Knot in geometrical optics

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We treat the geometrical optics as an Abelian $U(1)$ local gauge theory the same as the Abelian $U(1)$ Maxwell's gauge theory. We propose there exists a knot in a 3-dimensional Euclidean (flat) space of the geometrical optics (the eikonal equation) as a consequence there exists a knot in the Maxwell's theory in a vacuum. We formulate the Chern-Simons integral using an eikonal. We obtain the relation between the knot (the geometric optical helicity, an integer number) and the refractive index.

Keywords: *knot, geometrical optics, eikonal equation, Maxwell's theory, Abelian* $U(1)$ *local gauge theory, Chern-Simons integral, helicity, refractive index.*

It is commonly believed there exists no topological object in the linear theory, such as the Maxwell's theory. It is because of *a topological theory must be a non-linear theory*[1]. The existence of topological object, *a knot*, in the Maxwell's linear theory so far has not been well known[2]. How could a knot exist in the Maxwell's linear theory?

In the Maxwell's theory, the electromagnetic fields (*the set* of the solutions of Maxwell equations) in *vacuum* has *a subset field with a topological structure*[1]. Any electromagnetic field is *locally equal* (except in a zero measure set) but *globally different* to a subset field[1]. *The electromagnetic field is a linear field, but a subset field is a non-linear field. Both fields, the electromagnetic field and a subset of the electromagnetic field, are equal in the case of the weak field*[3]. It means that *a non-linear subset field theory* reduces to the Maxwell's linear theory in the case of the weak field. *A knot could exist in the Maxwell's theory because of the Maxwell's theory is the weak field limit*[3] *of a non-linear subset field theory.*

In this brief article, *we propose there exists a knot in the geometrical optics, as a solution of the eikonal equation.* The reason is, in fact, there exists a knot in the Maxwell's theory[1–3] and the geometrical optics (the eikonal equation) can be derived from the Maxwell's theory (Maxwell equations)[4–6]. We treat the geometrical optics as an Abelian $U(1)$ *local* gauge theory[7,8], the same as the Abelian $U(1)$ Maxwell's gauge theory. *To the best of our knowledge, the formulation of a knot in the geometrical optics has not been done yet*[1,2,9,10].

Let us consider *a map* of a subset field (consists of *a complex scalar field*) from a finite radius r to an infinite r which implies from the non-linear subset field to the linear field, the weak field. A scalar field has, *by definition*, the property that its value for a finite r depends on the magnitude and the direction of the position vector $\vec{r}$, but for an infinite r it is *well defined*[3] (depends on the magnitude of $\vec{r}$ only). In other words, for an infinite r, a scalar field is *isotropic*. Throughout this article we will work with the classical field.

The property of such a scalar field can be interpreted as a map $S^3 \to S^2$[1] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres, respectively. As maps of this kind *can be classified* in *homotopy classes*, labelled by *a topological invariant* called *the Hopf index*[1], *an integer number*. We see there exists (one) dimensional reduction in such map. We consider this dimensional reduction related to the isotropic (well defined) property of a scalar field for an infinite r. The property of a scalar field as a function of space seem likely in harmony with the property of space-time. The space-time could be locally non-isotropic, but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

In physics, the idea of a knot, *topologically stable matter*, had been proposed in 1868 by Lord Kelvin that *the atoms could be knots or links of vorticity lines of aether*[2]. *A knot is a smooth-embedding of a circle in* $\mathbb{R}^3$[10], a 3-dimensional Euclidean space[11]. Two knots are *equivalent* if one knot can be *deformed continuously* into the other *without crossing itself*[10].

In electrodynamics, *a knot could be formed by bending the electric and magnetic field lines* (the geometric concept of *magnetic lines of force* - those lines of force are today designated by the symbol $\vec{H}$, *the magnetic field* - is due to Faraday[12]) so that they could form *closed loops*[2]. *A set of closed loops in space forms a link*[13]. These closed loops can be *linked*[2] (although links do not actually need to be linked[14]). If two closed loops of field lines are *linked* then we have *a non-vanishing Gauss integral (Gauss linking integral). This linking could provide the topological structure*[2]. *The self-linking number (an integer number)* i.e. a non-vanishing Gauss integral describes *the knottedness*[2].

In mathematics, especially in algebraic topology, *a knot is defined by the Hopf index*[2]. The Hopf index is related to *the Hopf invariant*[1]. In turn, the Hopf invariant is related to *a non-trivial Hopf map*[15].

Suppose that we have a scalar field as a function of position vector, $\phi(\vec{r})$, with a property that, as we mentioned, can be interpreted using the non-trivial Hopf map written below[1,3]

$$\phi(\vec{r}) : S^3 \to S^2 \quad (1)$$

This non-trivial Hopf map is related to the Hopf invariant[15], $\mathcal{H}$, expressed as an integral[15–17]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \quad (2)$$

where ω is *a 1-form* on S^3[15]. The relation between the Hopf invariant and the Hopf index, h, can be written as[1]

$$\mathcal{H} = h\ \gamma^2 \quad (3)$$

where γ is *the total strength of the field*, that is the sum of the strengths of all *the tubes* formed by the integral lines of electric and magnetic fields[1].

The Hopf invariant have a deep relationship with *the Abelian Chern-Simons action*[15] *(the Chern-Simons integral)* and *self-helicity* in magnetohydrodynamics[15]. The Abelian Chern-Simons integral related to *the electromagnetic helicity*[2], h_{em}, can be written as[18,19]

$$h_{em} = S_{CS} = \int_{\mathbb{R}^3} \varepsilon^{\alpha\mu\nu}\ \vec{A}_\alpha\ \vec{F}_{\mu\nu}\ d^3x \quad (4)$$

where S_{CS} denotes the (Abelian) Chern-Simons action, $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$[19] denote a 3-dimensional space, $\vec{A}_\alpha$ is *the $U(1)$ gauge potential*[19] and $\vec{F}_{\mu\nu}$ is *the $U(1)$ gauge field tensor*[19] (the field strength tensor) written below[20]

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \quad (5)$$

Using the scalar field, ϕ, the $U(1)$ field strength can be written as[1]

$$\vec{F}_{\mu\nu} = f_{\mu\nu}(\phi) = \frac{1}{2\pi i} \frac{\partial_\mu \phi^*\ \partial_\nu \phi - \partial_\nu \phi^*\ \partial_\mu \phi}{(1 + \phi^* \phi)^2} \quad (6)$$

where[1]

$$\phi = a\ e^{i2\pi\sigma} \quad (7)$$

We interpret that the scalar field as *the scalar component* of the gauge potential, $\vec{A}_\mu = (\phi, \vec{A})$, $(\mu = 0, 1, 2, 3)$, where the gauge potential is the component of the *field* strength. We consider a (7) as *the scalar amplitude* as the consequence ϕ is the scalar field. We write down the scalar field (7) (also we will see the gauge potential) in "the wave notation" to describe the wave point of view we follow throughout this article.

The equation (6) is valid for the weak field only. It means that *in the case of the weak field, i.e. $\phi^*, \phi << 1$, $f_{\mu\nu}(\phi)$ has a linear form.* But *in general, $f_{\mu\nu}(\phi)$ has a non-linear form. The nonlinearity of $f_{\mu\nu}(\phi)$ is shown by* $\phi^*\phi$. If the field is weak then $\phi^*\phi << 1$, so the denominator in eq.(6) *can be taken* as being equal to one and $f_{\mu\nu}(\phi)$ is *equivalent* to the Maxwell linear theory[1]. We interpret the Maxwell's theory in a vacuum is the same as the weak field theory due to the field is taken far away from the source (electric charge or current).

Any electromagnetic field is *locally equal to* a subset field (i.e. any electromagnetic field can be obtained by patching together subset fields), except in a zero measure set[1]. This means that *the difference* between the set of the subset fields and all the electromagnetic fields in the Maxwell's theory in a vacuum is *global* instead of local, since the subset fields obey *the topological quantum condition* that the electromagnetic helicity (consists of electric and magnetic helicities) is equal to *an integer number*[1].

In Ranada works[1,3], because of the subset fields have well-defined property at infinity, so the subset fields can be interpreted as maps $S^3 \to S^2$, after identifying, via stereographic projection, $\mathbb{R}^3 \cup \{\infty\}$ with the sphere S^3 and the complete complex plane $C \cup \{\infty\}$ with the sphere S^2. These maps can be classified in *homotopy classes*, labelled by the value of the corresponding Hopf indexes, *the topological invariants*[1,3]. The other names of the topological invariant are *the topological charge, the winding number* (*the degree* of a continuous mapping)[21]. In physics, *the topological charge which is independent to the space metric tensor is interpreted as energy*[22].

Let us consider the geometrical optics as the gauge theory where the relation between the $U(1)$ gauge potential and *the eikonal (phase)*[23], ψ_1, is given by[7]

$$\vec{A}_\alpha = \vec{A}_\alpha^{\ U(1)} = \vec{a}_\alpha\ e^{i\psi} = \vec{a}_\alpha\ e^{i\frac{f_\theta}{c}(\psi_1 - ct)} \quad (8)$$

$$\psi_1 = \int_{x_1}^{x_2} n\ dx \text{ (1-dimensional space)} \quad (9)$$

$$\vec{A}_\alpha = \vec{a}_\alpha\ e^{iX\left(\int_{x_1}^{x_2} n\ dx - ct\right)} \quad (10)$$

where $\vec{a}_\alpha$ is *four-vector amplitudo*, f_θ is angular frequency, c is the speed of light in vacuum, t time and n *the refractive index*, $X = f_\theta/c$. Note here the phase, ψ_1, obeys the *Fermat's principle* $\delta\psi_1 = 0$. Eq.(10) shows explicitly the relation between the $U(1)$ gauge potential and the refractive index.

We see from eqs.(5), (6), (10) there exists the implicit relation between the scalar field and the refractive index. It looks like

$$\partial_\mu \phi^*\ \partial_\nu \phi \sim \partial_\mu \left\{ \vec{a}_\nu\ e^{iX\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)} \right\} \quad (11)$$

where the scalar field is a function of the refractive index also.

The refractive index (9) is *the real* scalar function of coordinates with positive values, *the slowness* at a point[7]. We consider from eq.(11) that the slowness of the refractive index corresponds to the weakness of the scalar field. The space of the weak field approximately represents the vacuum space. The weaker scalar field (in the area of infinite radius from the source), the smaller refractive index.

By substituting eqs.(5), (10) into (4), in the case of the 3-dimensional space, we obtain the Abelian Chern-Simons integral expressed in the refractive index related to *the geometric optical helicity*, $h_{\rm go}$, as follow

$$\int_{\mathbb{R}^3} \varepsilon^{\alpha\mu\nu}\ \vec{a}_\alpha\ e^{iX\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)} \left\{\partial_\mu \left[\vec{a}_\nu\ e^{iX\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)}\right] - \partial_\nu \left[\vec{a}_\mu\ e^{iX\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)}\right]\right\} d^3x =\ h_{\rm go} \quad (12)$$

where we replaced the electromagnetic helicity (4) to the geometric optical helicity or *the geometric optical knot*. Both, h_{em} and h_{go}, are *integer numbers*.

Eq.(12) shows explicitly the relation between the geometric optical knot and the refractive index. The refractive index is typically supplied as *known input, given*, and we seek *the solution, the phase*, ψ_1[7]. It means that *the geometric optical knot as an integer restricts the choice of the value of refractive index, so it makes the phase becomes singular*. *This phase singularity*[24] *where the phase is undefined*[24] *is the geometric optical knot solution*. In our case, *the geometric optical knot could exist in the weak field only*. We consider eq.(12) as a topological quantum condition[1].

So far, we formulate the theoretical existence of the geometric optical knot. *Does the electromagnetic (geometric optical) knot exist in universe or laboratory? Ball lightning*[25], probably, is an electromagnetic knot in universe[26]. *Tokamaks* and devices constructed to produce *fireball* are two possible laboratory settings to observe ball lightning[26]. *Knot of light* may be generated using *tightly focused circularly polarized laser beams*[27] or via *holographic metasurfaces*[24]. *We could observe the phase singularity at a location where the phase is undefined and the intensity of the field is zero*[24].

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