A Proof of the Erdös-Straus Conjecture

Zhang Tianshu

Emails: chinazhangtianshu@126.com; xinshijizhang@hotmail.com

Zhanjiang city, Guangdong province, China

Abstract

In this article, we classify positive integers step by step, and use the formulation to represent a certain class therein until all classes.

First, divide all integers ≥2 into 8 kinds, and formulate each of 7 kinds therein into a sum of 3 unit fractions.

For the unsolved kind, again divide it into 3 genera, and formulate each of 2 genera therein into a sum of 3 unit fractions.

For the unsolved genus, further divide it into 5 sorts, and formulate each of 3 sorts therein into a sum of 3 unit fractions.

For two unsolved sorts i.e. $4/(49+120c)$ and $4/(121+120c)$ where $c\geq 0$, let us depend on logical deduction to prove them separately.

AMS subject classification: 11D72; 11D45; 11P81

Keywords: Erdös-Straus conjecture; Diophantine equation; unit fraction

1. Introduction

The Erdös-Straus conjecture relates to Egyptian fractions. In 1948, Paul Erdös conjectured that for any integer n≥2, there are invariably

4/n=1/x+1/y+1/z, where x, y and z are positive integers; [1].

Later, Ernst G. Straus further conjectured that x, y and z satisfy x≠y, y≠z and z≠x, because there are convertible formulas 1/2r+1/2r=1/(r+1)+ 1/r(r+1) and 1/(2r+1)+1/(2r+1) =1/(r+1)+1/(r+1)(2r+1) where r≥1; [2].

Thus, the Erdös conjecture and the Straus conjecture are equivalent from each other, and they are called the Erdös-Straus conjecture collectively.

As a general rule, the Erdös-Straus conjecture states that for every integer n≥2, there are positive integers x, y and z, such that 4/n=1/x+1/y+1/z. Yet it remains a conjecture that has neither is proved nor disproved; [3].

2. Divide integers≥2 into 8 kinds and formulate 7 kinds therein

First, divide integers≥2 into 8 kinds, i.e. 8k+1with k≥1, and 8k+2, 8k+3, 8k+4, 8k+5, 8k+6, 8k+7, 8k+8, where k≥0, and arrange them as follows:

K\n:	8k+1,	8k+2,	8k+3,	8k+4,	8k+5,	8k+6,	8k+7,	8k+8
0,	①,	2,	3,	4,	5,	6,	7,	8,
1,	9,	10,	11,	12,	13,	14,	15,	16,
2,	17,	18,	19,	20,	21,	22,	23,	24,
3,	25,	26,	27,	28,	29,	30,	31,	32,
…,	…,	…,	…,	…,	…,	…,	…,	…,

Excepting n=8k+1, formulate each of other 7 kinds into 1/x+1/y+1/z:

(1) For n=8k+2, there are 4/(8k+2)=1/(4k+1)+1/(4k+2)+1/(4k+1)(4k+2);

(2) For n=8k+3, there are 4/(8k+3)=1/(2k+2)+1/(2k+1)(2k+2)+1/(2k+1)(8k+3);

(3) For n=8k+4, there are $4/(8k+4)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+1)(2k+2)$;

(4) For n=8k+5, there are $4/(8k+5)=1/(2k+2)+1/(8k+5)(2k+2)+1/(8k+5)(k+1)$;

(5) For n=8k+6, there are $4/(8k+6)=1/(4k+3)+1/(4k+4)+1/(4k+3)(4k+4)$;

(6) For n=8k+7, there are $4/(8k+7)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+2)(8k+7)$;

(7) For n=8k+8, there are $4/(8k+8)=1/(2k+4)+1/(2k+2)(2k+3)+1/(2k+3)(2k+4)$.

By this token, n as above 7 kinds of integers be suitable to the conjecture.

3. Divide the unsolved kind into 3 genera and formulate 2 genera therein

For the unsolved kind n=8k+1 with k≥1, we divide it by 3 and get 3 genera: (1) the remainder is 0 when k=1+3t; (2) the remainder is 2 when k=2+3t; (3) the remainder is 1 when k=3+3t, where t≥0, as listed below.

k:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15, ...
8k+1:	9,	17,	25,	33,	41,	49,	57,	65,	73,	81,	89,	97,	105,	113,	121, ...
The remainder:	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1, ...

Excepting the genus (3), the author formulates other 2 genera as follows:

(8) For (8k+1)/3 to the remainder=0 when k=1+3t with t≥0, there are $4/(8k+1)=1/(8k+1)/3+1/(8k+2)+1/(8k+1)(8k+2)$. Since there are $(8k+1)/3=8t+3$, so we confirm that (8k+1)/3 in the equation is an integer.

(9) For (8k+1)/3 to the remainder=2 when k=2+3t with t≥0, there are $4/(8k+1)=1/(8k+2)/3+1/(8k+1)+1/(8k+1)(8k+2)/3$. Since there are $(8k+2)/3=8t+6$, so we confirm that (8k+2)/3 in the equation is an integer.

4. Divide the unsolved genus into 5 sorts and formulate 3 sorts therein

For the unsolved genus (8k+1)/3 to the remainder=1 when k=3+3t with t≥0, i.e. 8k+1=25, 49, 73, 97, 121 etc. let us divide them into 5 sorts: 25+120c, 49+120c, 73+120c, 97+120c and 121+120c where c≥0, as listed below.

C\n:	25+120c,	49+120c,	73+120c,	97+120c,	121+120c,
0,	25,	49,	73,	97,	121,
1,	145,	169,	193,	217,	241,
2,	265,	289,	313,	337,	361,
…,	…,	…,	…,	…,	…,

Excepting n=49+120c and n=121+120c, formulate other 3 sorts as follows:

(10) For n=25+120c, there are 4/(25+120c)=1/(25+120c)+1/(50+240c)+1/(10+48c);

(11) For n=73+120c, there are 4/(73+120c)=1/(73+120c)(10+15c)+1/(20+30c)+ 1/(73+120c)(4+6c) ;

(12) For n=97+120c, there are 4/(97+120c)=1/(25+30c)+1/(97+120c)(50+60c)+ 1/(97+120c)(10+12c).

For each of listed above 12 equations which express part 4/n= 1/x+1/y+1/z, please each reader self to make a check respectively.

5. Proving the sort 4/(49+120c)=1/x+1/y+1/z

For a proof of the sort 4/(49+120c), it means that when c is equal to each of positive integers plus 0, there always are $4/(49+120c)=1/x+1/y+1/z$.

After c is given any value, $4/(49+120c)$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below:

$4/(49+120c)$

$= 1/(13+30c) + 3/(13+30c)(49+120c)$

$= 1/(14+30c) + 7/(14+30c)(49+120c)$

$= 1/(15+30c) + 11/(15+30c)(49+120c)$

…

$= 1/(13+\alpha+30c) + (3+4\alpha)/(13+\alpha+30c)(49+120c)$, where $\alpha\geq 0$ and $c\geq 0$

…

As listed above, it is observed that we can first let $1/(13+\alpha+30c)=1/x$.

In addition to $1/(13+\alpha+30c)=1/x$, we will go to prove $(3+4\alpha)/(13+\alpha+30c)(49+120c) = 1/y +1/z$, where $c\geq 0$ and $\alpha\geq 0$, *ut infra*.

Proof· First, let us analyse $3+4\alpha$ on the place of numerator. We can be seen that except $3+4\alpha$ as one numerator, it can also be expressed as the sum of an even number plus an odd number to act as two numerators, i.e. $(4\alpha+3)$, $(4\alpha+2)+1$, $(4\alpha+1)+2$, $(4\alpha)+3$, $(4\alpha-1)+4$, $(4\alpha-2)+5$, $(4\alpha-3)+6$, ...

If there are two addends on the place of numerator, then, these two addends are regarded as two matching numerators, and two matching numerators are denoted by ψ and φ, also, there is $\psi>\varphi$ between them.

In numerators of a denominator, largest ψ is denoted as ψ_1. In is obvious that ψ_1 matches with smallest φ, so there are $\psi_1=4\alpha+2$ and smallest $\varphi=1$.

Second, let us look at (13+α+30c)(49+120c) as the denominator, in reality, it merely needs us to take 13+α+30c as the denominator, and still reserve 49+120c for later.

In the fraction (4α+3)/(13+α+30c), let each α be assigned a number for each time, according to the order α= 0, 1, 2, 3, ...

Then, the denominator of the fraction (4α+3)/(13+α+30c) is able to be assigned into infinite more consecutive positive integers. As the value of α goes up, accordingly, numerators are getting more and more, and new adding numerators for each time are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, 13+α+30c as denominators and 4α+3, ψ and φ as numerators are listed below.

13+α+30c,	α,	(4α+3),	(4α+2)+1,	(4α+1)+2,	(4α)+3,	(4α-1)+4,	(4α-2)+5,	(4α-3)+6, ...	
13+30c,	0,	3,	2+1,	1+2					
14+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6	
15+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...	
16+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...	
17+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...	
...,	...,	...,	...,	...,	...,	...,	...,	...,	...

As can be seen from the list above, every denominator as (13+α+30c) corresponds with two special matching numerators as ψ_1 and 1, from this, we get the unit fraction $1/(13+\alpha+30c)$.

For the unit fraction $1/(13+\alpha+30c)$, multiply its denominator by $49+120c$ reserved in the front, then, we get the unit fraction $1/(13+\alpha+30c)(49+120c)$, and let $1/(13+\alpha+30c)(49+120c)$ be equal to $1/y$.

After that, we go to prove that $\psi_1/(13+\alpha+30c)$ is an unit fraction, namely prove that $(4\alpha+2)/(13+\alpha+30c)$ is an unit fraction.

Since $4\alpha+2$ as numerators be even numbers, such that the denominators $(13+\alpha+30c)$ must be even numbers. Only in this case, it is going to be able to reduce the fraction. Thus, α in the fraction $(4\alpha+2)/(13+\alpha+30c)$ must be odd numbers.

After assign odd numbers 1, 3, 5 and otherwise to α and each resulting fraction divided by 2, the fraction $(4\alpha+2)/(13+\alpha+30c)$ is turned into the fraction $(3+4t)/(k+15c)$ identically, where $c\geq0$, $t\geq0$ and $k\geq7$. The point is that $3+4t$ and $k+15c$ after the valuations coexist within a fraction in the sense that they have same ordinal number in order from small to large, i.e. $(3+4t)/(k+15c)=3/(7+15c)$, $7/(8+15c)$, $11/(9+15c)$, ...

Such being the case, let us divide the numerator and denominator of the fraction $(3+4t)/(k+15c)$ by $3+4t$, then, we get a temporary indeterminate unit fraction, and its denominator is $(k+15c)/(3+4t)$ and its numerator is 1.

Thus, we are necessary to prove that $(k+15c)/(3+4t)$ as the denominator can be a positive integer in which case $c\geq0$, $t\geq0$ and $k\geq7$.

For the fraction (k+15c)/(3+4t), due to k≥7, k+15c after the valuations are infinite more consecutive positive integers, while 3+4t =3, 7, 11 and otherwise positive odd numbers. The key is that each number of 3+4t after the valuations can seek its integer's multiples within infinite more consecutive positive integers of k+15c, in which case c equals each of positive integers plus 0.

As is known to all, there is a positive integer that contains the odd factor 2n+1 within 2n+1 consecutive positive integers, where n=1, 2, 3, ...

Like that, there is a positive integer that contains the odd factor 3+4t within 3+4t consecutive positive integers of k+15c, whatever odd number 3+4t is equal to. It is obvious that a fraction that consists of such a positive integer as the numerator and 3+4t as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer. That is to say, (k+15c)/(3+4t) as the denominator of the aforementioned temporary indeterminate unit fraction can become a positive integer, and represent the positive integer as μ. Then, in this case, the fraction (3+4t)/(k+15c) is expressed as 1/μ.

For the unit fraction 1/μ, multiply its denominator by 49+120c reserved in the front, then, we get the unit fraction 1/μ(49+120c), and let 1/μ(49+120c) be equal to 1/z.

If $3+4\alpha$ serve as one numerator, then, we can still prove $(3+4\alpha)/(13+\alpha+30c)(49+120c)=1/y$ by the same principles and methods as in the proof concerning $\psi_1/(13+\alpha+30c)(49+120c)=1/z$.

When $3+4\alpha$ serve as one numerator and from this get an unit fraction, we can multiply the denominator of the unit fraction by 2 to make a sum of two identical unit fractions, afterwards, convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r=1/(r+1)+1/r(r+1)$. Thus it can be seen, $(3+4\alpha)/(13+\alpha+30c)(49+120c)$ is absolutely able to be expressed into a sum of two each other's -distinct unit fractions, where $c\geq 0$ and $\alpha\geq 0$.

To sum up, we have proved $4/(49+120c)=1/x+1/y+1/z$, where $c\geq 0$.

6. Proving the sort 4/(121+120c)=1/x+1/y+1/z

The proof in this section is exactly similar to that in the section 5. Namely, for a proof of the sort $4/(121+120c)$, it means that when c is equal to each of positive integers plus 0, there always are $4/(121+120c)=1/x+1/y+1/z$.

After c is given any value, $4/(121+120c)$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below.

$4/(121+120c)$

$= 1/(31+30c) + 3/(31+30c)(121+120c)$,

$= 1/(32+30c) + 7/(32+30c)(121+120c)$,

$= 1/(33+30c) + 11/(33+30c)(121+120c)$,

…

$= 1/(31+\alpha+30c) + (3+4\alpha)/(31+\alpha+30c)(121+120c)$, where $\alpha \geq 0$ and $c \geq 0$.

…

As listed above, it is observed that we can first let $1/(31+\alpha+30c)=1/x$.

In addition to $1/(31+\alpha+30c)=1/x$, we will go to prove $(3+4\alpha)/(31+\alpha+30c)(121+120c) =1/y +1/z$, where $c \geq 0$ and $\alpha \geq 0$, *ut infra*.

Proof· First, let us analyse $3+4\alpha$ on the place of numerator. We can be seen that except $3+4\alpha$ as one numerator, it can also be expressed as the sum of an even number and an odd number to act as two numerators, i.e. $(4\alpha+3)$, $(4\alpha+2)+1$, $(4\alpha+1)+2$, $(4\alpha)+3$, $(4\alpha-1)+4$, $(4\alpha-2)+5$, $(4\alpha-3)+6$, ...

If there are two addends on the place of numerator, then, these two addends are regarded as two matching numerators, and two matching numerators are denoted by ψ and φ, also, there is $\psi>\varphi$ between them.

In numerators of a denominator, largest ψ is denoted as ψ_1. In is obvious that ψ_1 matches with smallest φ, so there are $\psi_1=4\alpha+2$ and smallest $\varphi=1$.

Second, let us look at $(31+\alpha+30c)(121+120c)$ as the denominator, in reality, it merely needs us to take $31+\alpha+30c$ as the denominator, and still reserve $121+120c$ for later.

In the fraction $(4\alpha+3)/(31+\alpha+30c)$, let each α be assigned a number for each time, according to the order α= 0, 1, 2, 3, ...

Then, the denominator of the fraction $(4\alpha+3)/(31+\alpha+30c)$ is able to be assigned into infinite more consecutive positive integers. As the value of α goes up, accordingly, numerators are getting more and more, and new adding numerators for each time are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, $31+\alpha+30c$ as denominators and $4\alpha+3$, ψ and φ as numerators are listed below.

$31+\alpha+30c$,	α,	$(4\alpha+3)$,	$(4\alpha+2)+1$,	$(4\alpha+1)+2$,	$(4\alpha)+3$,	$(4\alpha-1)+4$,	$(4\alpha-2)+5$,	$(4\alpha-3)+6$,	...
31+30c,	0,	3,	2+1,	1+2					
32+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6	
33+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...	
34+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...	
35+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...	
...,	...,	...,	...,	...,	...,	...,	...,	...,	...

As can be seen from the list above, every denominator as $(31+\alpha+30c)$ corresponds with two special matching numerators as ψ_1 and 1, from this, we get the unit fraction $1/(31+\alpha+30c)$.

For the unit fraction $1/(31+\alpha+30c)$, multiply its denominator by $121+120c$ reserved in the front, then, we get the unit fraction $1/(31+\alpha+30c)(121+120c)$, and let $1/(31+\alpha+30c)(121+120c)$ be equal to $1/y$.

After that, we go to prove that $\psi_1/(31+\alpha+30c)$ is an unit fraction, namely prove that $(4\alpha+2)/(31+\alpha+30c)$ is an unit fraction.

Since $4\alpha+2$ as numerators be even numbers, such that the denominators

(31+α+30c) must be even numbers. Only in this case, it is going to be able to reduce the fraction. Thus, α in the fraction (4α+2)/(31+α+30c) must be odd numbers.

After assign odd numbers 1, 3, 5 and otherwise to α and each resulting fraction divided by 2, the fraction (4α+2)/(31+α+30c) is turned into the fraction (3+4t)/(m+15c) identically, where c≥0, t≥0 and m≥16. The point is that 3+4t and m+15c after the valuations coexist within a fraction in the sense that they have same ordinal number in order from small to large, i.e. (3+4t)/(m+15c)=3/(16+15c), 7/(17+15c), 11/(18+15c), ...

Such being the case, let us divide the numerator and denominator of the fraction (3+4t)/(m+15c) by 3+4t, then, we get a temporary indeterminate unit fraction, and its denominator is (m+15c)/(3+4t) and its numerator is 1. Thus, we are necessary to prove that (m+15c)/(3+4t) as the denominator can be a positive integer in which case c≥0, t≥0 and m≥16.

For the fraction (m+15c)/(3+4t), due to m≥16, m+15c after the valuations are infinite more consecutive positive integers, while 3+4t=3, 7, 11 and otherwise positive odd numbers. The key is that each number of 3+4t after the valuations can seek its integer's multiples within infinite more consecutive positive integers of m+15c, in which case c equals each of positive integers plus 0.

As is known to all, there is a positive integer that contains the odd factor

2n+1 within 2n+1 consecutive positive integers, where n=1, 2, 3, ...

Like that, there is a positive integer that contains the odd factor 3+4t within 3+4t consecutive positive integers of m+15c,whatever odd number 3+4t is equal to. It is obvious that a fraction that consists of such a positive integer as the numerator and 3+4t as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer.

That is to say, (m+15c)/(3+4t) as the denominator of the aforementioned temporary indeterminate unit fraction can become a positive integer, and represent the positive integer as λ. Then, in this case, the fraction (3+4t)/(m+15c) is expressed as $1/\lambda$.

For the unit fraction $1/\lambda$, multiply its denominator by 121+120c reserved in the front, then, we get the unit fraction $1/\lambda(121+120c)$, and let $1/\lambda(121+120c)$ be equal to 1/z.

If $3+4\alpha$ serve as one numerator, then, we can still prove $(3+4\alpha)/(31+\alpha+30c)(121+120c) = 1/y$ by the same principles and methods as in the proof concerning $\psi_1/(31+\alpha+30c)(121+120c)=1/z$.

When $3+4\alpha$ serve as one numerator and from this get an unit fraction, we can multiply the denominator of the unit fraction by 2 to make a sum of two identical unit fractions, afterwards, convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r=1/(r+1)+1/r(r+1)$.

Thus it can be seen, $(3+4\alpha)/(31+\alpha+30c)(121+120c)$ is absolutely able to be expressed into a sum of two each other's -distinct unit fractions, where $c\geq 0$ and $\alpha\geq 0$.

To sum up, we have proved $4/(121+120c)=1/x+1/y+1/z$, where $c\geq 0$.

The proof was thus brought to a close. As a consequence, the Erdös-Straus conjecture is tenable.

References

[1] J. W. Sander, *On 4/x=1/x+1/y+1/z and Iwaniec' Half Dimensional Sieve*; Journal of Number Theory, Volume 46, Issue 2, February 1994, Pages 123-136;

https://www.sciencedirect.com/science/article/pii/S0022314X84710080?via%3Dihub

[2] Wolfram Math world, the web's most extensive mathematics resource;

https://mathworld.wolfram.com/Erdos-StrausConjecture.html

[3] Konstantine Zelator, *An ancient Egyptian problem: the diophantine equation 4/n=1/x+1/y+1/z, n>or=2*; arXiv: 0912.2458;

https://arxiv.org/search/?query=on+4%2Fn%3D1%2Fx%2B1%2Fy%2B1%2Fz&searchtype=title&source=header

The Conflict of Interest Statement

The author declares no conflicts of interest regarding the publication of this article. The headline, subtitles, all sentences and illustrations in this article are original and have not been published or used elsewhere.