What Makes Goldbach's Conjecture Correct

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Abstract

A direct proof shows Goldbach's conjecture is correct. It is as simple as can be imagined. A table consisting of two rows is used. The lower row counts from 0 to any n and and the top row counts down from $2n$ to n. All columns will have all numbers that add to $2n$. Using the idea of a sieve, all composites are crossed out and only columns with primes are left. Without loss of generality, an example shows that primes, ones that sum to $2n$ will always be left in such columns.

Introduction

Hardy and Apostol spend some time on Goldbach's conjecture [1, 2]. The conjecture has it that every even number can be expressed as the sum of two primes. And indeed it is fascinating to try it on some even numbers and quickly find some instances.

Various angles for finding examples are possible. One can just add any two odd primes and the result will be even. So $3 + 5 = 8$, $5 + 7 = 10$, and so on. This will give lots of even sums fast. If one allows, which the conjecture does, non distinct primes then we can add $3+3 = 6$ and $5+5 = 10$ and start to sense that, indeed, you might just get all evens.

Thence to the central rub with this conjecture. You get lots and lots of pairs that sum to ever larger evens. A plethora of evidence starts accumulating and one can quickly lose sight of the goal of proving it is generally true. Things inevitably get complicated and the schemes get more and more elaborate; and annoyingly, every now and again extremely simple. At least that was my experience.

Here is a scheme for the latter leading to the former. An even can be expressed in the form $2n - 2 + 2$. So take $44 = 44 - 2 + 2 = 42 + 2$ and start subtracting from 42 and adding to 2; immediately $42 - 1 = 41$ and $2 + 1 = 3$ and both 41 and 3 are primes. Keep going and you will have to get all composite and prime combinations. But how to you know you will ever get two primes at the same time? Thence to ever more elaborate considerations of say expressing each number using all primes via a division algorithm. So

$$44 = 0_2^{22} 2_3^{14} 4_5^{8} 2_7^{6} 0_{11}^{4} 5_{13}^{3} 5_{13}^{3} 10_{17}^{2} 6_{19}^{2} 21_{23}^{1} 15_{29}^{1} 13_{31}^{1} 37_7^{1} 3_{41}^{1} 1_{43}^{1} \tag{1}$$

and for any even eventually one will get *exponents, the multiples* of one and prime pairs seem to emerge. We see, for example, 13_{31}^1, 37_7^1, and 3_{41}^1 in (1) and these translate to winning pairs of primes: $\{(13, 31), (37, 7), (3, 41)\}$ that all sum to 44. But do we have assurances that the remainders, the 13, 37, and 3 will always or at least one time be prime(s)?

Thence to the frustration of seeing such hopeful *evidence* without getting closer to a proof.

Pulled both ways between easy ways to get them all and difficult ways that seem to give lots of granularity, like (1), that seems to suggest something complicated might work – well frustration and obsession seem to wax. You scratch your head a lot.

All of this is to say how one can forget the general intuition: it must be something very simple. Hint: expand your ideas out from just the primes and just the odds and odd primes and consider all numbers. Use a sieve. Here goes.

A Sieve

Given an even $2n$, we know $2 \ldots n$ has lots of early primes and $n + 1 \ldots 2n$ has at least one prime per Bertrand's postulate. Use two rows to count up to $2n$ with the lower row consisting of $1 \ldots n$ and the top row consisting of $n \ldots 2n$, counting up from right to left. Scratch off all but the first prime multiples of the first and second rows. What's left are primes on the first row and any primes on the second row line up with lower row primes because they survived the sieve. These pairs are odd primes that sum to $2n$.

Here's is an example: Table 1. This procedure is called a sieve. The Greek mathematician Eratosthenes used it to find primes. We are essentially doing the same thing.

~~20~~	19	~~18~~	17	~~16~~	~~15~~	~~14~~	13	~~12~~	~~11~~	10
~~0~~	~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~

Table 1: The survivors of the lower row index the survivors of the top row.

The Theory for the Sieve

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
1		44	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22
2		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
3	2																							
4	3																							
5	5																							

Figure 1: Caption for goldbach-tableFor44-527-120

How do we know that the primes on the bottom row will line up with those of the top? Using (1), we'll demonstrate the logic. Consider the primes less than $\sqrt{44}$: $P(44) = \{2, 3, 5\}$. Any number between 2 and 44 that doesn't have a factor from $P(44)$ will be a prime [2]. Using (1), we know that

$$44 = 2 \cdot 22 + 0 \ = 3 \cdot 14 + 2 \text{ and } 44 = 5 \cdot 8 + 4.$$

Let

$$GC(2) = \{(a, b) | a = 2(22 - m) \text{ and } b = 2m + 0 \text{ with } 0 \leq m \leq 22\}$$

$$GC(3) = \{(a, b) | a = 3(14 - m) \text{ and } b = 3m + 2 \text{ with } 0 \leq m \leq 14\}$$

and

$$GC(5) = \{(a, b) | a = 3(8 - m) \text{ and } b = 3m + 4 \text{ with } 0 \leq m \leq 8\}.$$

The ordered pairs in these sets correspond to the columns in a sieve for 44: Figure 1. That is

$$GC(2)=\{(44,0),(42,2),(40,4),(38,6),(36,8),(34,10),(32,12),(30,14),(28,16),(26,18),(24,20),(22,22)\},$$

$$GC(3)=\{(42,2),(39,5),(36,8),(33,11),(30,14),(27,17),(24,20),(21,23)\},$$

and

$$GC(5)=\{(40,4),(35,9),(30,14),(25,19),(20,24),(15,29),(10,34),(5,39)\}.$$

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
1		44	43	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22
2		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
3	2	x		x		x		x		x		x		x		x		x		x		x		x
4	3			x			x			x			x			x			x			x	x	
5	5					x	x				x	x				x	x				x			

Figure 2: Caption for goldbach-tableFor44-573-122-last

Using an “x” to cross reference the two tables, Figure 2 shows that four columns remain. These columns have no numbers divisible by one or more of $P(44)$ and so give two prime numbers, as needed. In more technical jargon, we are using the principle of cross classification as developed in Apostol [1]. This procedure works for any $2n$.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[2] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.