Helmholtzian Invariants

The Helmholtzian operator factorization is:

$\mathbf{J} \equiv D_B D_A \mathbf{f} = ((\Box - |m|^2))\mathbf{f}$

where:

$$D_B \equiv \begin{pmatrix} -D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0 & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \quad \& \quad D_A \equiv \begin{pmatrix} -D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

Theorem I.1: For m_i constants

and:

$D_i^{+} \equiv (\partial_i + m_i)$, $D_i^{-} \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^{+} & 0 \\ 0 & D_i^{-} \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^{-} & 0 \\ 0 & D_i^{+} \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^{-} \\ D_i^{+} & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^{+} \\ D_i^{-} & 0 \end{pmatrix}$$

and:

$$D_B \equiv \begin{pmatrix} -D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0 & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \quad \& \quad D_A \equiv \begin{pmatrix} -D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

then:

$$D_B D_A = \begin{pmatrix} \left(D_0 D_0^{\Updownarrow} + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) & \left(D_0 D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow}\right) & \left(-D_0 D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_0^{\Updownarrow} + D_1 D_3^{\Updownarrow}\right) & (D_0 D_1 - D_3^{\Leftrightarrow} D \cdots \\ \left(D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_0^{\Updownarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2 D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_0 D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_0^{\Updownarrow} + D_2 D_3^{\Leftrightarrow}\right) & (D_3^{\Leftrightarrow} D_1 + D_0 D \cdots \\ \left(-D_2^{\Leftrightarrow} D_0^{\Updownarrow} - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_1^{\Leftrightarrow} + D_3 D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_0 D_0^{\Updownarrow} + D_3 D_3^{\Updownarrow}\right) & (-D_2^{\Leftrightarrow} D_1 + D_1^{\Leftrightarrow} \cdots \\ \left(D_1^{\Updownarrow} D_0^{\Updownarrow} - D_2^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Updownarrow} D_2^{\Leftrightarrow} - D_0^{\Updownarrow} D_1^{\Updownarrow}\right) & \left(D_1^{\Updownarrow} D_3^{\Leftrightarrow} + D_2^{\Updownarrow} D_0^{\Updownarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_2^{\Leftrightarrow} + D_2^{\Updownarrow} D_1^{\Leftrightarrow} + D_3^{\Updownarrow} D_0^{\Updownarrow} - D_0^{\Updownarrow} D_3^{\Updownarrow}\right) & (D_1^{\Updownarrow} D_1 + D_2^{\Updownarrow} D \cdots \end{pmatrix}$$

$$D_A D_B = \begin{pmatrix} \left(D_0^{\Updownarrow} D_0 + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) & \left(-D_0^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow} D_0 - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow}\right) & \left(D_0^{\Updownarrow} D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow} D_0 + D_1 D_3^{\Updownarrow}\right) & (D_0^{\Updownarrow} D_1 - D_3^{\Leftrightarrow} D_2 \cdots \\ \left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0^{\Updownarrow} D_0 + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2 D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow} D_2^{\Leftrightarrow} - D_0^{\Updownarrow} D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_0 + D_2 D_3^{\Updownarrow}\right) & (-D_3^{\Leftrightarrow} D_1 + D_0^{\Updownarrow} D \cdots \\ \left(D_2^{\Leftrightarrow} D_0 - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_1^{\Leftrightarrow} + D_3 D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_0^{\Updownarrow} D_0 + D_3 D_3^{\Updownarrow}\right) & (D_2^{\Leftrightarrow} D_1 - D_1^{\Leftrightarrow} D \cdots \\ \left(D_1^{\Updownarrow} D_0 + D_2^{\Updownarrow} D_3^{\Leftrightarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_1^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_3^{\Leftrightarrow} + D_2^{\Updownarrow} D_0 + D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_2^{\Leftrightarrow} + D_2^{\Updownarrow} D_1^{\Leftrightarrow} + D_3^{\Updownarrow} D_0 - D_0 D_3^{\Updownarrow}\right) & (D_1^{\Updownarrow} D_1 + D_2^{\Updownarrow} D_2 \cdots \end{pmatrix}$$

Proof:

$$D_B D_A = \begin{pmatrix} -D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0 & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \begin{pmatrix} -D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

$$\Rightarrow D_B D_A = \begin{pmatrix} \left(D_0 D_0^{\Updownarrow} + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) & \left(D_0 D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow}\right) & \left(-D_0 D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_0^{\Updownarrow} + D_1 D_3^{\Updownarrow}\right) & (D_0 D_1 - D_3^{\Leftrightarrow} \cdots \\ \left(D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_0^{\Updownarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2 D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_0 D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_0^{\Updownarrow} + D_2 D_3^{\Leftrightarrow}\right) & (D_3^{\Leftrightarrow} D_1 + D_0 \cdots \\ \left(-D_2^{\Leftrightarrow} D_0^{\Updownarrow} - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_1^{\Leftrightarrow} + D_3 D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_0 D_0^{\Updownarrow} + D_3 D_3^{\Updownarrow}\right) & (-D_2^{\Leftrightarrow} D_1 + D \cdots \\ \left(D_1^{\Updownarrow} D_0^{\Updownarrow} - D_2^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Updownarrow} D_2^{\Leftrightarrow} - D_0^{\Updownarrow} D_1^{\Updownarrow}\right) & \left(D_1^{\Updownarrow} D_3^{\Leftrightarrow} + D_2^{\Updownarrow} D_0^{\Updownarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_2^{\Leftrightarrow} + D_2^{\Updownarrow} D_1^{\Leftrightarrow} + D_3^{\Updownarrow} D_0^{\Updownarrow} - D_0^{\Updownarrow} D_3^{\Updownarrow}\right) & (D_1^{\Updownarrow} D_1 + D_2^{\Updownarrow} \cdots \end{pmatrix}$$

$$D_A D_B = \begin{pmatrix} -D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix} \begin{pmatrix} -D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0 & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

$$\Rightarrow D_A D_B = \begin{pmatrix} \left(D_0^{\Updownarrow} D_0 + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) & \left(-D_0^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow} D_0 - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow}\right) & \left(D_0^{\Updownarrow} D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow} D_0 + D_1 D_3^{\Updownarrow}\right) & (D_0^{\Updownarrow} D_1 - D_3^{\Leftrightarrow} \cdots \\ \left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0^{\Updownarrow} D_0 + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2 D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow} D_2^{\Leftrightarrow} - D_0^{\Updownarrow} D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_0 + D_2 D_3^{\Updownarrow}\right) & (-D_3^{\Leftrightarrow} D_1 + \cdots \\ \left(D_2^{\Leftrightarrow} D_0 - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_1^{\Leftrightarrow} + D_3 D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_0^{\Updownarrow} D_0 + D_3 D_3^{\Updownarrow}\right) & (D_2^{\Leftrightarrow} D_1 - \cdots \\ \left(D_1^{\Updownarrow} D_0 + D_2^{\Updownarrow} D_3^{\Leftrightarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_1^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_3^{\Leftrightarrow} + D_2^{\Updownarrow} D_0 + D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow} D_2^{\Leftrightarrow} + D_2^{\Updownarrow} D_1^{\Leftrightarrow} + D_3^{\Updownarrow} D_0 - D_0 D_3^{\Updownarrow}\right) & (D_1^{\Updownarrow} D_1 + D \cdots \end{pmatrix}$$

□

Lemma I.2.1.1-(**BAc1r1**): For m_i constants
and:

$D_i^{+} \equiv (\partial_i + m_i) \quad , \quad D_i^{-} \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^{+} & 0 \\ 0 & D_i^{-} \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^{-} & 0 \\ 0 & D_i^{+} \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^{-} \\ D_i^{+} & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^{+} \\ D_i^{-} & 0 \end{pmatrix}$$

then:

$$\left(D_0 D_0^{\Updownarrow} + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) = \begin{pmatrix} (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 + m_3 m_3 + m_2 m_2 + m_1 m_1) & 0 \\ 0 & (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 \end{pmatrix}$$

Proof:

$$\left(D_0 D_0^{\Updownarrow} + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) = \left(\begin{pmatrix} (\partial_0 + m_0) & 0 \\ 0 & (\partial_0 - m_0) \end{pmatrix}\begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} + \begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_3 - \\ (\partial_3 + m_3) & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} (\partial_0 + m_0)(\partial_0 - m_0) + (\partial_3 + m_3)(\partial_3 - m_3) + (\partial_2 + m_2)(\partial_2 - m_2) + (\partial_1 + m_1)(\partial_1 - m_1) & \\ 0 & (\partial_0 - m_0)(\partial_0 + \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 + m_3 m_3 + m_2 m_2 + m_1 m_1) & \\ 0 & (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (\end{pmatrix}$$

□

Lemma I.2.1.2-(**BAc1r2**): For m_i constants:
and:

$D_i^{+} \equiv (\partial_i + m_i) \quad , \quad D_i^{-} \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^{+} & 0 \\ 0 & D_i^{-} \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^{-} & 0 \\ 0 & D_i^{+} \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^{-} \\ D_i^{+} & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^{+} \\ D_i^{-} & 0 \end{pmatrix}$$

then:

$$\left(D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_0 D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) = \left(\begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix}\begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} - \begin{pmatrix} (\partial_0 + m_0) & 0 \\ 0 & (\partial_0 - m_0) \end{pmatrix}\begin{pmatrix} 0 & (\partial_3 - \\ (\partial_3 + m_3) & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} -(\partial_1 - m_1)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_1 - m_1) & (\partial_3 - m_3)(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_3 - m_3) \\ (\partial_3 + m_3)(\partial_0 - m_0) - (\partial_0 - m_0)(\partial_3 + m_3) & -(\partial_1 + m_1)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_1 + m_1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.1.3-(**BAc1r3**): For m_i constants:
and:

$D_i^{+} \equiv (\partial_i + m_i) \quad , \quad D_i^{-} \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^{+} & 0 \\ 0 & D_i^{-} \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^{-} & 0 \\ 0 & D_i^{+} \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^{-} \\ D_i^{+} & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^{+} \\ D_i^{-} & 0 \end{pmatrix}$$

then:

$$\left(-D_2^{\Leftrightarrow} D_0^{\Updownarrow} - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_2^{\Leftrightarrow} D_0^{\Updownarrow} - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0 D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) = -\begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix}\begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} - \begin{pmatrix} 0 & (\partial_1 - m_1) \\ (\partial_1 + m_1) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_3 - \\ (\partial_3 + m_3) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_1 - m_1)(\partial_3 + m_3) + (\partial_3 + m_3)(\partial_1 - m_1) & -(\partial_2 - m_2)(\partial_0 + m_0) + (\partial_0 + m_0)(\partial_2 - m_2) \\ -(\partial_2 + m_2)(\partial_0 - m_0) + (\partial_0 - m_0)(\partial_2 + m_2) & -(\partial_1 + m_1)(\partial_3 - m_3) + (\partial_3 - m_3)(\partial_1 + m_1) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (m_1 m_3 - m_3 m_1) & -\partial_2 m_0 + m_2\partial_0 - \partial_0 m_{2n} + m_0\partial_2 + (m_2 m_0 - m_0 m_2) \\ (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (m_2 m_0 - m_0 m_2) & +\partial_1 m_3 - m_1\partial_3 + \partial_3 m_{1n} - m_3\partial_1 + (m_1 m_3 - m_3 m_1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.1.4-(**BAc1r4**): For m_i constants:
and:

$D_i^{+} \equiv (\partial_i + m_i) \quad , \quad D_i^{-} \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} \ , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} \ , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} \ , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_1^{\Updownarrow}D_0^{\Updownarrow} - D_2^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_1^{\Updownarrow}D_0^{\Updownarrow} - D_2^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Updownarrow}\right) = \left(\begin{pmatrix} D_{1m} & 0 \\ 0 & D_{1m}^+ \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix}\begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} D_{1m}D_{0n}^- - D_{0m}^-D_{1n}^- & -D_{2m}^-D_{3n}^- + D_{3m}^-D_{2n}^- \\ -D_{2m}^+D_{3n}^+ + D_{3m}^+D_{2n}^+ & D_{1m}^+D_{0n}^+ - D_{0m}^+D_{1n}^+ \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_1 - m_1)(\partial_0 - m_0) - (\partial_0 - m_0)(\partial_1 - m_1) & -(\partial_2 - m_{2m})(\partial_3 - m_3) + (\partial_3 - m_3)(\partial_2 - m_2) \\ -(\partial_2 + m_2)(\partial_3 + m_3) + (\partial_3 + m_3)(\partial_2 + m_2) & (\partial_1 + m_1)(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_1 + m_1) \end{pmatrix}$$

$$= \left(\begin{matrix} (+m_1 - m_{1m})\,\partial_0 + \left(+m_0 \ - m_0\right)\partial_1 + (+m_1m_0 - m_0m_1) & (+m_3 - m_3)\,\partial_2 + \left(m_2 \ - m_2\right)\partial_3 + (-m_2m_3 + m_3 \\ \left(+m_3 \ - m_3\right)\partial_2 + (+m_2 - m_2)\,\partial_3 + (-m_2m_3 + m_3m_2) & (+m_1 - m_1)\partial_0 + (m_0 - m_{0m})\,\partial_1 + (+m_1m_0 - m_0 \end{matrix}\right.$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.2.1-(**BAc2r1**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i) \quad , \quad D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} \ , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} \ , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} \ , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_0D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_0^{\Updownarrow} - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_0D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_0^{\Updownarrow} - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}\right) = \left(\begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} -D_{2m}^-D_{1n}^+ + D_{1m}^+D_{2n}^- & D_{0m}^+D_{3n}^- - D_{3m}^-D_{0n}^+ \\ D_{0m}^-D_{3n}^+ - D_{3m}^+D_{0n}^- & -D_{2m}^+D_{1n}^- + D_{1m}^-D_{2n}^+ \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_2 - m_2)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_2 - m_2) & (\partial_0 + m_0)(\partial_3 - m_3) - (\partial_3 - m_3)(\partial_0 + m_0) \\ (\partial_0 - m_0)(\partial_3 + m_3) - (\partial_3 + m_3)(\partial_0 - m_0) & -(\partial_2 + m_2)(\partial_1 - m_1) + (\partial_1 - m_1)(\partial_2 + m_2) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_2m_1 - m_1m_2) & (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_0m_3 + m_3m_0) \\ (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_0m_3 + m_3m_0) & (+m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_2m_1 - m_1m_2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.2.2-(**BAc2r2**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i) \quad , \quad D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} \ , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} \ , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} \ , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}\right) = \left(\begin{matrix} (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) - (m_3m_3 + m_0m_0 + m_1m_1 + m_2m_2) & 0 \\ 0 & (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) - (m_3m_3 + m_0m \end{matrix}\right.$$

Proof:

$$\left(D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}\right) = \left(\begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} D_{3m}^-D_{3n}^+ + D_{0m}^+D_{0n}^- + D_{1m}^-D_{1n}^+ + D_{2m}^+D_{2n}^- & 0 \\ 0 & D_{3m}^+D_{3n}^- + D_{0m}^-D_{0n}^+ + D_{1m}^-D_{1n}^+ + D_{2m}^-D_{2n}^+ \end{pmatrix}$$

$$= \left(\begin{matrix} (\partial_3 - m_3)(\partial_3 + m_3) + (\partial_0 + m_0)(\partial_0 - m_0) + (\partial_1 - m_1)(\partial_1 + m_1) + (\partial_2 + m_2)(\partial_2 - m_2) & \\ 0 & (\partial_3 + m_3)(\partial_3 - \end{matrix}\right.$$

$$= \left(\begin{matrix} (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) + (m_3 - m_3)\partial_3 + (m_0 - m_{0n})\partial_0 + (m_1 - m_1)\partial_{1n} + (m_2 - m_2)\partial_2 - (m_3m_3 + m \\ 0 \end{matrix}\right.$$

$$= \begin{pmatrix} (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) - (m_3m_3 + m_0m_0 + m_1m_1 + m_2m_2) & 0 \\ 0 & (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) - (m_3m_3 \end{pmatrix}$$

□

Lemma I.2.2.3-(**BAc2r3**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}$, $D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}$, $D_i^{\Leftarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$, $D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$

then:

$$\left(-D_2^{\Leftarrow}D_3^{\Leftarrow} + D_1^{\Leftarrow}D_0^{\Updownarrow} - D_0D_1^{\Leftarrow} + D_3D_2^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_2^{\Leftarrow}D_3^{\Leftarrow} + D_1^{\Leftarrow}D_0^{\Updownarrow} - D_0D_1^{\Leftarrow} + D_3D_2^{\Updownarrow}\right) = \left(-\begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} -D_{2m}^-D_{3n}^+ + D_{3m}^+D_{2n}^- & D_{1m}^-D_{0n}^+ - D_{0m}^+D_{1n}^- \\ D_{1m}^+D_{0n}^- - D_{0m}^-D_{1n}^+ & -D_{2m}^+D_{3n}^- + D_{3m}^-D_{2n}^+ \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_2 - m_2)(\partial_3 + m_3) + (\partial_3 + m_3)(\partial_2 - m_2) & (\partial_1 - m_1)(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_1 - m_1) \\ (\partial_1 + m_1)(\partial_{0n} - m_0) - (\partial_0 - m_0)(\partial_1 + m_1) & -(\partial_2 + m_2)(\partial_3 - m_3) + (\partial_3 - m_3)(\partial_2 + m_2) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_2 + (+m_2 - m_2)\partial_3 + (+m_2m_3 - m_3m_2) & (+m_1 - m_1)\partial_0 + (+m_0 - m_0)\partial_1 + (-m_1m_0 + m_0m_1) \\ (+m_1 - m_1)\partial_0 + (+m_0 - m_0)\partial_1 + (-m_1m_0 + m_0m_1) & (+m_3 - m_3)\partial_2 + (+m_2 - m_2)\partial_3 + (+m_2m_3 - m_3m_2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.2.4-(**BAc2r4**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}$, $D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}$, $D_i^{\Leftarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$, $D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$

then:

$$\left(D_1^{\Updownarrow}D_3^{\Leftarrow} + D_2^{\Updownarrow}D_0^{\Updownarrow} - D_3^{\Updownarrow}D_1^{\Leftarrow} - D_0^{\Updownarrow}D_2^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_1^{\Updownarrow}D_3^{\Leftarrow} + D_2^{\Updownarrow}D_0^{\Updownarrow} - D_3^{\Updownarrow}D_1^{\Leftarrow} - D_0^{\Updownarrow}D_2^{\Updownarrow}\right) = \left(\begin{pmatrix} D_{1m}^- & 0 \\ 0 & D_{1m}^+ \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix}\right.$$

$$= \begin{pmatrix} D_{2m}^-D_{0n}^- - D_{0m}^-D_{2n}^- & D_{1m}^-D_{3n}^- - D_{3m}^-D_{1n}^- \\ D_{1m}^+D_{3n}^+ - D_{3m}^+D_{1n}^+ & D_{2m}^+D_{0n}^+ - D_{0m}^+D_{2n}^+ \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2 - m_2)(\partial_0 - m_0) - (\partial_0 - m_0)(\partial_2 - m_2) & (\partial_1 - m_1)(\partial_3 - m_3) - (\partial_3 - m_3)(\partial_1 - m_1) \\ (\partial_1 + m_1)(\partial_3 + m_3) - (\partial_3 + m_3)(\partial_1 + m_1) & (\partial_2 + m_2)(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_2 + m_2) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (m_2m_0 - m_0m_2) & (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (m_1m_3 - m_3m_1) \\ (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (m_1m_3 - m_3m_1) & (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (m_2m_0 - m_0m_2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.3.1-(**BAc3r1**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}$, $D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}$, $D_i^{\Leftarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}$, $D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$

then:

$$\left(-D_0D_2^{\Leftarrow} - D_3^{\Leftarrow}D_1^{\Leftarrow} + D_2^{\Leftarrow}D_0^{\Updownarrow} + D_1D_3^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_0D_2^{\Leftarrow} - D_3^{\Leftarrow}D_1^{\Leftarrow} + D_2^{\Leftarrow}D_0^{\Updownarrow} + D_1D_3^{\Updownarrow}\right) = \left(-\begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix}\begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix}\right.$$

$$= \begin{pmatrix} -D^-_{3m}D^+_{1n} + D^+_{1m}D^-_{3n} & -D^+_{0m}D^-_{2n} + D^-_{2m}D^+_{0n} \\ -D^-_{0m}D^+_{2n} + D^+_{2m}D^-_{0n} & -D^+_{3m}D^-_{1n} + D^-_{1m}D^+_{3n} \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_3 - m_3)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_3 - m_3) & -(\partial_0 + m_0)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_0 + m_0) \\ -(\partial_0 - m_0)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_0 - m_0) & -(\partial_3 + m_{3m})(\partial_1 - m_1) + (\partial_1 - m_1)(\partial_3 + m_3) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (m_3m_1 - m_1m_3) & (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (m_0m_2 - m_2m_0) \\ (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (m_0m_2 - m_2m_0) & (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (m_3m_1 - m_1m_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.3.2-(**BAc3r2**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix},\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix},\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_0D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0^{\Updownarrow} + D_2D_3^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_0D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0^{\Updownarrow} + D_2D_3^{\Updownarrow}\right) = \left(-\begin{pmatrix} 0 & D^-_{3m} \\ D^+_{3m} & 0 \end{pmatrix}\begin{pmatrix} 0 & D^-_{2n} \\ D^+_{2n} & 0 \end{pmatrix} + \begin{pmatrix} D^+_{0m} & 0 \\ 0 & D^-_{0m} \end{pmatrix}\begin{pmatrix} 0 & D^-_{1n} \\ D^+_{1n} & 0 \end{pmatrix} - \begin{pmatrix} 0 & D^-_{1m} \\ D^+_{1m} & 0 \end{pmatrix}\begin{pmatrix} D^-_{0n} & 0 \\ 0 & D^+_{0n} \end{pmatrix}\right.$$

$$= \begin{pmatrix} -D^-_{3m}D^+_{2n} + D^+_{2m}D^-_{3n} & D^+_{0m}D^-_{1n} - D^-_{1m}D^+_{0n} \\ D^-_{0m}D^+_{1n} - D^+_{1m}D^-_{0n} & -D^+_{3m}D^-_{2n} + D^-_{2m}D^+_{3n} \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_3 - m_3)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_3 - m_3) & (\partial_0 + m_0)(\partial_1 - m_1) - (\partial_1 - m_1)(\partial_0 + m_0) \\ (\partial_0 - m_0)(\partial_{1n} + m_1) - (\partial_1 + m_1)(\partial_0 - m_0) & -(\partial_3 + m_3)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_3 + m_3) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_2 + (m_2 - m_2)\partial_3 + (+m_3m_2 - m_2m_3) & (+m_1 - m_1)\partial_0 + (m_0 - m_0)\partial_1 + (-m_0m_1 + m_1m_0) \\ (+m_1 - m_1)\partial_0 + (m_0 - m_0)\partial_1 + (-m_0m_1 + m_1m_0) & (+m_2 - m_3)\partial_2 + (m_3 - m_2)\partial_3 + (+m_3m_2 - m_2m_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.3.3-(**BAc3r3**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix},\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix},\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_3D_3^{\Updownarrow}\right) = \begin{pmatrix} (\partial_2\partial_2 + \partial_{1n}\partial_{1n} + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m_1m_1 + m_0m_0 + m_3m_3) & 0 \\ 0 & (\partial_2\partial_2 + \partial_1\partial_{1n} + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m \end{pmatrix}$$

Proof:

$$\left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_3D_3^{\Updownarrow}\right) = \left(\begin{pmatrix} 0 & D^-_{2m} \\ D^+_{2m} & 0 \end{pmatrix}\begin{pmatrix} 0 & D^-_{2n} \\ D^+_{2n} & 0 \end{pmatrix} + \begin{pmatrix} 0 & D^-_{1m} \\ D^+_{1m} & 0 \end{pmatrix}\begin{pmatrix} 0 & D^-_{1n} \\ D^+_{1n} & 0 \end{pmatrix} + \begin{pmatrix} D^+_{0m} & 0 \\ 0 & D^-_{0m} \end{pmatrix}\begin{pmatrix} D^-_{0n} & 0 \\ 0 & D^+_{0n} \end{pmatrix}\right.$$

$$= \begin{pmatrix} D^-_{2m}D^+_{2n} + D^-_{1m}D^+_{1n} + D^+_{0m}D^-_{0n} + D^+_{3m}D^-_{3n} & 0 \\ 0 & D^+_{2m}D^-_{2n} + D^+_{1m}D^-_{1n} + D^-_{0m}D^+_{0n} + D^-_{3m}D^+_{3n} \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2 - m_2)(\partial_2 + m_2) + (\partial_1 - m_1)(\partial_1 + m_1) + (\partial_0 + m_0)(\partial_0 - m_0) + (\partial_3 + m_3)(\partial_3 - m_3) & \\ 0 & (\partial_2 + m_2)(\partial_2 - \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2\partial_2 + \partial_1\partial_{1n} + \partial_0\partial_0 + \partial_3\partial_3) + (m_2 - m_2)\partial_2 + (m_1 - m_{1m})\partial_1 + (m_0 - m_0)\partial_0 + (m_3 - m_3)\partial_3 - (m_2m_2 + m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m_1m_1 + m_0m_0 + m_3m_3) & 0 \\ 0 & (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 \end{pmatrix}$$

□

Lemma I.2.3.4-(**BAc3r4**): For m_i constants:
and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix},\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix},\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow}+D_2^{\Updownarrow}D_1^{\Leftrightarrow}+D_3^{\Updownarrow}D_0^{\Updownarrow}-D_0^{\Updownarrow}D_3^{\Updownarrow}\right)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$\left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow}+D_2^{\Updownarrow}D_1^{\Leftrightarrow}+D_3^{\Updownarrow}D_0^{\Updownarrow}-D_0^{\Updownarrow}D_3^{\Updownarrow}\right)=\left(-\begin{pmatrix}D_{1m}^{-}&0\\0&D_{1m}^{+}\end{pmatrix}\begin{pmatrix}0&D_{2n}^{-}\\D_{2n}^{+}&0\end{pmatrix}+\begin{pmatrix}D_{2m}^{-}&0\\0&D_{2m}^{+}\end{pmatrix}\begin{pmatrix}0&D_{1n}^{-}\\D_{1n}^{+}&0\end{pmatrix}+\begin{pmatrix}D_{3m}^{-}&0\\0&D_{3m}^{+}\end{pmatrix}\begin{pmatrix}D_{0n}^{-}&0\\0&D_{0n}^{+}\end{pmatrix}\right.$$

$$=\begin{pmatrix}D_{3m}^{-}D_{0n}^{-}-D_{0m}^{-}D_{3n}^{-}&-D_{1m}^{-}D_{2n}^{-}+D_{2m}^{-}D_{1n}^{-}\\-D_{1m}^{+}D_{2n}^{+}+D_{2m}^{+}D_{1n}^{+}&D_{3m}^{+}D_{0n}^{+}-D_{0m}^{+}D_{3n}^{+}\end{pmatrix}$$

$$=\left(\begin{matrix}\partial_3\partial_0-\partial_3m_0-m_3\partial_0+m_3m_0-\partial_0\partial_3+\partial_0m_3+m_0\partial_3-m_0m_3&-\partial_1\partial_2+\partial_1m_2+m_1\partial_2-m_{1m}m_2+\partial_2\partial_1\\-\partial_1\partial_2-\partial_1m_2-m_{1m}\partial_2-m_1m_2+\partial_{2m}\partial_1+\partial_2m_1+m_2\partial_1+m_2m_1&\partial_3\partial_0+\partial_3m_0+m_3\partial_0+m_3m_0-\partial_0\partial_3-\end{matrix}\right.$$

$$=\begin{pmatrix}+(m_3-m_3)\partial_0+(m_0-m_0)\partial_3+(+m_3m_0-m_0m_3)&+(m_2-m_2)\partial_1+(m_1-m_1)\partial_2+(-m_1m_2+m_2m_1)\\+(m_2-m_2)\partial_1+(m_1-m_{1m})\partial_2+(-m_1m_2+m_2m_1)&+(m_3-m_3)\partial_0+(m_0-m_0)\partial_3+(+m_3m_0-m_0m_3)\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

□

Lemma I.2.4.1-(**BAc4r1**): For m_i constants:

and:

$D_i^{+}\equiv(\partial_i+m_i)\quad,\quad D_i^{-}\equiv(\partial_i-m_i)$

$$D_i\equiv\begin{pmatrix}D_i^{+}&0\\0&D_i^{-}\end{pmatrix}\ ,\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^{-}&0\\0&D_i^{+}\end{pmatrix}\ ,\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0&D_i^{-}\\D_i^{+}&0\end{pmatrix}\ ,\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0&D_i^{+}\\D_i^{-}&0\end{pmatrix}$$

then:

$$(D_0D_1-D_3^{\Leftrightarrow}D_2+D_2^{\Leftrightarrow}D_3-D_1D_0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$(D_0D_1-D_3^{\Leftrightarrow}D_2+D_2^{\Leftrightarrow}D_3-D_1D_0)=\left(\begin{pmatrix}D_{0m}^{+}&0\\0&D_{0m}^{-}\end{pmatrix}\begin{pmatrix}D_{1n}^{+}&0\\0&D_{1n}^{-}\end{pmatrix}-\begin{pmatrix}0&D_{3m}^{-}\\D_{3m}^{+}&0\end{pmatrix}\begin{pmatrix}D_{2n}^{+}&0\\0&D_{2n}^{-}\end{pmatrix}+\begin{pmatrix}0&D_{2m}^{-}\\D_{2m}^{+}&0\end{pmatrix}\begin{pmatrix}D_{3n}^{+}&0\\0&D_{3n}^{-}\end{pmatrix}\right.$$

$$=\begin{pmatrix}D_{0m}^{+}D_{1n}^{+}-D_{1m}^{+}D_{0n}^{+}&-D_{3m}^{-}D_{2n}^{-}+D_{2m}^{-}D_{3n}^{-}\\-D_{3m}^{+}D_{2n}^{+}+D_{2m}^{+}D_{3n}^{+}&D_{0m}^{-}D_{1n}^{-}-D_{1m}^{-}D_{0n}^{-}\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_0+m_0)(\partial_1+m_1)-(\partial_1+m_1)(\partial_0+m_0)&-(\partial_3-m_3)(\partial_2-m_2)+(\partial_2-m_2)(\partial_3-m_3)\\-(\partial_3+m_3)(\partial_2+m_2)+(\partial_{2m}+m_2)(\partial_3+m_3)&(\partial_0-m_0)(\partial_1-m_1)-(\partial_1-m_1)(\partial_0-m_0)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_1-m_{1m})\partial_0+(m_0-m_0)\partial_1+(+m_0m_1-m_1m_0)&(+m_3-m_3)\partial_2+(m_2-m_2)\partial_3+(-m_3m_2+m_2m_3)\\(+m_3-m_3)\partial_2+(m_{2m}-m_2)\partial_3+(-m_3m_2+m_2m_3)&(+m_1-m_1)\partial_0+(m_{0n}-m_0)\partial_1+(+m_0m_1-m_1m_0)\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

□

Lemma I.2.4.2-(**BAc4r2**): For m_i constants:

and:

$D_i^{+}\equiv(\partial_i+m_i)\quad,\quad D_i^{-}\equiv(\partial_i-m_i)$

$$D_i\equiv\begin{pmatrix}D_i^{+}&0\\0&D_i^{-}\end{pmatrix}\ ,\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^{-}&0\\0&D_i^{+}\end{pmatrix}\ ,\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0&D_i^{-}\\D_i^{+}&0\end{pmatrix}\ ,\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0&D_i^{+}\\D_i^{-}&0\end{pmatrix}$$

then:

$$(D_3^{\Leftrightarrow}D_1+D_0D_2-D_1^{\Leftrightarrow}D_3-D_2D_0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$(D_3^{\Leftrightarrow}D_1+D_0D_2-D_1^{\Leftrightarrow}D_3-D_2D_0)=\left(\begin{pmatrix}0&D_{3m}^{-}\\D_{3m}^{+}&0\end{pmatrix}\begin{pmatrix}D_{1n}^{+}&0\\0&D_{1n}^{-}\end{pmatrix}+\begin{pmatrix}D_{0m}^{+}&0\\0&D_{0m}^{-}\end{pmatrix}\begin{pmatrix}D_{2n}^{+}&0\\0&D_{2n}^{-}\end{pmatrix}-\begin{pmatrix}0&D_{1m}^{-}\\D_{1m}^{+}&0\end{pmatrix}\begin{pmatrix}D_{3n}^{+}&0\\0&D_{3n}^{-}\end{pmatrix}\right.$$

$$=\begin{pmatrix}D_{0m}^{+}D_{2n}^{+}-D_{2m}^{+}D_{0n}^{+}&D_{3m}^{-}D_{1n}^{-}-D_{1m}^{-}D_{3n}^{-}\\D_{3m}^{+}D_{1n}^{+}-D_{1m}^{+}D_{3n}^{+}&D_{0m}^{-}D_{2n}^{-}-D_{2m}^{-}D_{0n}^{-}\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_0+m_0)(\partial_2+m_2)-(\partial_2+m_2)(\partial_0+m_{0n})&(\partial_3-m_3)(\partial_1-m_1)-(\partial_1-m_1)(\partial_3-m_3)\\(\partial_3+m_3)(\partial_1+m_1)-(\partial_1+m_{1m})(\partial_3+m_3)&(\partial_0-m_0)(\partial_2-m_2)-(\partial_{2m}-m_2)(\partial_0-m_0)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_2-m_2)\partial_0+(m_0-m_0)\partial_2+(m_0m_2-m_2m_{0n})&(+m_3-m_3)\partial_1+(m_1-m_1)\partial_3+(m_3m_1-m_1m_3)\\(+m_3-m_3)\partial_1+(m_1-m_1)\partial_3+(m_3m_1-m_{1m}m_3)&(+m_0-m_0)\partial_2+(m_2-m_2)\partial_0+(m_0m_2-m_2m_0)\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

□

Lemma I.2.4.3-(**BAc4r3**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$(-D_2^{\Leftrightarrow} D_1 + D_1^{\Leftrightarrow} D_2 + D_0 D_3 - D_3 D_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$(-D_2^{\Leftrightarrow} D_1 + D_1^{\Leftrightarrow} D_2 + D_0 D_3 - D_3 D_0) = \Bigg(-\begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix}\begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix}\begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} D_{3n}^+ & 0 \\ 0 & D_{3n}^-$$

$$= \begin{pmatrix} D_{0m}^+ D_{3n}^+ - D_{3m}^+ D_{0n}^+ & -D_{2m}^- D_{1n}^- + D_{1m}^- D_{2n}^- \\ -D_{2m}^+ D_{1n}^+ + D_{1m}^+ D_{2n}^+ & D_{0m}^- D_{3n}^- - D_{3m}^- D_{0n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0 + m_0)(\partial_3 + m_3) - (\partial_3 + m_3)(\partial_0 + m_0) & -(\partial_2 - m_2)(\partial_1 - m_1) + (\partial_1 - m_1)(\partial_2 - m_2) \\ -(\partial_{2m} + m_2)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_2 + m_2) & (\partial_0 - m_0)(\partial_3 - m_3) - (\partial_3 - m_3)(\partial_0 - m_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (m_0 m_3 - m_3 m_0) & (+m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (m_1 m_2 - m_{2m} m_1) \\ (+m_1 - m_1)\partial_2 + (m_2 - m_{2m})\partial_1 + (m_1 m_2 - m_2 m_1) & (+m_3 - m_3)\partial_0 + (m_{0n} - m_0)\partial_3 + (m_0 m_3 - m_3 m_0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.2.4.4-(**BAc4r4**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_1^{\Updownarrow} D_1 + D_2^{\Updownarrow} D_2 + D_3^{\Updownarrow} D_3 + D_0^{\Updownarrow} D_0\right) = \begin{pmatrix} (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1 m_1 + m_2 m_2 + m_3 m_3 + m_0 m_0) & 0 \\ 0 & (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1 m_1 + m_2 m_2 +$$

Proof:

$$\left(D_1^{\Updownarrow} D_1 + D_2^{\Updownarrow} D_2 + D_3^{\Updownarrow} D_3 + D_0^{\Updownarrow} D_0\right) = \Bigg(\begin{pmatrix} D_{1m}^- & 0 \\ 0 & D_{1m}^+ \end{pmatrix}\begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} + \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix}\begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} + \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix}\begin{pmatrix} D_{3n}^+ & 0 \\ 0 & D_{3n}^-$$

$$= \begin{pmatrix} D_{1m}^- D_{1n}^+ + D_{2m}^- D_{2n}^+ + D_{3m}^- D_{3n}^+ + D_{0m}^- D_{0n}^+ & 0 \\ 0 & D_{1m}^+ D_{1n}^- + D_{2m}^+ D_{2n}^- + D_{3m}^+ D_{3n}^- + D_{0m}^+ D_{0n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_1 - m_1)(\partial_1 + m_1) + (\partial_2 - m_{2m})(\partial_2 + m_2) + (\partial_{3m} - m_3)(\partial_3 + m_3) + (\partial_0 - m_0)(\partial_0 + m_0) \\ 0 & (\partial_1 + m_1)(\partial_1$$

$$= \begin{pmatrix} (\partial_{1n}\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) + (m_1 - m_1)\partial_1 + (m_2 - m_2)\partial_2 + (m_3 - m_3)\partial_3 + (m_{0n} - m_0)\partial_0 - (m_1 m_1 + m \\ 0$$

$$= \begin{pmatrix} (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1 m_1 + m_2 m_2 + m_3 m_3 + m_0 m_0) & 0 \\ 0 & (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1 m_1$$

□

Lemma I.3.1.1-(**ABc1r1**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_0^{\Updownarrow} D_0 + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) = \begin{pmatrix} (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 + m_3 m_3 + m_2 m_2 + m_1 m_1) & 0 \\ 0 & (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 + m_3 m$$

Proof:

$$\left(D_0^{\Updownarrow} D_0 + D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_1 D_1^{\Updownarrow}\right) = \Bigg(\begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix}\begin{pmatrix} (\partial_0 + m_{0m}) & 0 \\ 0 & (\partial_0 - m_0) \end{pmatrix} + \begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_3 - \\ (\partial_3 + m_3) & 0$$

$$= \begin{pmatrix} (\partial_0 - m_0)(\partial_0 + m_0) + (\partial_3 - m_3)(\partial_3 + m_3) + (\partial_2 - m_2)(\partial_{2m} + m_{2m}) + (\partial_1 + m_1)(\partial_1 - m_1) \\ 0 & (\partial_0 + m_0)(\partial_0$$

$$= \left(\begin{array}{l} (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) + (m_0 - m_0)\partial_0 + (m_3 - m_3)\partial_3 + (m_2 - m_2)\partial_2 + (m_1 - m_1)\partial_1 - (m_0 m_0 + m_3 \\ 0 \end{array} \right.$$

$$= \left(\begin{array}{ll} (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 + m_3 m_3 + m_2 m_2 + m_1 m_1) & 0 \\ 0 & (\partial_0\partial_0 + \partial_3\partial_3 + \partial_2\partial_2 + \partial_1\partial_1) - (m_0 m_0 \end{array} \right.$$

□

Lemma I.3.1.2-(**ABc1r2**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow} \right) = \left(-\begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix} \begin{pmatrix} (\partial_0 + m_0) & 0 \\ 0 & (\partial_0 - m_0) \end{pmatrix} + \begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} \left(\begin{array}{ll} 0 & (\partial_{3m} \\ (\partial_3 + m_3) & \end{array} \right. \right.$$

$$= \begin{pmatrix} -(\partial_1 - m_1)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_1 - m_1) & -(\partial_3 - m_3)(\partial_0 - m_0) + (\partial_0 - m_0)(\partial_{3m} - m_3) \\ -(\partial_3 + m_3)(\partial_0 + m_0) + (\partial_0 + m_0)(\partial_3 + m_3) & -(\partial_1 + m_1)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_1 + m_1) \end{pmatrix}$$

$$= \begin{pmatrix} (m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_1 m_{2m} - m_2 m_1) & (m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_3 m_0 + m_0 m_3) \\ (m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_3 m_0 + m_0 m_3) & (m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_1 m_2 - m_2 m_1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.1.2-(**ABc1r2**)**a**: For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_3^{\Leftrightarrow} D_0 + D_0^{\Updownarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow} \right) = \left(-\begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix} \begin{pmatrix} (\partial_0 + m_0) & 0 \\ 0 & (\partial_0 - m_0) \end{pmatrix} + \begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} \left(\begin{array}{ll} 0 & (\partial_3 \\ (\partial_3 + m_3) & \end{array} \right. \right.$$

$$= \begin{pmatrix} -(\partial_1 - m_1)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_1 - m_1) & -(\partial_3 - m_3)(\partial_0 - m_0) + (\partial_0 - m_0)(\partial_3 - m_3) \\ -(\partial_3 + m_3)(\partial_0 + m_0) + (\partial_0 + m_0)(\partial_3 + m_3) & -(\partial_1 + m_1)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_1 + m_1) \end{pmatrix}$$

$$= \left(\begin{array}{ll} (+m_2 - m_2)\partial_1 + (+m_1 - m_1)\partial_2 + (+m_1 m_2 - m_2 m_1) & (+m_3 - m_3)\partial_0 + (+m_{0n} - m_0)\partial_3 + (-m_3 m_{0n} + m_0 \\ (+m_3 - m_{3m})\partial_0 + (+m_0 - m_0)\partial_3 + (-m_3 m_0 + m_0 m_3) & (+m_2 - m_2)\partial_1 + (+m_1 - m_1)\partial_2 + (+m_1 m_2 - m_2 \end{array} \right.$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.1.3-(**ABc1r3**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_2^{\Leftrightarrow} D_0 - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_2^{\Leftrightarrow} D_0 - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow} \right) = \left(\left(\begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} - \begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} - \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \right. \right.$$

$$= \begin{pmatrix} -D_{1n}^- D_{3m}^+ + D_{3n}^+ D_{1m}^- & D_{2n}^- D_{0m}^- - D_{0n}^- D_{2m}^- \\ D_{2n}^+ D_{0m}^+ - D_{0n}^+ D_{2m}^+ & -D_{1n}^+ D_{3m}^- + D_{3n}^- D_{1m}^+ \end{pmatrix}$$

$$= \left(\begin{matrix} -(\partial_1 - m_1)(\partial_3 + m_3) + (\partial_3 + m_3)(\partial_1 - m_1) & (\partial_2 - m_2)(\partial_0 - m_0) - (\partial_0 - m_0)(\partial_2 - m_2) \\ (\partial_2 + m_2)(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_{2m} + m_2) & -(\partial_1 + m_1)(\partial_3 - m_3) + (\partial_3 - m_3)(\partial_1 + m_1) \end{matrix} \right.$$

$$= \left(\begin{matrix} (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (+m_1 m_3 - m_3 m_1) & (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (+m_2 m_0 - m_0 m_2) \\ (+m_2 - m_2)\partial_0 + (m_0 - m_0)\partial_2 + (+m_2 m_0 - m_0 m_2) & (+m_3 - m_3)\partial_1 + (m_{1m} - m_1)\partial_3 + (+m_1 m_3 - m_3 m_1) \end{matrix} \right.$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.1.4-(**ABc1r4**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} ,\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} ,\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} ,\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_1^{\Updownarrow} D_0 + D_2^{\Updownarrow} D_3^{\Leftrightarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_1^{\Updownarrow} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(D_1^{\Updownarrow} D_0 + D_2^{\Updownarrow} D_3^{\Leftrightarrow} - D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0 D_1^{\Updownarrow} \right) = \left(\begin{pmatrix} D_{1n}^- & 0 \\ 0 & D_{1n}^+ \end{pmatrix} \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} + \begin{pmatrix} D_{2n}^- & 0 \\ 0 & D_{2n}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} - \begin{pmatrix} D_{3n}^- & 0 \\ 0 & D_{3n}^+ \end{pmatrix} \left(\begin{matrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{matrix} \right. \right.$$

$$= \begin{pmatrix} D_{1n}^- D_{0m}^+ - D_{0n}^+ D_{1m}^- & D_{2n}^- D_{3m}^- - D_{3n}^- D_{2m}^- \\ D_{2n}^+ D_{3m}^+ - D_{3n}^+ D_{2m}^+ & D_{1n}^+ D_{0m}^- - D_{0n}^- D_{1m}^+ \end{pmatrix}$$

$$= \left(\begin{matrix} (\partial_1 - m_{1n})(\partial_0 + m_0) - (\partial_0 + m_0)(\partial_1 - m_1) & (\partial_2 - m_2)(\partial_3 - m_3) - (\partial_3 - m_3)(\partial_2 - m_2) \\ (\partial_2 + m_2)(\partial_{3m} + m_3) - (\partial_3 + m_3)(\partial_2 + m_2) & (\partial_1 + m_1)(\partial_0 - m_0) - (\partial_0 - m_0)(\partial_1 + m_1) \end{matrix} \right.$$

$$= \left(\begin{matrix} (+m_1 - m_1)\partial_0 + (m_0 - m_0)\partial_1 + (-m_1 m_{0m} + m_0 m_1) & (+m_3 - m_3)\partial_2 + (m_2 - m_2)\partial_3 + (+m_2 m_3 - m_3 m_2) \\ (+m_3 - m_3)\partial_2 + (m_2 - m_2)\partial_3 + (+m_2 m_3 - m_3 m_2) & (+m_1 - m_{1m})\partial_0 + (m_0 - m_0)\partial_1 + (-m_1 m_0 + m_0 m_1) \end{matrix} \right.$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.2.1-(**ABc2r1**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} ,\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} ,\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} ,\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_0^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow} D_0 - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_0^{\Updownarrow} D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow} D_0 - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow} \right) = -\begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix} \begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix} \left(\begin{matrix} (\partial_0 + m_0) & 0 \\ 0 & (\partial_0 - \end{matrix} \right.$$

$$= \left(\begin{matrix} -(\partial_2 - m_2)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_{2m} - m_2) & -(\partial_0 - m_0)(\partial_3 - m_3) + (\partial_3 - m_3)(\partial_0 - m_0) \\ -(\partial_0 + m_0)(\partial_3 + m_3) + (\partial_3 + m_3)(\partial_0 + m_0) & -(\partial_2 + m_2)(\partial_1 - m_{1m}) + (\partial_1 - m_1)(\partial_2 + m_2) \end{matrix} \right.$$

$$= \left(\begin{matrix} (+m_2 - m_2)\partial_1 + (+m_1 - m_1)\partial_2 + (+m_2 m_{1m} - m_1 m_2) & (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_0 m_3 + m_3 m \\ (+m_3 - m_3)\partial_0 + (-m_0 + m_0)\partial_3 + (-m_0 m_3 + m_3 m_0) & (-m_2 + m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_2 m_1 - m_1 m \end{matrix} \right.$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.2.2-(**ABc2r2**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} ,\ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} ,\ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} ,\ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_3^{\Leftrightarrow} D_3^{\Leftrightarrow} + D_0^{\Updownarrow} D_0 + D_1^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_2 D_2^{\Updownarrow} \right) = \left(\begin{matrix} (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_{1n} + \partial_2\partial_2) - (m_3 m_3 + m_0 m_0 + m_1 m_1 + m_2 m_2) & 0 \\ 0 & (\partial_3\partial_3 + \partial_0\partial_0 + \partial_1\partial_1 + \partial_2\partial_2) - (m_3 m_3 + m_0 \end{matrix} \right.$$

Proof:

$$\left(D_3^{\Leftrightarrow}D_3^{\Leftrightarrow}+D_0^{\Updownarrow}D_0+D_1^{\Leftrightarrow}D_1^{\Leftrightarrow}+D_2D_2^{\Updownarrow}\right)=\begin{pmatrix}0&(\partial_3-m_3)\\(\partial_3+m_3)&0\end{pmatrix}\begin{pmatrix}0&(\partial_3-m_3)\\(\partial_3+m_3)&0\end{pmatrix}+\begin{pmatrix}(\partial_0-m_0)&0\\0&(\partial_0+m_0)\end{pmatrix}\begin{pmatrix}(\partial_0+m_0)&0\\0&(\partial_0-m_0\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_3-m_3)(\partial_3+m_3)+(\partial_0-m_0)(\partial_0+m_0)+(\partial_1-m_1)(\partial_1+m_1)+(\partial_2+m_2)(\partial_2-m_2)&\\0&(\partial_3+m_3)(\partial_3-\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_3\partial_3+\partial_0\partial_0+\partial_{1n}\partial_1+\partial_2\partial_2)+(m_0-m_0)\partial_0+(m_1-m_1)\partial_{1n}+(m_2-m_2)\partial_2+(m_3-m_3)\partial_3-(m_3m_3+\\0\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_3\partial_3+\partial_0\partial_0+\partial_1\partial_1+\partial_2\partial_2)-(m_3m_3+m_0m_0+m_1m_1+m_2m_2)&0\\0&(\partial_3\partial_3+\partial_0\partial_0+\partial_1\partial_1+\partial_2\partial_2)-(m_3m\end{pmatrix}$$

□

Lemma I.3.2.3-(**ABc2r3**): For m_i constants:

and:

$D_i^+\equiv(\partial_i+m_i)\quad,\quad D_i^-\equiv(\partial_i-m_i)$

$$D_i\equiv\begin{pmatrix}D_i^+&0\\0&D_i^-\end{pmatrix},\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^-&0\\0&D_i^+\end{pmatrix},\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0&D_i^-\\D_i^+&0\end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0&D_i^+\\D_i^-&0\end{pmatrix}$$

then:

$$\left(-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow}-D_1^{\Leftrightarrow}D_0+D_0^{\Updownarrow}D_1^{\Leftrightarrow}+D_3D_2^{\Updownarrow}\right)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$\left(-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow}-D_1^{\Leftrightarrow}D_0+D_0^{\Updownarrow}D_1^{\Leftrightarrow}+D_3D_2^{\Updownarrow}\right)=-\begin{pmatrix}0&(\partial_2-m_2)\\(\partial_2+m_2)&0\end{pmatrix}\begin{pmatrix}0&(\partial_3-m_3)\\(\partial_{3m}+m_3)&0\end{pmatrix}-\begin{pmatrix}0&(\partial_1-m_1)\\(\partial_1+m_1)&0\end{pmatrix}\begin{pmatrix}(\partial_0+m_0)&0\\0&(\partial_0-\end{pmatrix}$$

$$=\begin{pmatrix}-(\partial_2-m_2)(\partial_3+m_3)+(\partial_3+m_3)(\partial_2-m_2)&-(\partial_1-m_1)(\partial_0-m_0)+(\partial_0-m_0)(\partial_1-m_1)\\-(\partial_1+m_1)(\partial_0+m_0)+(\partial_0+m_0)(\partial_1+m_{1m})&-(\partial_2+m_2)(\partial_3-m_3)+(\partial_3-m_3)(\partial_2+m_2)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_3-m_3)\partial_2+(m_2-m_2)\partial_3+(+m_2m_3-m_3m_2)&(+m_1-m_1)\partial_0+(m_{0m}-m_0)\partial_1+(-m_1m_0+m_0m_1\\(+m_1-m_1)\partial_0+(m_0-m_0)\partial_1+(-m_1m_0+m_0m_1)&(+m_3-m_3)\partial_2+(m_2-m_2)\partial_3+(+m_2m_3-m_3m_2\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

□

Lemma I.3.2.4-(**ABc2r4**): For m_i constants:

and:

$D_i^+\equiv(\partial_i+m_i)\quad,\quad D_i^-\equiv(\partial_i-m_i)$

$$D_i\equiv\begin{pmatrix}D_i^+&0\\0&D_i^-\end{pmatrix},\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^-&0\\0&D_i^+\end{pmatrix},\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0&D_i^-\\D_i^+&0\end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0&D_i^+\\D_i^-&0\end{pmatrix}$$

then:

$$\left(-D_1^{\Updownarrow}D_3^{\Leftrightarrow}+D_2^{\Updownarrow}D_0+D_3^{\Updownarrow}D_1^{\Leftrightarrow}-D_0D_2^{\Updownarrow}\right)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$\left(-D_1^{\Updownarrow}D_3^{\Leftrightarrow}+D_2^{\Updownarrow}D_0+D_3^{\Updownarrow}D_1^{\Leftrightarrow}-D_0D_2^{\Updownarrow}\right)=\begin{pmatrix}(\partial_1-m_1)&0\\0&(\partial_1+m_1)\end{pmatrix}\begin{pmatrix}0&(\partial_3-m_3)\\(\partial_3+m_3)&0\end{pmatrix}+\begin{pmatrix}(\partial_2-m_2)&0\\0&(\partial_2+m_2)\end{pmatrix}\begin{pmatrix}(\partial_0+m_0)&0\\0&(\partial_0-m_0)\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_2-m_2)(\partial_0+m_0)-(\partial_0+m_0)(\partial_2-m_2)&(\partial_1-m_1)(\partial_3-m_3)-(\partial_3-m_3)(\partial_1-m_1)\\(\partial_1+m_1)(\partial_3+m_3)-(\partial_3+m_3)(\partial_1+m_1)&(\partial_2+m_2)(\partial_0-m_0)-(\partial_0-m_0)(\partial_2+m_2)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_2-m_2)\partial_0+(m_0-m_0)\partial_2+(-m_2m_0+m_0m_2)&(+m_3-m_3)\partial_1+(m_1-m_1)\partial_3+(+m_1m_3-m_3m_1)\\(+m_3-m_3)\partial_1+(m_1-m_{1m})\partial_3+(+m_1m_3-m_3m_1)&(+m_2-m_2)\partial_{0n}+(m_0-m_0)\partial_2+(-m_2m_0+m_0m_2)\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

□

Lemma I.3.3.1-(**ABc3r1**): For m_i constants:

and:

$D_i^+\equiv(\partial_i+m_i)\quad,\quad D_i^-\equiv(\partial_i-m_i)$

$$D_i\equiv\begin{pmatrix}D_i^+&0\\0&D_i^-\end{pmatrix},\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^-&0\\0&D_i^+\end{pmatrix},\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0&D_i^-\\D_i^+&0\end{pmatrix},\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0&D_i^+\\D_i^-&0\end{pmatrix}$$

then:

$$\left(D_0^{\Updownarrow}D_2^{\Leftrightarrow}-D_3^{\Leftrightarrow}D_1^{\Leftrightarrow}-D_2^{\Leftrightarrow}D_0+D_1D_3^{\Updownarrow}\right)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Proof:

$$\left(D_0^{\Updownarrow}D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow}D_0 + D_1D_3^{\Updownarrow}\right) = \begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix}\begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_1 - m_1 \\ (\partial_1 + m_1) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_3 - m_3)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_3 - m_3) & (\partial_0 - m_0)(\partial_2 - m_2) - (\partial_2 - m_2)(\partial_0 - m_0) \\ (\partial_0 + m_0)(\partial_2 + m_2) - (\partial_2 + m_2)(\partial_0 + m_{0m}) & -(\partial_3 + m_3)(\partial_1 - m_1) + (\partial_1 - m_1)(\partial_3 + m_3) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_1 + (m_1 - m_1)\partial_3 + (+m_3m_1 - m_1m_3) & (+m_2 - m_2)\partial_0 + (m_{0m} - m_0)\partial_2 + (+m_0m_2 - m_2m_0 \\ (+m_2 - m_2)\partial_0 + (m_0 - m_{0m})\partial_2 + (+m_0m_2 - m_2m_{0m}) & (+m_{3m} - m_3)\partial_1 + (m_{1m} - m_1)\partial_3 + (+m_3m_1 - m_1m_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.3.2-(**ABc3r2**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0 + D_2D_3^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0 + D_2D_3^{\Updownarrow}\right) = -\begin{pmatrix} 0 & (\partial_3 - m_3) \\ (\partial_3 + m_3) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix} - \begin{pmatrix} (\partial_0 - m_0) & 0 \\ 0 & (\partial_0 + m_0) \end{pmatrix}\begin{pmatrix} 0 & (\partial_1 - \\ (\partial_1 + m_1) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -(\partial_3 - m_3)(\partial_2 + m_2) + (\partial_2 + m_2)(\partial_3 - m_3) & -(\partial_0 - m_0)(\partial_1 - m_1) + (\partial_1 - m_1)(\partial_0 - m_0) \\ -(\partial_0 + m_0)(\partial_1 + m_1) + (\partial_1 + m_1)(\partial_0 + m_0) & -(\partial_3 + m_3)(\partial_2 - m_2) + (\partial_2 - m_2)(\partial_3 + m_3) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_2 + (m_2 - m_2)\partial_3 + (+m_3m_2 - m_2m_3) & (+m_1 - m_1)\partial_0 + (m_0 - m_{0m})\partial_1 + (-m_0m_1 + m_1m \\ (+m_1 - m_1)\partial_0 + (m_0 - m_0)\partial_1 + (-m_0m_1 + m_1m_{0m}) & (+m_3 - m_3)\partial_2 + (m_2 - m_2)\partial_3 + (+m_3m_2 - m_2m \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.3.3-(**ABc3r3**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_3D_3^{\Updownarrow}\right) = \begin{pmatrix} (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m_1m_1 + m_0m_0 + m_3m_3) & 0 \\ 0 & (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m_1m \end{pmatrix}$$

Proof:

$$D_{An}D_{Bm} \Rightarrow \left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_3D_3^{\Updownarrow}\right) = \begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix}\begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (\partial_1 - m_1) \\ (\partial_1 + m_1) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2 - m_2)(\partial_2 + m_2) + (\partial_1 - m_1)(\partial_1 + m_1) + (\partial_0 - m_0)(\partial_0 + m_0) + (\partial_3 + m_3)(\partial_3 - m_{3m}) & \\ 0 & (\partial \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) + (m_2 - m_2)\partial_2 + (m_1 - m_1)\partial_1 + (m_0 - m_0)\partial_0 + (m_3 - m_3)\partial_3 - \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial_3) - (m_2m_2 + m_1m_1 + m_0m_0 + m_3m_3) & \\ 0 & (\partial_2\partial_2 + \partial_1\partial_1 + \partial_0\partial_0 + \partial_3\partial \end{pmatrix}$$

□

Lemma I.3.3.4-(**ABc3r4**): For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \ D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \ D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \ D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0 - D_0D_3^{\Updownarrow}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Proof:

$$\left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow}+D_2^{\Updownarrow}D_1^{\Leftrightarrow}+D_3^{\Updownarrow}D_0-D_0D_3^{\Updownarrow}\right)=-\begin{pmatrix}(\partial_1-m_1) & 0\\ 0 & (\partial_1+m_1)\end{pmatrix}\begin{pmatrix}0 & (\partial_2-m_2)\\ (\partial_2+m_2) & 0\end{pmatrix}+\begin{pmatrix}(\partial_2-m_2) & 0\\ 0 & (\partial_2+m_2)\end{pmatrix}\begin{pmatrix}0 & (\partial_1-m_1\\ (\partial_1+m_1) & 0\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_3-m_3)(\partial_0+m_0)-(\partial_0+m_0)(\partial_3-m_3) & -(\partial_1-m_1)(\partial_2-m_2)+(\partial_2-m_2)(\partial_1-m_1)\\ -(\partial_1+m_1)(\partial_2+m_2)+(\partial_2+m_2)(\partial_1+m_1) & (\partial_3+m_3)(\partial_{0m}-m_0)-(\partial_0-m_0)(\partial_3+m_3)\end{pmatrix}$$

$$=\begin{pmatrix}+(m_3-m_3)\partial_0+(m_0-m_0)\partial_3+(-m_3m_0+m_0m_3) & +(m_2-m_2)\partial_1+(m_1-m_1)\partial_2+(-m_1m_2+m_2m_1)\\ +(m_2-m_2)\partial_1+(m_1-m_1)\partial_2+(-m_1m_2+m_2m_1) & +(m_3-m_3)\partial_0+(m_0-m_0)\partial_3+(-m_3m_{0m}+m_0m_3)\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

□

Lemma I.3.4.1-(**ABc4r1**): For m_i constants:
and:

$D_i^{+}\equiv(\partial_i+m_i)\quad,\quad D_i^{-}\equiv(\partial_i-m_i)$

$D_i\equiv\begin{pmatrix}D_i^{+} & 0\\ 0 & D_i^{-}\end{pmatrix}\ ,\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^{-} & 0\\ 0 & D_i^{+}\end{pmatrix}\ ,\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0 & D_i^{-}\\ D_i^{+} & 0\end{pmatrix}\ ,\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0 & D_i^{+}\\ D_i^{-} & 0\end{pmatrix}$

then:

$$\left(D_0^{\Updownarrow}D_1+D_3^{\Leftrightarrow}D_2-D_2^{\Leftrightarrow}D_3-D_1D_0^{\Updownarrow}\right)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Proof:

$$\left(D_0^{\Updownarrow}D_1+D_3^{\Leftrightarrow}D_2-D_2^{\Leftrightarrow}D_3-D_1D_0^{\Updownarrow}\right)=\begin{pmatrix}(\partial_0-m_0) & 0\\ 0 & (\partial_0+m_0)\end{pmatrix}\begin{pmatrix}(\partial_1+m_1) & 0\\ 0 & (\partial_1-m_1)\end{pmatrix}+\begin{pmatrix}0 & (\partial_3-m_3)\\ (\partial_3+m_3) & 0\end{pmatrix}\begin{pmatrix}(\partial_2+m_2) & 0\\ 0 & (\partial_{2m}-m_2)\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_0-m_0)(\partial_1+m_1)-(\partial_1+m_1)(\partial_0-m_0) & (\partial_3-m_3)(\partial_2-m_2)-(\partial_2-m_2)(\partial_3-m_3)\\ (\partial_3+m_3)(\partial_{2m}+m_2)-(\partial_2+m_2)(\partial_{3m}+m_3) & (\partial_0+m_0)(\partial_1-m_1)-(\partial_1-m_1)(\partial_{0m}+m_0)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_1-m_1)\partial_0+(m_0-m_0)\partial_1+(-m_0m_1+m_1m_0) & (+m_3-m_3)\partial_2+(m_2-m_2)\partial_3+(+m_3m_2-m_2m_3)\\ (+m_3-m_3)\partial_2+(m_2-m_2)\partial_3+(+m_3m_2-m_2m_3) & (+m_1-m_1)\partial_0+(m_0-m_0)\partial_1+(-m_0m_1+m_1m_0)\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

□

Lemma I.3.4.2-(**ABc4r2**): For m_i constants:
and:

$D_i^{+}\equiv(\partial_i+m_i)\quad,\quad D_i^{-}\equiv(\partial_i-m_i)$

$D_i\equiv\begin{pmatrix}D_i^{+} & 0\\ 0 & D_i^{-}\end{pmatrix}\ ,\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^{-} & 0\\ 0 & D_i^{+}\end{pmatrix}\ ,\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0 & D_i^{-}\\ D_i^{+} & 0\end{pmatrix}\ ,\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0 & D_i^{+}\\ D_i^{-} & 0\end{pmatrix}$

then:

$$\left(-D_3^{\Leftrightarrow}D_1+D_0^{\Updownarrow}D_2+D_1^{\Leftrightarrow}D_3-D_2D_0^{\Updownarrow}\right)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Proof:

$$\left(-D_3^{\Leftrightarrow}D_1+D_0^{\Updownarrow}D_2+D_1^{\Leftrightarrow}D_3-D_2D_0^{\Updownarrow}\right)=-\begin{pmatrix}0 & (\partial_3-m_3)\\ (\partial_3+m_3) & 0\end{pmatrix}\begin{pmatrix}(\partial_1+m_1) & 0\\ 0 & (\partial_1-m_1)\end{pmatrix}+\begin{pmatrix}(\partial_0-m_0) & 0\\ 0 & (\partial_0+m_0)\end{pmatrix}\begin{pmatrix}(\partial_2+m_2) & 0\\ 0 & (\partial_{2m}-m\end{pmatrix}$$

$$=\begin{pmatrix}(\partial_0-m_0)(\partial_2+m_2)-(\partial_2+m_2)(\partial_{0m}-m_0) & -(\partial_3-m_3)(\partial_1-m_1)+(\partial_1-m_1)(\partial_3-m_3)\\ -(\partial_3+m_3)(\partial_1+m_1)+(\partial_1+m_1)(\partial_3+m_3) & (\partial_0+m_0)(\partial_{2m}-m_2)-(\partial_2-m_2)(\partial_0+m_0)\end{pmatrix}$$

$$=\begin{pmatrix}(+m_2-m_2)\partial_0+(m_0-m_0)\partial_2+(-m_0m_{2m}+m_2m_0) & (+m_3-m_3)\partial_1+(m_1-m_1)\partial_3+(-m_3m_1+m_1m_3\\ (+m_3-m_3)\partial_1+(m_1-m_1)\partial_{3n}+(-m_3m_1+m_1m_3) & (+m_2-m_2)\partial_0+(m_0-m_0)\partial_2+(-m_0m_2+m_2m_{0m}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

□

Lemma I.3.4.3-(**ABc4r3**): For m_i constants:
and:

$D_i^{+}\equiv(\partial_i+m_i)\quad,\quad D_i^{-}\equiv(\partial_i-m_i)$

$D_i\equiv\begin{pmatrix}D_i^{+} & 0\\ 0 & D_i^{-}\end{pmatrix}\ ,\ D_i^{\Updownarrow}\equiv\begin{pmatrix}D_i^{-} & 0\\ 0 & D_i^{+}\end{pmatrix}\ ,\ D_i^{\Leftrightarrow}\equiv\begin{pmatrix}0 & D_i^{-}\\ D_i^{+} & 0\end{pmatrix}\ ,\ D_i^{\Leftrightarrow\Updownarrow}\equiv\begin{pmatrix}0 & D_i^{+}\\ D_i^{-} & 0\end{pmatrix}$

then:

$$\left(D_2^{\Leftrightarrow}D_1-D_1^{\Leftrightarrow}D_2+D_0^{\Updownarrow}D_3-D_3D_0^{\Updownarrow}\right)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Proof:

$$\left(D_2^{\Leftrightarrow}D_1 - D_1^{\Leftrightarrow}D_2 + D_0^{\Updownarrow}D_3 - D_3D_0^{\Updownarrow}\right) = \begin{pmatrix} 0 & (\partial_2 - m_2) \\ (\partial_2 + m_2) & 0 \end{pmatrix}\begin{pmatrix} (\partial_1 + m_1) & 0 \\ 0 & (\partial_1 - m_1) \end{pmatrix} - \begin{pmatrix} 0 & (\partial_1 - m_1) \\ (\partial_1 + m_1) & 0 \end{pmatrix}\begin{pmatrix} (\partial_2 + m_2) & 0 \\ 0 & (\partial_2 - m_2) \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_0 - m_0)(\partial_3 + m_3) - (\partial_3 + m_3)(\partial_0 - m_0) & (\partial_2 - m_2)(\partial_1 - m_1) - (\partial_1 - m_1)(\partial_{2m} - m_2) \\ (\partial_2 + m_2)(\partial_1 + m_1) - (\partial_1 + m_1)(\partial_2 + m_2) & (\partial_0 + m_0)(\partial_{3m} - m_3) - (\partial_3 - m_3)(\partial_0 + m_0) \end{pmatrix}$$

$$= \begin{pmatrix} (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_0m_3 + m_3m_0) & (+m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_2m_1 - m_1m_2) \\ (+m_2 - m_2)\partial_1 + (m_1 - m_1)\partial_2 + (+m_2m_1 - m_1m_2) & (+m_3 - m_3)\partial_0 + (m_0 - m_0)\partial_3 + (-m_0m_3 + m_{3n}m_0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

□

Lemma I.3.4.4-(**ABc4r4**): For m_i constants:
and:
$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$$\left(D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D_2 + D_3^{\Updownarrow}D_3 + D_0D_0^{\Updownarrow}\right) = \begin{pmatrix} (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1m_1 + m_2m_2 + m_3m_3 + m_0m_0) & 0 \\ 0 & (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1m_1 + m_2m_2 + \end{pmatrix}$$

Proof:

$$\left(D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D_2 + D_3^{\Updownarrow}D_3 + D_0D_0^{\Updownarrow}\right) = \begin{pmatrix} (\partial_1 - m_1) & 0 \\ 0 & (\partial_1 + m_1) \end{pmatrix}\begin{pmatrix} (\partial_1 + m_1) & 0 \\ 0 & (\partial_1 - m_1) \end{pmatrix} + \begin{pmatrix} (\partial_2 - m_2) & 0 \\ 0 & (\partial_2 + m_2) \end{pmatrix}\begin{pmatrix} (\partial_2 + m_2) & 0 \\ 0 & (\partial_{2m} - m_2) \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_1 - m_1)(\partial_1 + m_1) + (\partial_2 - m_2)(\partial_2 + m_2) + (\partial_3 - m_3)(\partial_{3m} + m_3) + (\partial_0 + m_0)(\partial_{0m} - m_0) & \\ 0 & (\partial_1 + m_1)(\partial_1 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) + (m_1 - m_1)\partial_1 + (m_2 - m_2)\partial_2 + (m_3 - m_3)\partial_3 + (m_0 - m_0)\partial_0 - (m_1m_1 + m_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1m_1 + m_2m_2 + m_3m_3 + m_0m_0) & 0 \\ 0 & (\partial_1\partial_1 + \partial_2\partial_2 + \partial_3\partial_3 + \partial_0\partial_0) - (m_1m_1 \end{pmatrix}$$

□

Theorem II.1: Given: **B-factors** & **A-factors**: D_B & D_A ,
BA-factorizations & **AB-factorizations** are equal: $D_BD_A = D_AD_B$;
i.e.: **BA-factorizations** & **AB-factorizations** are commutative.

Proof:
From Theorem I.1:

$$D_BD_A = \begin{pmatrix} \left(D_0D_0^{\Updownarrow} + D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1D_1^{\Updownarrow}\right) & \left(D_0D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_0^{\Updownarrow} - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}\right) & \left(-D_0D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_0^{\Updownarrow} + D_1D_3^{\Updownarrow}\right) & \left(D_0D_1 - D_3^{\Leftrightarrow}D \right. \\ \left(D_3^{\Leftrightarrow}D_0^{\Updownarrow} - D_0D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_2D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_0D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0^{\Updownarrow} + D_2D_3^{\Leftrightarrow}\right) & \left(D_3^{\Leftrightarrow}D_1 + D_0D \right. \\ \left(-D_2^{\Leftrightarrow}D_0^{\Updownarrow} - D_1^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0D_2^{\Leftrightarrow} + D_3D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0^{\Updownarrow} - D_0D_1^{\Leftrightarrow} + D_3D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_3D_3^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow}D_1 + D_1^{\Leftrightarrow} \right. \\ \left(D_1^{\Updownarrow}D_0^{\Updownarrow} - D_2^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Updownarrow}\right) & \left(D_1^{\Updownarrow}D_3^{\Leftrightarrow} + D_2^{\Updownarrow}D_0^{\Updownarrow} - D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0^{\Updownarrow}D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0^{\Updownarrow} - D_0^{\Updownarrow}D_3^{\Updownarrow}\right) & \left(D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D \right. \end{pmatrix}$$

$$D_AD_B = \begin{pmatrix} \left(D_0^{\Updownarrow}D_0 + D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1D_1^{\Updownarrow}\right) & \left(-D_0^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow}D_0 - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}\right) & \left(D_0^{\Updownarrow}D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow}D_0 + D_1D_3^{\Updownarrow}\right) & \left(D_0^{\Updownarrow}D_1 - D_3^{\Leftrightarrow}D \right. \\ \left(-D_3^{\Leftrightarrow}D_0 + D_0^{\Updownarrow}D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_2D_1^{\Updownarrow}\right) & \left(D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0 + D_2D_3^{\Updownarrow}\right) & \left(-D_3^{\Leftrightarrow}D_1 + D_0^{\Updownarrow}D \right. \\ \left(D_2^{\Leftrightarrow}D_0 - D_1^{\Leftrightarrow}D_3^{\Leftrightarrow} - D_0^{\Updownarrow}D_2^{\Leftrightarrow} + D_3D_1^{\Updownarrow}\right) & \left(-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0 + D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_3D_2^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_3D_3^{\Updownarrow}\right) & \left(D_2^{\Leftrightarrow}D_1 - D_1^{\Leftrightarrow}D \right. \\ \left(D_1^{\Updownarrow}D_0 + D_2^{\Updownarrow}D_3^{\Leftrightarrow} - D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0D_1^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow}D_3^{\Leftrightarrow} + D_2^{\Updownarrow}D_0 + D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0D_2^{\Updownarrow}\right) & \left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0 - D_0D_3^{\Updownarrow}\right) & \left(D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D \right. \end{pmatrix}$$

So, from Lemmas I.2 & I.3:

$$\Rightarrow D_BD_A = \begin{pmatrix} \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} \end{pmatrix}$$

$$\Rightarrow D_AD_B = \begin{pmatrix} \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [\Box - |m|^2] & 0 \\ 0 & [\Box - |m|^2] \end{pmatrix} \end{pmatrix}$$

$$= [\Box - |m|^2] \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 \end{pmatrix}$$

$$= D_BD_A$$

where: $\Box \equiv \sum_{j=0}^{4-1} \partial_j^2$ & $|m|^2 \equiv \sum_{j=0}^{4-1} m_j^2$

□

Theorem III.1: For m_i constants:
and:
$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

then:

$D_iD_j^{\Updownarrow} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$	$D_iD_j^{\Leftrightarrow} = D_j^{\Leftrightarrow}D_i^{\Updownarrow}$	$D_i^{\Updownarrow}D_j^{\Leftrightarrow} = D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i = D_j^{\Updownarrow}D_i^{\Leftrightarrow}$
$D_i^{\Updownarrow}D_j = D_jD_i^{\Updownarrow} = D_i^{\Leftrightarrow}D_j^{\Leftrightarrow}$	$D_i^{\Updownarrow}D_j^{\Updownarrow} = D_j^{\Updownarrow}D_i^{\Updownarrow}$	$D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i = D_i^{\Updownarrow}D_j^{\Leftrightarrow} = D_j^{\Updownarrow}D_i^{\Leftrightarrow}$
$D_iD_j^{\Updownarrow} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$	$D_iD_j = D_jD_i$	

Proof:

$$D_iDj \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} D_i^+D_j^+ & 0 \\ 0 & D_i^-D_j^- \end{pmatrix} = DjD_i$$

$$D_iD_j^{\Updownarrow} \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} D_i^+D_j^- & 0 \\ 0 & D_i^-D_j^+ \end{pmatrix} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$$

$$D_iD_j^{\Leftrightarrow} \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_i^+D_j^- \\ D_i^-D_j^+ & 0 \end{pmatrix} = D_j^{\Leftrightarrow}D_i^{\Updownarrow}$$

$$D_i^{\Updownarrow}D_j \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} D_i^-D_j^+ & 0 \\ 0 & D_i^+D_j^- \end{pmatrix} = D_jD_i^{\Updownarrow} = D_i^{\Leftrightarrow}D_j^{\Leftrightarrow}$$

$$D_i^{\Updownarrow}D_j^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} D_i^-D_j^- & 0 \\ 0 & D_i^+D_j^+ \end{pmatrix} = D_j^{\Updownarrow}D_i^{\Updownarrow}$$

$$D_i^{\Updownarrow}D_j^{\Leftrightarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^- \\ D_i^+D_j^+ & 0 \end{pmatrix} = D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i$$
$$= D_j^{\Updownarrow}D_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} D_j^+ & 0 \\ 0 & D_j^- \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^- \\ D_i^+D_j^+ & 0 \end{pmatrix} = D_j^{\Leftrightarrow}D_i = D_i^{\Updownarrow}D_j^{\Leftrightarrow}$$
$$= D_j^{\Updownarrow}D_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j^{\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} D_j^- & 0 \\ 0 & D_j^+ \end{pmatrix} = \begin{pmatrix} 0 & D_i^-D_j^+ \\ D_i^+D_j^- & 0 \end{pmatrix} = D_jD_i^{\Leftrightarrow}$$

$$D_i^{\Leftrightarrow}D_j^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_j^- \\ D_j^+ & 0 \end{pmatrix} = \begin{pmatrix} D_i^-D_j^+ & 0 \\ 0 & D_i^+D_j^- \end{pmatrix} = D_i^{\Updownarrow}D_j = D_jD_i^{\Updownarrow}$$
$$= D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$$

□

definiton 1: For m_i constants:
and:
$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

and:

$$D_B \equiv \begin{pmatrix} -D_0 & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0 & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix} \;\&\; D_A \equiv \begin{pmatrix} -D_0^{\Updownarrow} & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix}$$

then:

$$\widetilde{D_B} \equiv \begin{pmatrix} -D_0^{\Updownarrow} & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix} \;\&\; \widetilde{D_A} \equiv \begin{pmatrix} -D_0 & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0 & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

(ie.: $\widetilde{D_B}$ & $\widetilde{D_A}$ are transformations of D_B & D_A where D_0 & $D_0^{\Updownarrow}$ are exchanged)

Theorem IV.1: For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix} , \; D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix} , \; D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix} , \; D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

and:

$$\widetilde{D_B} \equiv \begin{pmatrix} -D_0^{\Updownarrow} & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix} \;\&\; \widetilde{D_A} \equiv \begin{pmatrix} -D_0 & -D_3^{\Leftrightarrow} & D_2^{\Leftrightarrow} & -D_1 \\ D_3^{\Leftrightarrow} & -D_0 & -D_1^{\Leftrightarrow} & -D_2 \\ -D_2^{\Leftrightarrow} & D_1^{\Leftrightarrow} & -D_0 & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0^{\Updownarrow} \end{pmatrix}$$

then:

$$\widetilde{D_B}\widetilde{D_A} = \begin{pmatrix} (D_0^{\Updownarrow}D_0 + D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1D_1^{\Updownarrow}) & (D_0^{\Updownarrow}D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_0 - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}) & (-D_0^{\Updownarrow}D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_0 + D_1D_3^{\Updownarrow}) & (D_0^{\Updownarrow}D_1 - D_3^{\Leftrightarrow}D_2 \ldots \\ (D_3^{\Leftrightarrow}D_0 - D_0^{\Updownarrow}D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_2D_1^{\Updownarrow}) & (D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}) & (-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_0^{\Updownarrow}D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0 + D_2D_3^{\Updownarrow}) & (D_3^{\Leftrightarrow}D_1 + D_0^{\Updownarrow}D_2 \ldots \\ (-D_2^{\Leftrightarrow}D_0 - D_1^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0^{\Updownarrow}D_2^{\Leftrightarrow} + D_3D_1^{\Updownarrow}) & (-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0 - D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_3D_2^{\Updownarrow}) & (D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0^{\Updownarrow}D_0 + D_3D_3^{\Updownarrow}) & (-D_2^{\Leftrightarrow}D_1 + D_1^{\Leftrightarrow} \ldots \\ (D_1^{\Updownarrow}D_0 - D_2^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0D_1^{\Updownarrow}) & (D_1^{\Updownarrow}D_3^{\Leftrightarrow} + D_2^{\Updownarrow}D_0 - D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0D_2^{\Updownarrow}) & (-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0 - D_0D_3^{\Updownarrow}) & (D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D_2 \ldots \end{pmatrix}$$

$$\widetilde{D_A}\widetilde{D_B} = \begin{pmatrix} (D_0D_0^{\Updownarrow} + D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1D_1^{\Updownarrow}) & (-D_0D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow}D_0^{\Updownarrow} - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}) & (D_0D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow}D_0^{\Updownarrow} + D_1D_3^{\Updownarrow}) & (D_0D_1 - D_3^{\Leftrightarrow}D_2 \ldots \\ (-D_3^{\Leftrightarrow}D_0^{\Updownarrow} + D_0D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_2D_1^{\Updownarrow}) & (D_3^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2D_2^{\Updownarrow}) & (-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} - D_0D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0^{\Updownarrow} + D_2D_3^{\Updownarrow}) & (-D_3^{\Leftrightarrow}D_1 + D_0D_2 \ldots \\ (D_2^{\Leftrightarrow}D_0^{\Updownarrow} - D_1^{\Leftrightarrow}D_3^{\Leftrightarrow} - D_0D_2^{\Leftrightarrow} + D_3D_1^{\Updownarrow}) & (-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0^{\Updownarrow} + D_0D_1^{\Leftrightarrow} + D_3D_2^{\Updownarrow}) & (D_2^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_0D_0^{\Updownarrow} + D_3D_3^{\Updownarrow}) & (D_2^{\Leftrightarrow}D_1 - D_1^{\Leftrightarrow}D \ldots \\ (D_1^{\Updownarrow}D_0^{\Updownarrow} + D_2^{\Updownarrow}D_3^{\Leftrightarrow} - D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0^{\Updownarrow}D_1^{\Updownarrow}) & (-D_1^{\Updownarrow}D_3^{\Leftrightarrow} + D_2^{\Updownarrow}D_0^{\Updownarrow} + D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0^{\Updownarrow}D_2^{\Updownarrow}) & (-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0^{\Updownarrow} - D_0^{\Updownarrow}D_3^{\Updownarrow}) & (D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D_2 \ldots \end{pmatrix}$$

and:

$$\widetilde{D_B}\widetilde{D_A} = \widetilde{D_A}\widetilde{D_B} = [\Box - |m|^2] \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 \end{pmatrix}$$

$$= D_B D_A = D_A D_B$$

where: $\Box \equiv \sum_{j=0}^{4-1} \partial_j^2$ & $|m|^2 \equiv \sum_{j=0}^{4-1} m_j^2$

Proof:

By theorem III.1

$D_iD_j^{\Updownarrow} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$	$D_iD_j^{\Leftrightarrow} = D_j^{\Leftrightarrow}D_i^{\Updownarrow}$	$D_i^{\Updownarrow}D_j^{\Leftrightarrow} = D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i = D_j^{\Updownarrow}D_i^{\Leftrightarrow}$
$D_i^{\Updownarrow}D_j = D_jD_i^{\Updownarrow} = D_i^{\Leftrightarrow}D_j^{\Leftrightarrow}$	$D_i^{\Updownarrow}D_j^{\Updownarrow} = D_j^{\Updownarrow}D_i^{\Updownarrow}$	$D_i^{\Leftrightarrow}D_j = D_j^{\Leftrightarrow}D_i = D_i^{\Updownarrow}D_j^{\Leftrightarrow} = D_j^{\Updownarrow}D_i^{\Leftrightarrow}$
$D_iD_j^{\Updownarrow} = D_j^{\Updownarrow}D_i = D_j^{\Leftrightarrow}D_i^{\Leftrightarrow}$	$D_iD_j = D_jD_i$	

$\Downarrow$

$\widetilde{D_B}\widetilde{D_A}$

$\left(D_0^{\Updownarrow}D_0 + D_3^{\Updownarrow}D_3 + D_2^{\Updownarrow}D_2 + D_1^{\Updownarrow}D_1\right) = \sum_{j=0}^{4-3}\begin{pmatrix} D_j^-D_j^+ & 0 \\ 0 & D_j^+D_j^- \end{pmatrix}$	$\left(D_3^{\Updownarrow}D_3 + D_0^{\Updownarrow}D_0 + D_1^{\Updownarrow}D_1 + D_2^{\Updownarrow}D_2\right) = \sum_{j=0}^{4-3}\begin{pmatrix} D_j^-D_j^+ & 0 \\ 0 & D_j^+D_j^- \end{pmatrix}$	$(D_2^{\Updownarrow}D_2 + D_1^{\Updownarrow}D_1 + D_0^{\Updownarrow}D_0 + \ldots$

$\left(D_3^{\Leftrightarrow}D_0 - D_0^{\Updownarrow}D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_2D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(D_0^{\Updownarrow}D_3^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_0 - D_2^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_1D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_0^{\Updownarrow}D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow}D_1^{\Leftrightarrow} + D_2^{\Leftrightarrow}D_0 + D_1D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(D \ldots$
$\left(-D_2^{\Leftrightarrow}D_0 - D_1^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_0^{\Updownarrow}D_2^{\Leftrightarrow} + D_3D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_2^{\Leftrightarrow}D_3^{\Leftrightarrow} + D_1^{\Leftrightarrow}D_0 - D_0^{\Updownarrow}D_1^{\Leftrightarrow} + D_3D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_3^{\Leftrightarrow}D_2^{\Leftrightarrow} + D_0^{\Updownarrow}D_1^{\Leftrightarrow} - D_1^{\Leftrightarrow}D_0 + D_2D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(D \ldots$
$\left(D_1^{\Updownarrow}D_0 - D_2^{\Updownarrow}D_3^{\Leftrightarrow} + D_3^{\Updownarrow}D_2^{\Leftrightarrow} - D_0D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(D_1^{\Updownarrow}D_3^{\Leftrightarrow} + D_2^{\Updownarrow}D_0 - D_3^{\Updownarrow}D_1^{\Leftrightarrow} - D_0D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_1^{\Updownarrow}D_2^{\Leftrightarrow} + D_2^{\Updownarrow}D_1^{\Leftrightarrow} + D_3^{\Updownarrow}D_0 - D_0D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(- \ldots$

$$\begin{gathered}\Vert \\ \widetilde{D_A}\widetilde{D_B}\end{gathered}$$

$\left(D_0 D_0^{\Updownarrow} + D_3 D_3^{\Updownarrow} + D_2 D_2^{\Updownarrow} + D_1^{\Updownarrow} D_1\right) = \sum_{j=0}^{4-3} \begin{pmatrix} D_j^- D_j^+ & 0 \\ 0 & D_j^+ D_j^- \end{pmatrix}$	$\left(D_3 D_3^{\Updownarrow} + D_0 D_0^{\Updownarrow} + D_1 D_1^{\Updownarrow} + D_2 D_2^{\Updownarrow}\right) = \sum_{j=0}^{4-3} \begin{pmatrix} D_j^- D_j^+ & 0 \\ 0 & D_j^+ D_j^- \end{pmatrix}$	$\left(D_2 D_2^{\Updownarrow} + D_1 D_1^{\Updownarrow} + D_0 D_0^{\Updownarrow} + \ldots$

$\left(-D_3^{\Leftrightarrow} D_0^{\Updownarrow} + D_0 D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_2^{\Leftrightarrow} + D_2 D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_0 D_3^{\Leftrightarrow} + D_3^{\Leftrightarrow} D_0^{\Updownarrow} - D_2^{\Leftrightarrow} D_1^{\Leftrightarrow} + D_1 D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(D_0 D_2^{\Leftrightarrow} - D_3^{\Leftrightarrow} D_1^{\Leftrightarrow} - D_2^{\Leftrightarrow} D_0^{\Updownarrow} + D_1 D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(D\ldots$
$\left(D_2^{\Leftrightarrow} D_0^{\Updownarrow} - D_1^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_0 D_2^{\Leftrightarrow} + D_3 D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_2^{\Leftrightarrow} D_3^{\Leftrightarrow} - D_1^{\Leftrightarrow} D_0^{\Updownarrow} + D_0 D_1^{\Leftrightarrow} + D_3 D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_3^{\Leftrightarrow} D_2^{\Leftrightarrow} - D_0 D_1^{\Leftrightarrow} + D_1^{\Leftrightarrow} D_0^{\Updownarrow} + D_2 D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(-\ldots$
$\left(D_1^{\Updownarrow} D_0^{\Updownarrow} + D_2^{\Updownarrow} D_3^{\Leftrightarrow} - D_3^{\Updownarrow} D_2^{\Leftrightarrow} - D_0^{\Updownarrow} D_1^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_1^{\Updownarrow} D_3^{\Leftrightarrow} + D_2^{\Updownarrow} D_0^{\Updownarrow} + D_3^{\Updownarrow} D_1^{\Leftrightarrow} - D_0^{\Updownarrow} D_2^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$\left(-D_1^{\Updownarrow} D_2^{\Leftrightarrow} + D_2^{\Updownarrow} D_1^{\Leftrightarrow} + D_3^{\Updownarrow} D_0^{\Updownarrow} - D_0^{\Updownarrow} D_3^{\Updownarrow}\right) = \mathbf{0} + \mathbf{0}$	$(D\ldots$

So:

$$\widetilde{D_B}\widetilde{D_A} = \widetilde{D_A}\widetilde{D_B} = [\Box - |m|^2] \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 \end{pmatrix}$$

$$= D_B D_A = D_A D_B$$

where: $\Box \equiv \sum_{j=0}^{4-1} \partial_j^2 \;\;\&\;\; |m|^2 \equiv \sum_{j=0}^{4-1} m_j^2$

□

Theorem V.1: For m_i constants:

and:

$D_i^+ \equiv (\partial_i + m_i)$, $D_i^- \equiv (\partial_i - m_i)$

$$D_i \equiv \begin{pmatrix} D_i^+ & 0 \\ 0 & D_i^- \end{pmatrix}, \quad D_i^{\Updownarrow} \equiv \begin{pmatrix} D_i^- & 0 \\ 0 & D_i^+ \end{pmatrix}, \quad D_i^{\Leftrightarrow} \equiv \begin{pmatrix} 0 & D_i^- \\ D_i^+ & 0 \end{pmatrix}, \quad D_i^{\Leftrightarrow\Updownarrow} \equiv \begin{pmatrix} 0 & D_i^+ \\ D_i^- & 0 \end{pmatrix}$$

and:

$$\mathbf{f} \equiv \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix}, \qquad f^j \equiv \begin{pmatrix} f_+^j \\ f_-^j \end{pmatrix}$$

then:

$$\widetilde{D_B}\mathbf{f} = \begin{pmatrix} -D_0^{\Updownarrow} & D_3^{\Leftrightarrow} & -D_2^{\Leftrightarrow} & -D_1 \\ -D_3^{\Leftrightarrow} & -D_0^{\Updownarrow} & D_1^{\Leftrightarrow} & -D_2 \\ D_2^{\Leftrightarrow} & -D_1^{\Leftrightarrow} & -D_0^{\Updownarrow} & -D_3 \\ -D_1^{\Updownarrow} & -D_2^{\Updownarrow} & -D_3^{\Updownarrow} & D_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} -D_0^{\Updownarrow} f^1 + D_3^{\Leftrightarrow} f^2 - D_2^{\Leftrightarrow} f^3 - D_1 f^0 \\ -D_3^{\Leftrightarrow} f^1 - D_0^{\Updownarrow} f^2 + D_1^{\Leftrightarrow} f^3 - D_2 f^0 \\ D_2^{\Leftrightarrow} f^1 - D_1^{\Leftrightarrow} f^2 - D_0^{\Updownarrow} f^3 - D_3 f^0 \\ -D_1^{\Updownarrow} f^1 - D_2^{\Updownarrow} f^2 - D_3^{\Updownarrow} f^3 + D_0 f^0 \end{pmatrix} = \begin{pmatrix} E^1 - B_{\Updownarrow}^1 \\ E^2 - B_{\Updownarrow}^2 \\ E^3 - B_{\Updownarrow}^3 \\ -D_1^{\Updownarrow} f^1 - D_2^{\Updownarrow} f^2 - D_3^{\Updownarrow} f^3 + D_0 f^0 \end{pmatrix} = \mathbf{E} - \mathbf{B}_{\Updownarrow}$$

□

Note: the 0th component: $-D_1^{\Updownarrow} f^1 - D_2^{\Updownarrow} f^2 - D_3^{\Updownarrow} f^3 + D_0 f^0$ represents the guage of the electromagnetic-nuclear field.
Bote the variousness of the □ and field-potentials are harmonious with legacy electromagnetic field potentials and Helmholtz and Klein-Gordon differential equations under appropriate x^0 transformations
(such as: $\pm ict, -ct$, etc.)

REFERENCES

[1] Cassano, Claude.Michael ; "Reality is a Mathematical Model", 2010.
ISBN: 1468120921 ; http://www.amazon.com/dp/1468120921
ASIN: B0049P1P4C ;
http://www.amazon.com/Reality-Mathematical-Modelebook/dp/B0049P1P4C/ref=tmm_kin_swatch_0?_encoding=UTF8&sr=&qid
[2] Cassano, Claude.Michael ; "A Mathematical Preon Foundation for the Standard Model", 2011.
ISBN:1468117734 ; http://www.amazon.com/dp/1468117734
ASIN: B004IZLHI2 ; http://www.amazon.com/Mathematical-Preon-Foundation-Standardebook/dp/B004IZLHI2/ref=tmm_kin_swatch_0?_encoding=UTF8&sr=&qid=