A Proof of the Erdös-Straus Conjecture

Zhang Tianshu

Email: chinazhangtianshu@126.com;

xinshijizhang@hotmail.com

Zhanjiang city, Guangdong province, China

Abstract

In this article, the author classifies positive integers step by step, and use the formulation to represent a certain class therein up to all classes.

First, divide all integers ≥2 into 8 kinds, and formulate each of 7 kinds therein into a sum of 3 unit fractions.

For the unsolved kind, again divide it into 3 genera, and formulate each of 2 genera therein into a sum of 3 unit fractions.

For the unsolved genus, further divide it into 5 sorts, and formulate each of 3 sorts therein into a sum of 3 unit fractions.

For two unsolved sorts i.e. $4/(49+120c)$ and $4/(121+120c)$ where $c\geq 0$, let us depend on logical deduction to prove them separately.

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1. Introduction

The Erdös-Straus conjecture relates to Egyptian fractions. In 1948, Paul Erdös conjectured that for any integer n≥2, there are invariably

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$4/n=1/x+1/y+1/z$, where x, y and z are positive integers; [1].

Later, Ernst G. Straus further conjectured that x, y and z satisfy $x\neq y$, $y\neq z$ and $z\neq x$, because there are convertible formulas $1/2r+1/2r=1/(r+1)+1/r(r+1)$ and $1/(2r+1)+1/(2r+1)=1/(r+1)+1/(r+1)(2r+1)$ where $r\geq 1$; [2].

Thus, the Erdös conjecture and the Straus conjecture are equivalent from each other, and they are called the Erdös-Straus conjecture collectively.

As a general rule, the Erdös-Straus conjecture states that for every integer $n\geq 2$, there are positive integers x, y and z, such that $4/n=1/x+1/y+1/z$. Yet it remains a conjecture that has neither been proved nor disproved; [3].

2. Divide integers≥2 into 8 kinds and formulate 7 kinds therein

First, divide integers≥2 into 8 kinds, i.e. 8k+1with $k\geq 1$, and 8k+2, 8k+3, 8k+4, 8k+5, 8k+6, 8k+7, 8k+8, where $k\geq 0$, and arrange them as follows:

K\n:	8k+1,	8k+2,	8k+3,	8k+4,	8k+5,	8k+6,	8k+7,	8k+8
0,	①,	2,	3,	4,	5,	6,	7,	8,
1,	9,	10,	11,	12,	13,	14,	15,	16,
2,	17,	18,	19,	20,	21,	22,	23,	24,
3,	25,	26,	27,	28,	29,	30,	31,	32,
…,	…,	…,	…,	…,	…,	…,	…,	…,

Excepting n=8k+1, formulate each of other 7 kinds into $1/x+1/y+1/z$:

(1) For n=8k+2, there are $4/(8k+2)=1/(4k+1)+1/(4k+2)+1/(4k+1)(4k+2)$;

(2) For n=8k+3, there are $4/(8k+3)=1/(2k+2)+1/(2k+1)(2k+2)+1/(2k+1)(8k+3)$;

(3) For n=8k+4, there are 4/(8k+4)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+1)(2k+2);

(4) For n=8k+5, there are 4/(8k+5)=1/(2k+2)+1/(8k+5)(2k+2)+1/(8k+5)(k+1);

(5) For n=8k+6, there are 4/(8k+6)=1/(4k+3)+1/(4k+4)+1/(4k+3)(4k+4);

(6) For n=8k+7, there are 4/(8k+7)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+2)(8k+7);

(7) For n=8k+8, there are 4/(8k+8)=1/(2k+4)+1/(2k+2)(2k+3)+1/(2k+3)(2k+4).

By this token, n as above 7 kinds of integers be suitable to the conjecture.

3. Divide the unsolved kind into 3 genera and formulate 2 genera therein

For the unsolved kind n=8k+1 with k≥1, divide it by 3 and we get 3 genera to (1) the remainder 1, (2) the remainder 2 and (3) the remainder 0 as listed below, and "Remainder" in the list i.e. the remainder of (8k+1)/3.

k:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15, ...
8k+1:	9,	17,	25,	33,	41,	49,	57,	65,	73,	81,	89,	97,	105,	113,	121, ...
Remainder:	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1, ...

Excepting the genus (1), formulate other 2 genera as follows:

(8) For (8k+1)/3 to the remainder 0, there are 4/(8k+1)= 1/(8k+1)/3+1/(8k+2)+1/(8k+1)(8k+2) with k≥1. Let k=3t+1 with t≥0, then there are (8k+1)/3 =8t+3, thus confirm that (8k+1)/3 is an integer.

(9) For (8k+1)/3 to the remainder 2, there are 4/(8k+1)= 1/(8k+2)/3+1/(8k+1)+1/(8k+1)(8k+2)/3 with k≥2. Let k=3t+2 with t≥0, then there are (8k+2)/3=8t+6, thus confirm that (8k+2)/3 is an integer.

4. Divide the unsolved genus into 5 sorts and formulate 3 sorts therein

For the unsolved genus n= 8k+1 with k≥3, divide it by 3 to the remainder 1, i.e. 8k+1=25, 49, 73, 97, 121 etc. We divide it into 5 sorts, to wit 25+120c, 49+120c, 73+120c, 97+120c and 121+120c, where c≥0, as listed below.

C\n:	25+120c,	49+120c,	73+120c,	97+120c,	121+120c,
0,	25,	49,	73,	97,	121,
1,	145,	169,	193,	217,	241,
2,	265,	289,	313,	337,	361,
…,	…,	…,	…,	…,	…,

Excepting n=49+120c and 121+120c, formulate other 3 sorts as follows:

(10) For n=25+120c, there are 4/(25+120c)=1/(25+120c)+1/(50+240c)+1/(10+48c);

(11) For n=73+120c, there are 4/(73+120c)=1/(73+120c)(10+15c)+1/(20+30c)+ 1/(73+120c)(4+6c) ;

(12) For n=97+120c, there are 4/(97+120c)=1/(25+30c)+1/(97+120c)(50+60c)+ 1/(97+120c)(10+12c).

For each of listed above 12 equations which express 4/n=1/x+1/y+1/z, please each reader self to make a check respectively.

5. Proving the sort 4/(49+120c)=1/x+1/y+1/z

For a proof of the sort 4/(49+120c), it means that when c is equal to each of positive integers plus 0, there are 4/(49+120c)=1/x+1/y+1/z .

After c is endowed with any value, $4/(49+120c)$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below:

$4/(49+120c)$

$= 1/(13+30c) + 3/(13+30c)(49+120c)$

$= 1/(14+30c) + 7/(14+30c)(49+120c)$

$= 1/(15+30c) + 11/(15+30c)(49+120c)$

…

$= 1/(13+\alpha+30c) + (3+4\alpha)/(13+\alpha+30c)(49+120c)$, where $\alpha\geq 0$ and $c\geq 0$

…

As listed above, it is observed that we can first let $1/(13+\alpha+30c)=1/x$, so in addition to $1/(13+\alpha+30c)=1/x$, the author is going to prove $(3+4\alpha)/(13+\alpha+30c)(49+120c) = 1/y +1/z$, *ut infra*.

***Proof*·** When c=0, such as $\alpha=1$, there are $(3+4\alpha)/(13+\alpha+30c)(49+120c)$ $= 7/14\times 49 =1/99 +1/(98\times 99)$;

When c=1, such as $\alpha=9$, there are $(3+4\alpha)/(13+\alpha+30c)(49+120c) = 39/52\times 169 = 1/(4\times 169) + 1/(2\times 169)$.

This shows that when c=0 and 1, $(3+4\alpha)/(13+\alpha+30c)(49+120c)$ has been proved to be expressed by $1/y +1/z$ respectively.

In following paragraphs, when $c\geq 2$ and $\alpha\geq 0$, the author will prove $(3+4\alpha)/(13+\alpha+30c)(49+120c)=1/y+1/z$.

For the numerator $3+4\alpha$, excepting itself as an integer, we express also it into the sum of two integers, i.e. $1+(2+4\alpha)$, $2+(1+4\alpha)$, $3+(4\alpha)$, $(3+3\alpha)+\alpha$, $(2+3\alpha)+(1+\alpha)$, $(1+3\alpha)+(2+\alpha)$, $(3+\alpha)+3\alpha$, $(3+2\alpha)+2\alpha$ and $(2+2\alpha)+(1+2\alpha)$.

For the denominator $(13+\alpha+30c)(49+120c)$, in reality, merely need us to take $13+\alpha+30c$ as the denominator, and still reserve $49+120c$ for later.

Since $13+\alpha$ can express every integer≥ 13 due to $\alpha\geq 0$, of course, the denominator $13+\alpha+30c$ can also express every integer≥ 73 where $c\geq 2$.

That is to say, $13+\alpha$ can express infinite more consecutive integers≥ 13, and the denominator $13+\alpha+30c$ can also express infinite more consecutive integers≥ 73.

As stated, on the place of the numerator, there be either an integer i.e. $3+4\alpha$ or two addends which apart $3+4\alpha$ into.

Such being the case, once α is determined to be any one positive integer or 0, then there be only one or two definite integers as one or two numerators. On other, for infinite more consecutive integers≥ 73, either they are all greater than each numerator, or they involve infinite more consecutive integers which are greater than each numerator.

So, in these infinite more consecutive integers which are greater than each numerator, there are absolutely such integers whose each is integer's multiples of corresponding numerator.

It is obvious that each such proper fraction whose denominator is integer's multiples of corresponding numerator can be turned into an unit fraction via the reduction .

If 3+4α serve as an integer, and from this get an unit fraction, then we can multiply the denominator by 2 to make a sum of two identical unit fractions, afterwards, again convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r=1/(r+1)+1/r(r+1)$.

Therefore, $(3+4\alpha)/(13+\alpha+30c)$ can be expressed into a sum of two each other's -distinct unit fractions, where $c\geq2$ and $\alpha\geq0$.

Let a sum of two unit fractions which express $(3+4\alpha)/(13+\alpha+30c)$ into be written as $1/\mu+1/\nu$, where μ and ν are positive integers.

For $1/\mu+1/\nu$, multiply two denominators in them by 49+120c reserved in the front, then you get $1/\mu(49+120c)+1/\nu(49+120c)=1/y+1/z$.

To sum up, the author has proved $4/(49+120c)=1/x+1/y+1/z$, where $c\geq0$.

6. Proving the sort 4/(121+120c)=1/x+1/y+1/z

Overall, the proof in this section is similar to that in section 5.

For a proof of the sort $4/(121+120c)$, it means that when c is equal to each of positive integers plus 0, there are $4/(121+120c)=1/x+1/y+1/z$.

After c is endowed with any value, $4/(121+120c)$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below:

$4/(121+120c)$

$= 1/(31+30c) + 3/(31+30c)(121+120c)$,

$= 1/(32+30c) + 7/(32+30c)(121+120c)$,

$= 1/(33+30c) + 11/(33+30c)(121+120c)$,

…

$= 1/(31+\alpha+30c) + (3+4\alpha)/(31+\alpha+30c)(121+120c)$, where $\alpha \geq 0$ and $c \geq 0$.

…

As listed above, it is observed that we can first let $1/(31+\alpha+30c)=1/x$, so in addition to $1/(31+\alpha+30c)=1/x$, the author is going to prove $(3+4\alpha)/(31+\alpha+30c)(121+120c) = 1/y + 1/z$, *ut infra*.

***Proof*·** When c=0, such as $\alpha=2$, there are $(3+4\alpha)/(31+\alpha+30c)(121+120c) = 11/33\times121 = 1/(3\times11^2+1) + 1/(3\times11^2)(3\times11^2+1)$;

When c=1, such as $\alpha=2$, there are $(3+4\alpha)/(31+\alpha+30c)(121+120c) = 11/63\times241 = 1/(2\times3\times241) + 1/(2\times3^2\times7\times241)$.

This shows that when c=0 and 1, $(3+4\alpha)/(31+\alpha+30c)(121+120c)$ has been proved to be expressed by $1/y + 1/z$ respectively.

In following paragraphs, when $c \geq 2$ and $\alpha \geq 0$, the author will prove $(3+4\alpha)/(31+\alpha+30c)(121+120c)=1/y+1/z$.

For the numerator $3+4\alpha$, excepting itself as an integer, we express also it into the sum of two integers, i.e. $1+(2+4\alpha)$, $2+(1+4\alpha)$, $3+(4\alpha)$, $(3+3\alpha)+\alpha$, $(2+3\alpha)+(1+\alpha)$, $(1+3\alpha)+(2+\alpha)$, $(3+\alpha)+3\alpha$, $(3+2\alpha)+2\alpha$ and $(2+2\alpha)+(1+2\alpha)$.

For the denominator $(31+\alpha+30c)(121+120c)$, in reality, merely need us to take $31+\alpha+30c$ as the denominator, and still reserve $121+120c$ for later.

Since $31+\alpha$ can express every integer≥ 31 due to $\alpha\geq 0$, of course, the denominator $31+\alpha+30c$ can also express every integer≥ 91 where $c\geq 2$.

That is to say, $31+\alpha$ can express infinite more consecutive integers≥ 31, and the denominator $31+\alpha+30c$ can also express infinite more consecutive integers≥ 91.

As stated, on the place of the numerator, there be either an integer i.e. $3+4\alpha$, or two addends which apart $3+4\alpha$ into.

Such being the case, once α is determined to be any one positive integer or 0, then there be only one or two definite integers as one or two numerators. On other, for infinite more consecutive integers≥ 91, either they are all greater than each numerator, or they involve infinite more consecutive integers which are greater than each numerator.

So, in these infinite more consecutive integers which are greater than each numerator, there are absolutely such integers whose each is integer's multiples of corresponding numerator.

It is obvious that each such proper fraction whose denominator is integer's multiples of corresponding numerator can be turned into an unit fraction via the reduction .

If $3+4\alpha$ serve as an integer, and from this get an unit fraction, then we can multiply the denominator by 2 to make a sum of two identical unit fractions, afterwards, again convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r=1/(r+1)+1/r(r+1)$.

Therefore, $(3+4\alpha)/(31+\alpha+30c)$ can be expressed into a sum of two each other's -distinct unit fractions, where $c\geq 2$ and $\alpha\geq 0$.

Let a sum of two unit fractions which express $(3+4\alpha)/(31+\alpha+30c)$ into be written as $1/\beta+1/\xi$, where β and ξ are positive integers.

For $1/\beta+1/\xi$, multiply two denominators in them by $121+120c$ reserved in the front, then you get $1/\beta(121+120c)+1/\xi\,(121+120c)=1/y+1/z$.

To sum up, the author has proved $4/(121+120c) =1/x+1/y+1/z$,where $c\geq 0$.

The proof was thus brought to a close. As a consequence, the Erdös-Straus conjecture is tenable.

References

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