{u^{2n+1}}{2n+1}(a_n+b_n) \tag{11}$$

where

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)\left(-\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)\left(-\frac{3-\sqrt{5}}{2}\right)^n \quad ,n\ge 0 \tag{12}$$

$$b_n = \left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right)^n + \left(\frac{1-i\sqrt{3}}{2}\right)\left(\frac{1+i\sqrt{3}}{2}\right)^n \quad ,n\ge 0 \tag{13}$$

Remark: $i=\sqrt{-1}$.

Entry 6.

$$\frac{f(z)+f(-z)}{2} = \frac{1-z^2-z^4}{\left(1-z^2-z^4\right)^2-4z^2} = \sum_{n=0}^{\infty}c_{2n}\,z^{2n} \tag{14}$$

$$\frac{f(z)-f(-z)}{2} = \frac{2z}{\left(1-z^2-z^4\right)^2-4z^2} = \sum_{n=0}^{\infty}c_{2n+1}\,z^{2n+1} \tag{15}$$

Entry 7.

$$c_{2n} = \sum_{m=0}^{[n/2]}\sum_{k=m}^{n-m}2^{2n-2m-2k}\binom{2n-2m-k}{k}\binom{k}{m} \quad ,n\ge 0 \tag{16}$$

$$c_{2n+1} = \sum_{m=0}^{[n/2]}\sum_{k=m}^{n-m}2^{2n-2m-2k+1}\binom{2n-2m-k+1}{k}\binom{k}{m} \quad ,n\ge 0 \tag{17}$$

Entry 8.

$$u = \lim_{n\to\infty}\sqrt{\frac{c_{2n}}{c_{2n+2}}} \tag{18}$$

$$u = \lim_{n\to\infty}\sqrt{\frac{c_{2n+1}}{c_{2n+3}}} \tag{19}$$

Entry 9. If $u=\tan\theta$, then

$$\pi = 4\tan^{-1}\left(\frac{\tan(2\theta)}{2}\right)+4\tan^{-1}\left(\frac{\sin(2\theta)}{2}\right) \tag{20}$$

Entry 10. If $u = \tanh\theta$, then

$$\pi = 4\tan^{-1}\left(\frac{\tanh(2\theta)}{2}\right) + 4\tan^{-1}\left(\frac{\sinh(2\theta)}{2}\right) \tag{21}$$

Entry 11.

$$\pi = 4\sqrt{u}\sum_{n=0}^{\infty} u^n 2^{-2n} \sum_{k=0}^{[n/4]} 2^{4k} \sum_{m=k}^{\left[\frac{n-2k}{2}\right]} \binom{2n-4k-2m}{n-2k-m}\binom{n-2k-m}{m}\binom{m}{k}\frac{2^{2m}}{2n-4k-4m+1} \tag{22}$$

Entry 12.

$$\pi = 4\sqrt{2}\sum_{n=0}^{\infty} u^{n+1} 2^{-2n} \sum_{k=0}^{[n/4]} 2^{4k} \sum_{m=k}^{\left[\frac{n-2k}{2}\right]} \binom{2n-4k-4m}{n-2k-2m}\binom{n-2k-m}{m}\binom{m}{k}\frac{2^{4m}}{2n-4k-4m+1} \tag{23}$$

Entry 13.

$$\int_{-\infty}^{+\infty} f(xi)\,dx = 2\pi v \tag{24}$$

where

$$1424v^4 - 92v^2 + 16v - 1 = 0 \tag{25}$$

$$v = 0.1426... \tag{26}$$

Entry 14.

$$\pi = 6\sum_{n=0}^{\infty} u^{2n+1} \sum_{k=0}^{[n/2]} 2^{4k} \sum_{m=0}^{n-k} \binom{2m}{m}\binom{n-k+m}{n-k-m}\binom{n-k-m}{k}\frac{2^{-2m}}{2m+1} \tag{27}$$

Entry 15. For $a > 1/u = 2.4709...$, we have

$$2\pi v = \int_{-a}^{a} \frac{1}{1-2xi+x^2-x^4}\,dx - 2\sum_{n=0}^{\infty} \frac{(-1)^n a^{-2n-3}}{2n+3} s_{2n} \tag{28}$$

where

$$s_n = -s_{n-2} - 2s_{n-3} + s_{n-4} \tag{29}$$

$$\{s_n : n \geq 0\} = \{1, 0, 1, -2, 2, 4, 1, -10, 9, ...\} \tag{30}$$

and v is defined by (24),(25),(26) .

Entry 16.

$$\int_{\sqrt{3}}^{\infty} \frac{1}{x^4+x^2+2x-1}\,dx = \\ = \frac{1}{\sqrt{3}} - \frac{\pi}{6} - 2\sum_{n=0}^{\infty} (-1)^n 3^{-n-3} \sum_{k=0}^{n} \left(\frac{2}{\sqrt{3}}\right)^k \binom{n+1}{n-k} \sum_{m=0}^{k+1} \binom{k+1}{k-m+1} \frac{\left(-2\sqrt{3}\right)^{-m}}{2n+k+m+6} \tag{31}$$

Entry 17. If $u = e^{-\theta}$, then

$$\pi = 4\tan^{-1}\left(\frac{1}{2\cosh\theta}\right) + 4\tan^{-1}\left(\frac{1}{2\sinh\theta}\right) \tag{32}$$

Entry 18.

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}(u) + 4\tan^{-1}\left(\frac{u^2}{2+u^3}\right) \tag{33}$$

Entry 19.

$$c_n = \sum_{k=0}^{[n/4]} \sum_{m=0}^{\left[\frac{n-2k}{2}\right]} 2^{n-2k-2m} \binom{n-m-2k}{m} \binom{m}{k} \quad ,n \geq 0 \tag{34}$$

Entry 20.

$$4\sqrt{2}\int_0^{1/\sqrt{2}} f\left(-x^2\right)dx = \pi + \sum_{n=0}^{\infty} \frac{2^{-2n}}{4n+5} \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n-k+1}{k} F\left(n-k+2, 2n+\frac{5}{2}, 2n+\frac{7}{2}, -1\right) \tag{35}$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 21.

$$\int_{\sqrt{3}}^{\infty} \frac{\left(1+x^2\right)^3}{x^8+2x^6-x^4-4x^2-3}dx = \frac{\pi}{6} + \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{3^{-n}c_n}{2n+1} F\left(n+1, n+\frac{1}{2}, n+\frac{3}{2}, -\frac{1}{3}\right) \tag{36}$$

Remark: c_n is defined by (3),(4),(5). $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 22.

$$\pi = 4\sum_{n=0}^{\infty} u^{n+1} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{n-2k} + 4u^2 + 4\sum_{n=0}^{\infty} u^{2n+2} \sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{2k+1}{n-2k} \tag{37}$$

Entry 23. For $s^4+s^2-2s-1=0,\ s=1.1841...$, we have

$$\pi = 4\tan^{-1}\left(\frac{1}{s}\right) + 4\tan^{-1}\left(\frac{1}{s^3}\right) - 4\tan^{-1}\left(\frac{s}{s^2+1}\right) \tag{38}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{s^3}\right) + 4\tan\left(\frac{1}{s^3+2s}\right) \tag{39}$$

$$\pi = 4\tan^{-1}\left(\frac{s}{s+1}\right) + 4\tan^{-1}\left(\frac{1}{s^4+s^2}\right) \tag{40}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{s^3+s-1}\right) + 4\tan^{-1}\left(\frac{1}{s^4+s^2}\right) \tag{41}$$

Entry 24. For $z^4+z^2+2z-1=0, z=0.389...+i\cdot 1.390...$, we have

$$\pi = -4\tan^{-1}\left(\frac{1}{z^3}\right) - 4\tan^{-1}\left(\frac{1}{z^3+2z}\right) \tag{42}$$

Complex Plots

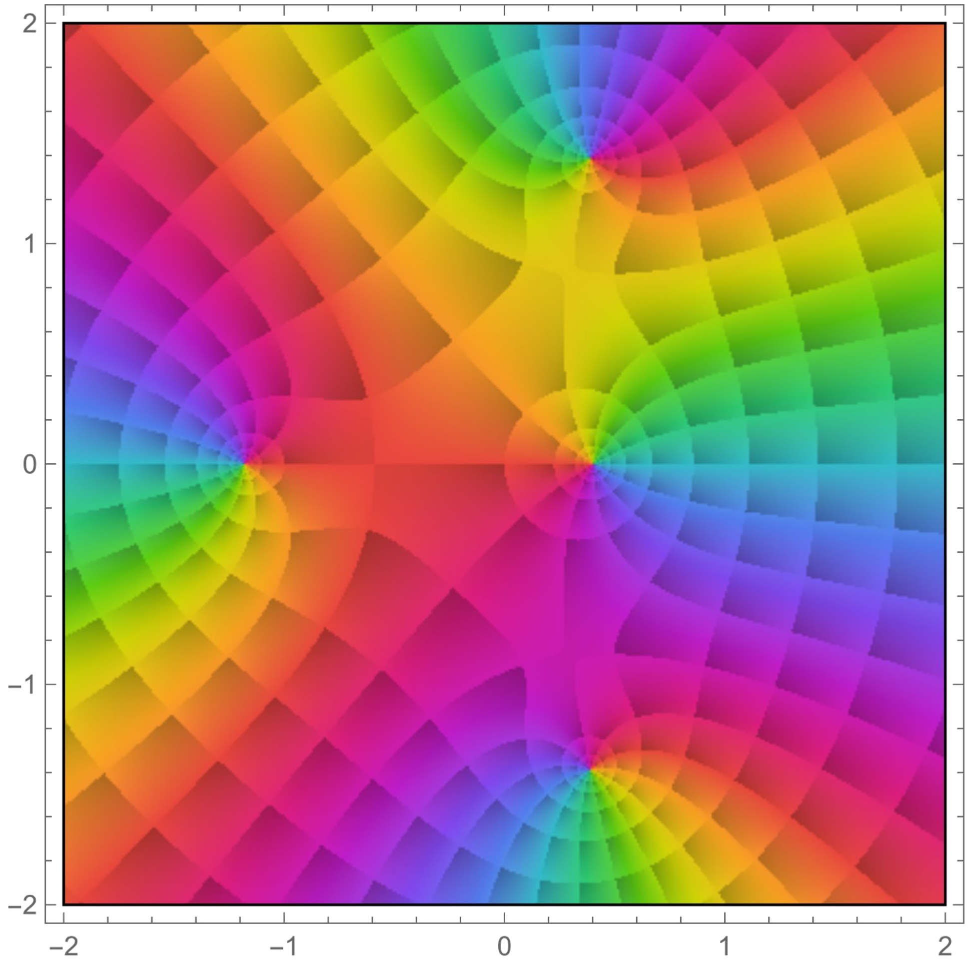

Fig. 1 : $f(z), z \in (-2-2i, 2+2i)$

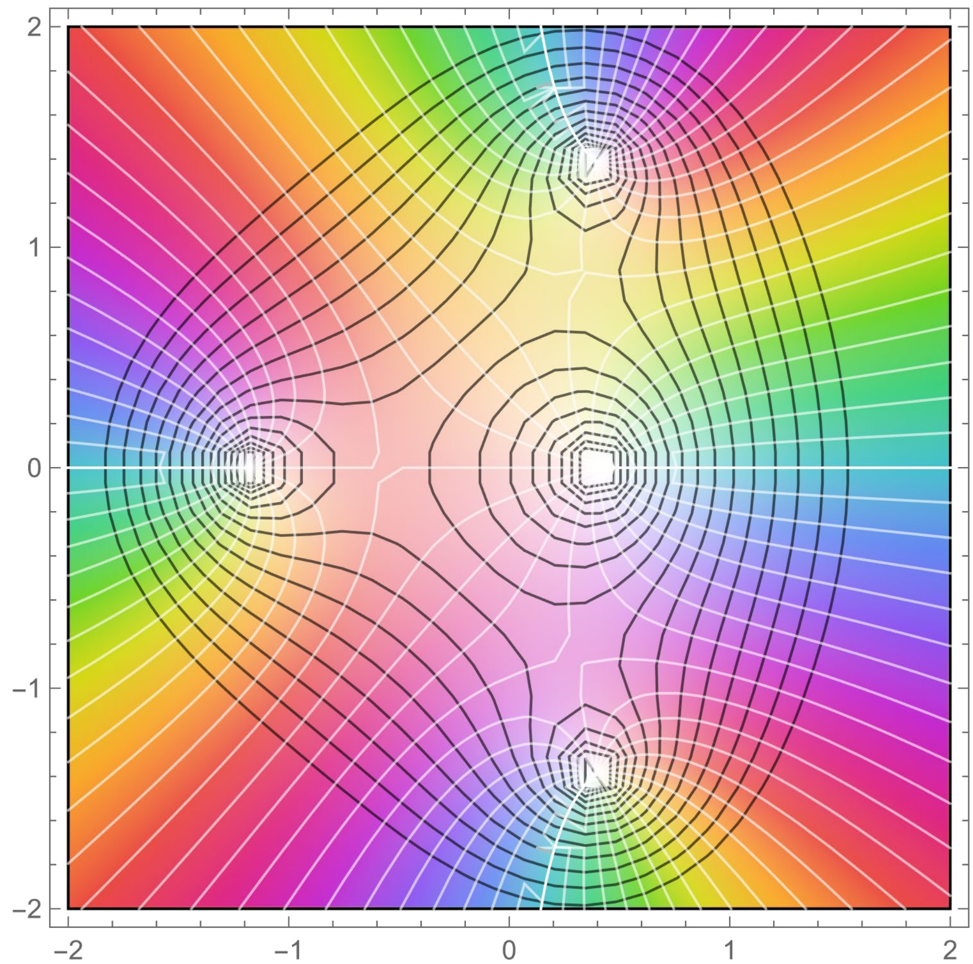

Fig. 2 : $f(z), z \in (-2-2i, 2+2i)$

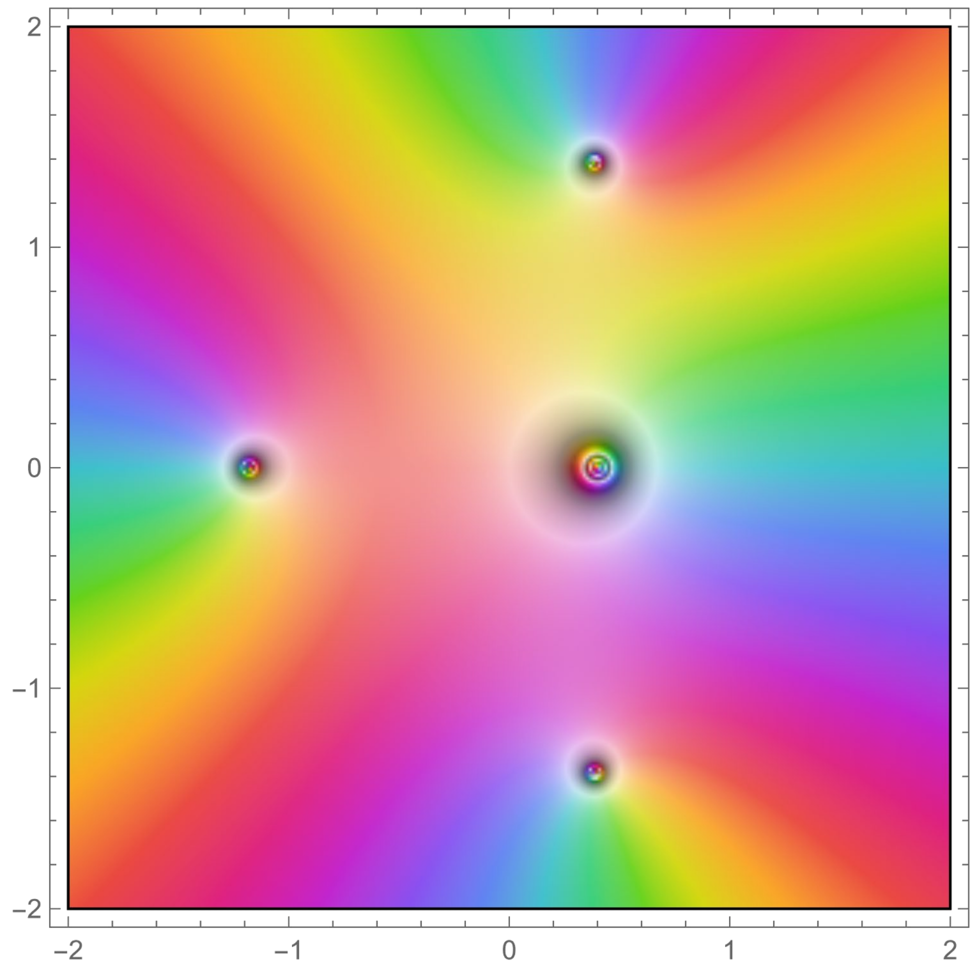

Fig. 3 : $f\left(z\right), z \in \left(-2-2i, 2+2i\right)$

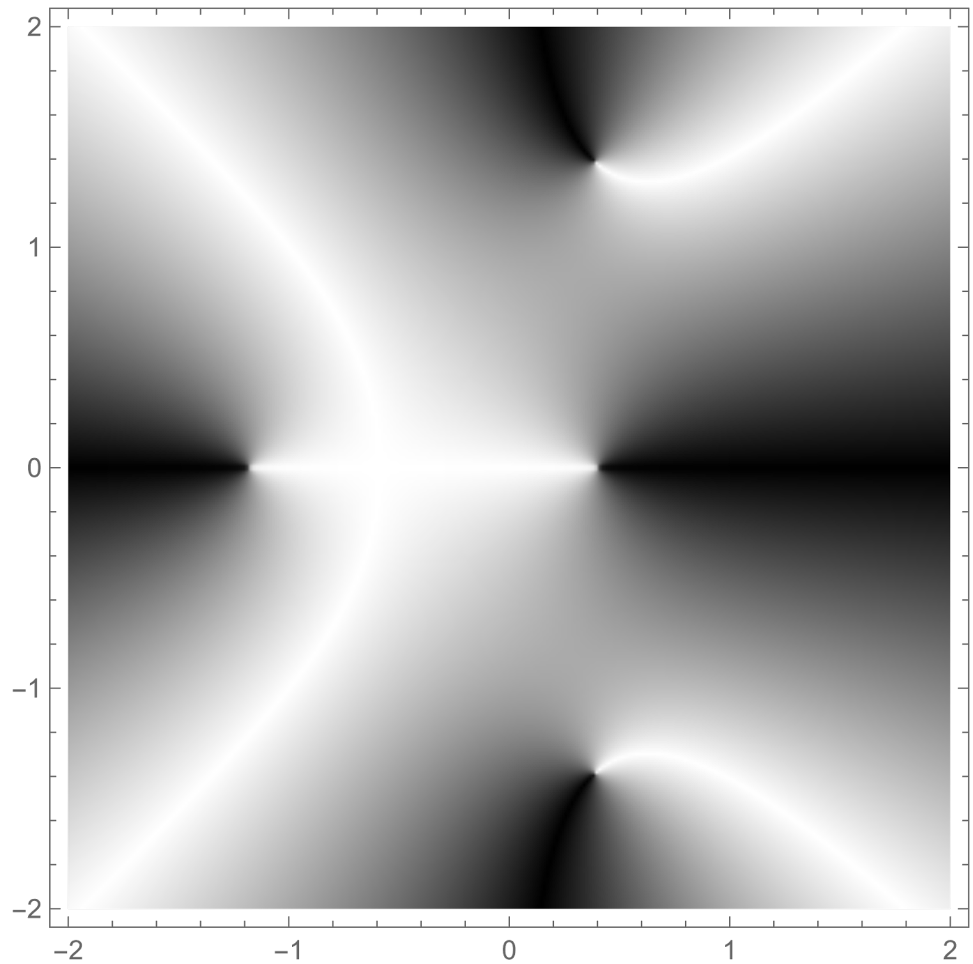

Fig. 4 : $f(z), z \in (-2-2i, 2+2i)$

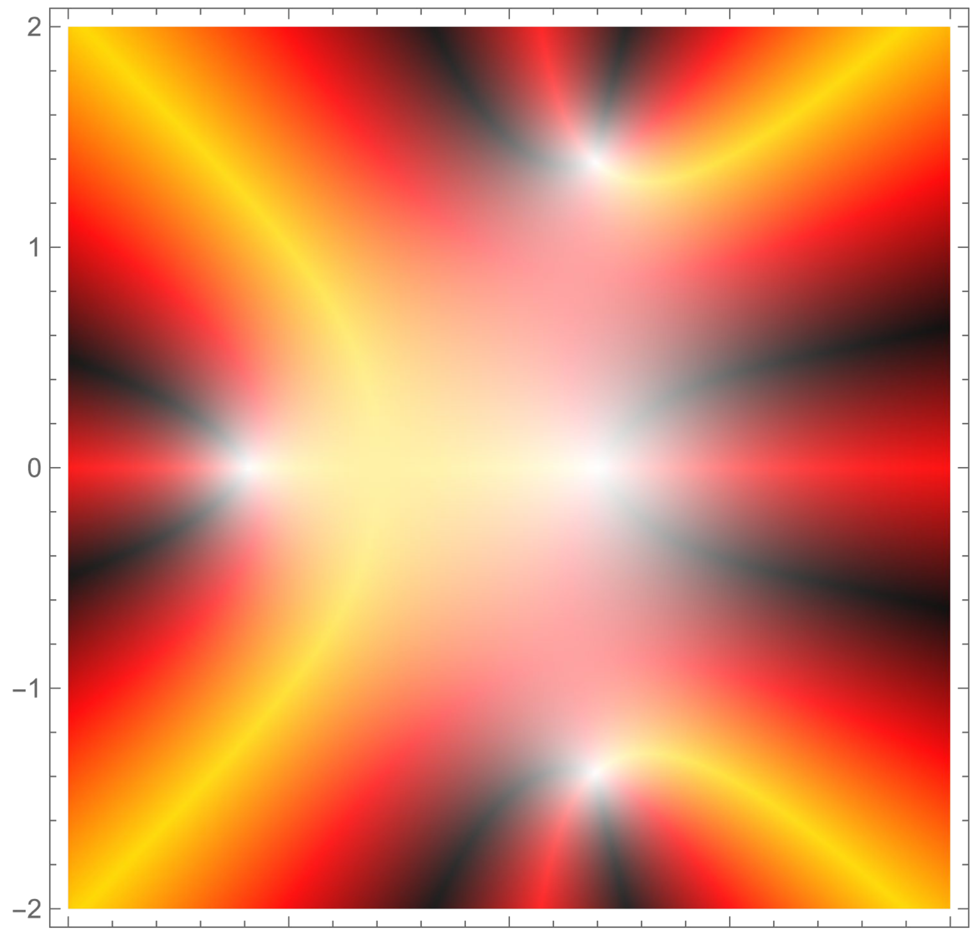

Fig. 5 : $f(z), z \in (-2-2i, 2+2i)$

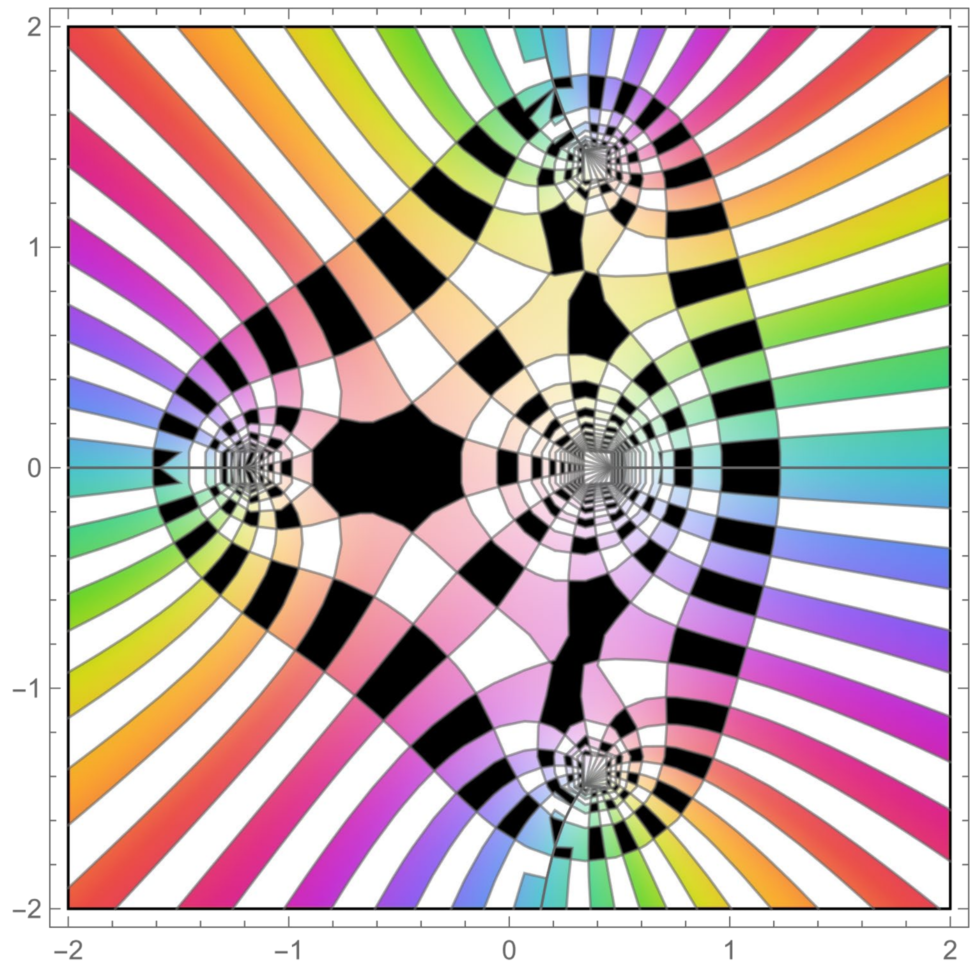

Fig. 6 : $f(z), z \in (-2-2i, 2+2i)$

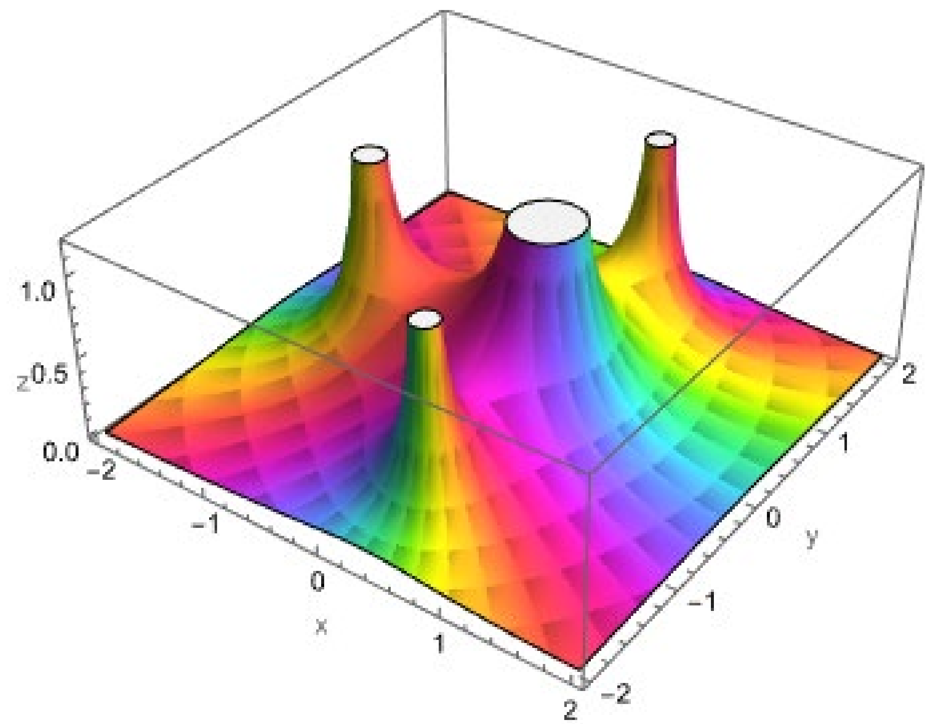

Fig. 7 : $f\left(z\right), z \in \left(-2-2i, 2+2i\right)$

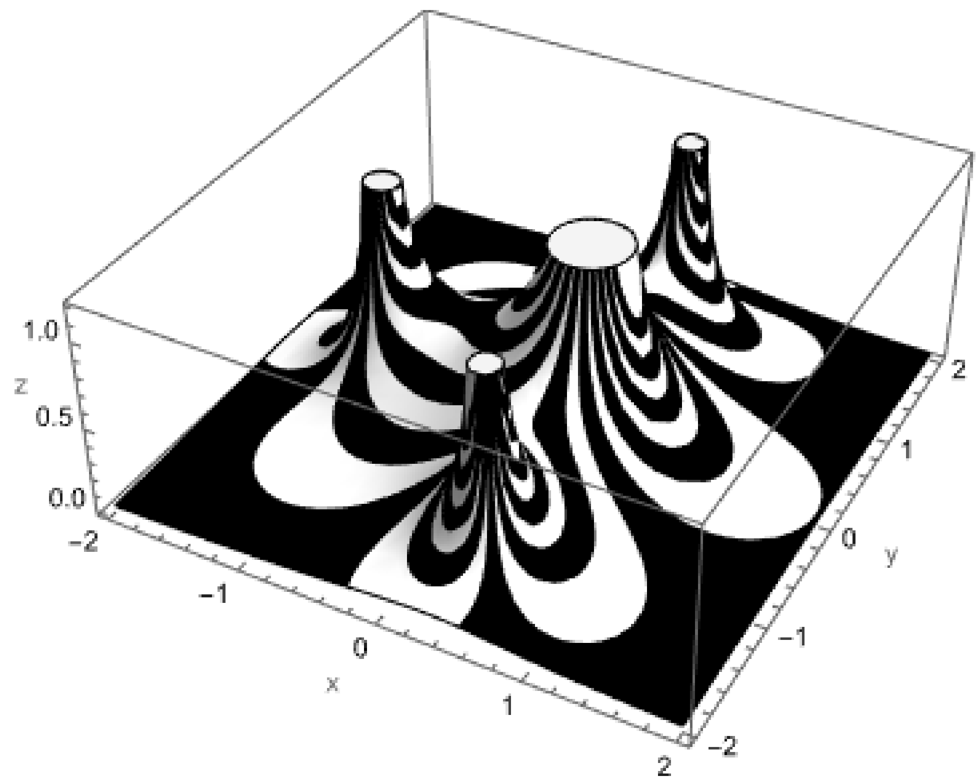

Fig. 8 : $f(z), z \in (-2-2i, 2+2i)$

References

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