What Makes Goldbach's Conjecture Correct

Timothy W. Jones

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Abstract

After a brief review of Goldbach's conjecture and certain mathematical highlights, we prove Goldbach's conjecture is true.

Introduction

Certainly Goldbach's conjecture is the ultimate in easily expressed and understood difficult number theory problems. What could be simpler than every even number is the sum of two primes?

Apostol spends some time in the beginning and end of his *Introduction to Analytic Number Theory* to give contemporary research results [1]. Chen's result is mentioned: allow the second number to have just two (not one) prime factor and the result is proven. It's a two page proof, not easy. The last chapter of Apostol is on *partitions* which he evolves, one can sense, from the mathematical frustration at getting no where with Goldbach's conjecture. I sense Waring's problem and the like are a kind of sour grape story. If we can't get anything concrete with the easiest sum of two primes, Goldbach what can be done with arbitrary sums of numbers to various powers?

Perhaps, like many open number theory problems, what drives researchers to write programs that test results on hard numbers (into the trillions) must be the biting, irritating sense that there is some easy explanation that we just can't yet see. For me the primordial example of mathematical puzzles resolved with solutions that eventually turn out to be thought of as simple, obvious and beyond reproach are at least two: Cantor's work on set theory and the positional number system.

The former has counter-intuitive elements. Consider that the limit of $(-1/n, 1/n)$ as n goes to infinity is the empty set. This despite the fact that for each $n > 0$ the number of points in these open intervals is uncountable [3]. So how can something that's uncountably infinite go to something that has no elements, zero, nada without passing through countably infinite and just plain finite first? Yet, these symbols, our minds do understand it, believe it, accept this counter-intuitive truth and we use it to build lots of great mathematics. The continuum hypotheses is really not an hypotheses anymore – there is not an in between $\aleph_0$ and $\aleph_1$.

The other wonder of mathematics, the other great success story is the positional number system and the use, great of late, of various number bases. We can add, subtract, multiply, and divide with relative alacrity. We can, using the binary number base, get machines to do these operations for us in blinding speed. Who invented the positional number system and the idea of various number bases? After much research, reading Dickson and exploring Jstor it isn't particularly clear that one person or that some school of thought came up with the idea. It may be that the idea is so fundamental, so basic historians, mathematicians themselves don't feel inclined to give credit to anyone or anything – its just too obviously the right way to conceive of all numbers – all natural numbers anyway. Reals, decimals are another story.

These two success stories lead to a solution to Goldbach's conjecture: its true.

Sets and Positions

One can express Goldbach's conjecture in this form: let $\{primes\}$ be all the primes between 3 and $2n - 3$, then

$$\frac{2n - \{primes\}}{\{primes\}} = 1 \tag{1}$$

is solvable. That is there exists $p_1, p_2 \in \{primes\}$ such that $p_1 + p_2 = 2n$.

Take the even number 108. Are there primes such that $(108 - p_1)/p_2$ is 1? Yes: 103 and 5.

Using $2 \cdot 9 = 18$, consider the set of odd numbers given by

$$\text{Odds}(18) = \{3, 5, 7, 9, 11, 13, 15\}$$

x	1	2	3	4	5	6	7
	C	P	P	C	P	P	P
Num(x)	15	13	11	9	7	5	3
P	P	P	C	P	P	C	P
Den(x)	3	5	7	9	11	13	15

Table 1: P=prime; C=Composite. Solutions are in columns with two Ps. Positions give solutions via a dot product. If the dot product is greater than 1, solutions exist.

and define the set

$$\{18\} \equiv \{18 - x | x \in \text{Odds}(18)\},$$

then

$$\{18\} = \text{Odds}(18) \text{ or, in general } \{2n\} = \text{Odds}(2\text{n}).$$

But as all primes are odds in Odd(2n) all primes will occur in $\{2n\}$.

We can define a function with a table. In Table 1, let $Num(x)$ and $Den(x)$ be functions defined for the numerator and the denominator of (1). For every prime in row 3 there is that same prime in row 5. So, for example, $Num(3) = 13$ and $Den(7) = 13$, so (1) is solvable, if primes exist between n and $2n$. But using Bertrand's postulate [2], at least one prime exists between n and $2n$.

A binary string giving 1 for odd primes and 0 for odd composite numbers combined with its reverse string using a dot product will always evaluate to more than 1 yielding the existence of the requisite two primes. So for the example $2n = 18$: $[0, 1, 1, 0, 1, 1, 1] \cdot [1, 1, 1, 0, 1, 1, 0] > 0$ indicates the existence of at least two of the requisite primes. As such sequences are independent of $2n$, the general case holds.

Diagonal Matrices

In this section we will explore (attempt) an induction proof that for every $n > 3$ there exists two prime numbers p_1 and p_2 (not necessarily distinct) such that $p_1 + p_2 = 2n$. Diagonal matrices are such that sums of two are sums of their respective diagonals. We can evolve a systematic way to enumerate cases that remain invariant relative to solving a given $2n$ for these primes.

In Table 2, we execute a sieve action. Write primes down diagonals and composite next to primes along rows. We will call these tables *rectangular composite* tables. As one moves from outer to inner, composites eliminate solutions. This table is for $2 * 9 = 18$. The outer most two, 3 and 15 are eliminated as 15 is next to the prime 13 and so is a composite. The next most outer pair, 5 and 13 are both primes, so that is a solution; continuing 7 and 11 are another solution. The center number is 9 and $9 + 9 = 18$, but 9 is a composite so it is eliminated. It is automatically eliminated as it is next to the prime 7, the position for composites.

3					
	5				
		7	9		
			11		
				13	15

Table 2: A rectangular composite table for $2n = 18$.

The next even is 20 and we will add 17 a prime to this table. As 17 is a prime, it will go on a separate row and immediately be paired with the prime 3: $3 + (2n - 3) = 2n$. In general, if the next $2n - 3$ is a prime, we will always have this solution: Table 3.

3_1					
	5_2				
		7_3	9_4		
			11_4		
				13_3	15_2
					17_1

Table 3: A rectangular composite table for $2n = 20$. Note when a prime row is added, one solution is immediate.

Another interesting idea for using diagonal matrices as solution sieves for Goldbach solutions is to make new columns for each number and place it on the row of the primes appropriate for its prime factorization, if it is composite. This establishes a predictive pattern and yields square matrices: Table 4. We will call these *prime square* tables.

3			9			15
	5					15
		7				
			0			
				11		
					13	
						0

Table 4: Prime square matrix for 18.

When a square prime diagonal is reversed and summed with this, call it, complement matrix we have solutions: Table 5. One has to ignore rows in either matrix with a 0 in them.

0			9			15
	13					15
		11				
			0			
				7		
					5	
						3

Table 5: Prime square complement matrix for 18.

Let's try the next $2 * 11 = 22$. Its rectangular composite table is given in Table 6. We immediately have two solutions: one per 19 being a prime and one per 11, the center being a prime. We have left the previous indices in to show how they will not change for the top left half diagonal and will adjust for the lower right. These composite rectangular matrices are the easiest to construct, but they don't yield as much information as prime square matrices.

We will add 17 and 19 to a prime square matrix: Tables 7, 8. The next iteration, using the periodicity suggested by the top rows, gives a 0 row as 21 is a composite: Table 9.

3_1						
	5_2					
		7_3	9_4			
			11_4			
				13_3	15_2	
					17_1	
						19_1

Table 6: Composite rectangular matrix for 22.

3			9			15	
	5					15	
		7					
			0				
				11			
					13		
						0	
							17

Table 7: Prime square matrix for 20.

3			9			15		
	5					15		
		7						
			0					
				11				
					13			
						0		
							17	
								19

Table 8: Prime square matrix for 22.

These prime square matrices seem to be invariant. When we add a composite or prime row the first half of the diagonal will not be adjusted. The

3			9			15			21
	5					15			
		7							21
			0						
				11					
					13				
						0			
							17		
								19	
									0

Table 9: Prime square matrix for 24.

second half, per Bertrand's postulate will always have a prime in it; per the asymmetry of prime distributions given by the composite indicators we know that not all zero columns for the prime square and its complement can be the same. These features show Goldbach's conjecture is true. In Tables 10 and 11 we add the next two odd numbers, 23, a prime and 25, a composite. We can predict both types based on the periodicity given in top rows: if the mod values are non-zero for all primes, the number is a new prime, if a zero mod value is to occur in a row, then the number is composite. If a prime, we have at least two solutions; if the number is composite, the induction proceeds as the zero rows are asymmetric.

3			9			15			21	
	5					15				
		7							21	
			0							
				11						
					13					
						0				
							17			
								19		
									0	
										23

Table 10: Prime square matrix for 26.

3			9			15			21		
	5					15					25
		7							21		
			0								
				11							
					13						
						0					
							17				
								19			
									0		
										23	
											0

Table 11: Prime square matrix for 28.

Conclusion

A pigeon hole argument also applies.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*. New York: Springer.

[2] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[3] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.