Some Facts about Relations and Operations of Algebras

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Abstract Let $\boldsymbol{A}$ be a σ-algebra. Suppose that Θ is a congruence of $\boldsymbol{A}$. Then Θ is a subalgebra of $\boldsymbol{A} \times \boldsymbol{A}$. If ϕ is an automorphism from $\boldsymbol{A}$ to $\boldsymbol{A}$, then $\langle \phi, \phi \rangle$ is an automorphism of $\boldsymbol{A} \times \boldsymbol{A}$. And it is obvious that $\langle \phi, \phi \rangle(\Theta)$ is a congruence of $\boldsymbol{A}$. Let $\boldsymbol{B}$ be a σ-algebra and ψ a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Then $\boldsymbol{B'} := \psi(A)$ is a subalgebra of $\boldsymbol{B}$. And $\langle \psi, \psi \rangle(\Theta)$ is a congruence of $\boldsymbol{B'}$. If ψ is an epimorphism, then $\langle \psi, \psi \rangle(\Theta)$ is a congruence of $\boldsymbol{B}$. Suppose that $\mathcal{A}$ is a category of all σ-algebras. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{A}$ and $\psi \colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism. Then the pullback $\boldsymbol{A} \sqcap_{\boldsymbol{B}} \boldsymbol{A}$ is isomorphic to a congruence of $\boldsymbol{A}$. An n-ary relation Φ of an algebra $\boldsymbol{A}$ is a subset of $\boldsymbol{A}^n$. If Φ satisfies some conditions, then Φ is a subalgebra of $\boldsymbol{A}^n$. The set of languages is a lattice. If $\boldsymbol{\Sigma}$ is the set of the compositions of the operations in a language σ, then $\boldsymbol{\Sigma}$ is an algebra.

Contents

1. Congruence

Let σ be an algebraic language[2]. Suppose that $\boldsymbol{A}$ is an algebra of language σ. Let $\boldsymbol{Con}\ \boldsymbol{A}$[4] be the set of all congruences on an algebra $\boldsymbol{A}$. If $\Theta \in \boldsymbol{Con}\ \boldsymbol{A}$, then Θ is a subalgebra of $\boldsymbol{A} \times \boldsymbol{A}$ (See [1]). Suppose that $\phi \colon \boldsymbol{A} \to \boldsymbol{A}$ is an automorphism[4]. Then

$$\langle \phi, \phi \rangle \colon \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A} \times \boldsymbol{A}$$

given by

$$\langle a, b \rangle \mapsto \langle \phi(a), \phi(b) \rangle \text{ for all } \langle a, b \rangle \in \boldsymbol{A} \times \boldsymbol{A}$$

is an automorphism of $\boldsymbol{A} \times \boldsymbol{A}$.

Proposition 1.1 (cf. [2, 4])**.** *Let ϕ be an automorphism of $\boldsymbol{A}$ and $\Theta \in \boldsymbol{Con}\ \boldsymbol{A}$. Then $\langle \phi, \phi \rangle(\Theta) \in \boldsymbol{Con}\ \boldsymbol{A}$. If Θ is fully invariant*[4]*, then $\langle \phi, \phi \rangle(\Theta) \subseteq \Theta$.*

Proof. Let $a, b, c \in \boldsymbol{A}$ with $(a, b), (b, c) \in \Theta$. Then $\langle \phi(a), \phi(b) \rangle$, $\langle \phi(b), \phi(a) \rangle$, $\langle \phi(a), \phi(a) \rangle$, $\langle \phi(a), \phi(c) \rangle$ are in $\langle \phi, \phi \rangle(\Theta)$. If Θ is fully invariant then $(\phi(a), \phi(b)) \in \Theta$. It follows $\langle \phi, \phi \rangle(\Theta) \subseteq \Theta$. An automorphism is compatible with the operations in language of $\boldsymbol{A}$. Hence $\langle \phi, \phi \rangle(\Theta)$ is a congruence. □

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Let $\boldsymbol{B}$ be a σ-algebra and ψ a homomorphism[4] from $\boldsymbol{A}$ to $\boldsymbol{B}$. Then the image of ψ is a subalgebra[4] of $\boldsymbol{B}$. And we have a homomorphism

$$\langle \psi, \psi \rangle : \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{B} \times \boldsymbol{B}$$

given by

$$\langle a, a' \rangle \mapsto \langle \psi(a), \psi(a') \rangle \text{ for all } \langle a, a' \rangle \in \boldsymbol{A} \times \boldsymbol{A}$$

If ψ is an epimorphism[4], then $\langle \psi, \psi \rangle$ is an epimorphism.

Proposition 1.2 (cf. [2, 4])**.** *Let $\boldsymbol{A}, \boldsymbol{B}$ be σ-algebras, $\Theta \in \boldsymbol{Con}\, \boldsymbol{A}$ and $\psi : \boldsymbol{A} \to \boldsymbol{B}$ a homomorphism. Suppose that $\boldsymbol{B'}$ is the image of ψ. Then $\langle \psi, \psi \rangle(\Theta) \in \boldsymbol{Con}\, \boldsymbol{B'}$. If ψ is an epimorphism, then $\langle \psi, \psi \rangle(\Theta) \in \boldsymbol{Con}\, \boldsymbol{B}$.*

Proof. Let $a, b \in \boldsymbol{A}$. If $\langle a, b \rangle \in \Theta$ with $\psi(a) = \psi(b)$, then $\langle \psi(a), \psi(b) \rangle \in \langle \psi, \psi \rangle(\Theta)$. Then the proof is similar to proposition 1.1 □

2. Congruence and Pullback

In a category, if a pullback[3] of $A \xrightarrow{f} B \xleftarrow{f} A$ exists, then it is called kernel(See [3]). If $\boldsymbol{A}, \boldsymbol{B}$ are σ-algebras and $\psi : \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, then the pullback of $\boldsymbol{A} \xrightarrow{\psi} \boldsymbol{B} \xleftarrow{\psi} \boldsymbol{A}$ is isomorphic to the kernel of ψ. A kernel of homomorphism is a congruence (See [4]). On the other hand, for every $\Theta \in \boldsymbol{Con}\, \boldsymbol{A}$, the congruence Θ is a kernel of $\boldsymbol{A} \to \boldsymbol{A}/\Theta$ (See [4]).

Proposition 2.1 (cf. [2–4])**.** *Suppose that $\mathcal{A}$ is the category of σ-algebras. Every congruence of a σ-algebra is a pullback in $\mathcal{A}$.*

Proof. It is obvious. □

3. n-ary Relation and Operation

Let $\boldsymbol{A}$ be a algebra of language σ. Suppose that $\boldsymbol{A}$ has an n-ary operation f and an n-ary relation Φ. Let g be an m-ary operation of $\boldsymbol{A}$ and $a_{ij} \in \boldsymbol{A}$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Definition 3.1 (cf. [2, 4])**.** Suppose that $\langle a_{1j}, a_{2j}, \ldots, a_{nj} \rangle \in \Phi$ for $1 \leq j \leq m$. If

$$\langle g(a_{11}, a_{12}, \ldots, a_{1m}), g(a_{21}, a_{22}, \ldots, a_{2m}), \ldots, g(a_{n1}, a_{n2}, \ldots, a_{nm}) \rangle \in \Phi$$

then we say that Φ is **compatible with** g.

Definition 3.2 (cf. [2, 4])**.** If

$$f(g(a_{11}, a_{12}, \ldots, a_{1m}), g(a_{21}, a_{22}, \ldots, a_{2m}), \ldots, g(a_{n1}, a_{n2}, \ldots, a_{nm}))$$
$$= g(f(a_{11}, a_{21}, \ldots, a_{n1}), f(a_{12}, a_{22}, \ldots, a_{n2}), \ldots, f(a_{1m}, a_{21}, \ldots, a_{nm}))$$

then we say that f is **distributive with** g.

An n-ary relation of $\boldsymbol{A}$ is a subset of $\boldsymbol{A}^n$ (See [2]). Hence if $\langle a_1, a_2, \ldots, a_n \rangle \in \Phi$, then $f(a_1, a_2, \ldots, a_n) \in \boldsymbol{A}$. And $f(\Phi)$ is a subset of $\boldsymbol{A}$.

Proposition 3.1. *If Φ is compatible with all operations of $\boldsymbol{A}$ and f is distributive with all operations of $\boldsymbol{A}$, then $f(\Phi)$ is a subalgebra of $\boldsymbol{A}$.*

Lemma 3.1. *If Φ is compatible with all operations of $\boldsymbol{A}$, then Φ is a subalgebra of $\boldsymbol{A}^n$.*

Proof. It is obvious. □

Lemma 3.2. *If f is distributive with all operations of $\boldsymbol{A}$, then f induces a homomorphism*

$$\tilde{f}: \boldsymbol{A}^n \to \boldsymbol{A}$$

given by

$$\langle a_1, \ldots a_n \rangle \mapsto f(a_1, \ldots a_n)$$

Proof. It is obvious. □

Proof of proposition 3.1. That $f(\Phi)$ is the image of the $\tilde{f}$ restricted to the subalgebra Φ. □

4. n-ary Relation and Homomorphism

Suppose that $\mathcal{S}$ is the category[3] of τ-algebras. Suppose that the algebras in $\mathcal{S}$ have n-ary relation Φ, and that Φ is compatible with all operations of algebras in $\mathcal{S}$. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{S}$, $\alpha: \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism[2]. Then $\Phi_A, \Phi_B \in \mathcal{S}$. Suppose that α preserves Φ. Then the homomorphism α induces a homomorphism

$$\vec{\alpha}: \Phi_A \to \Phi_B$$

given by

$$\langle a_1, \ldots, a_n \rangle \mapsto \langle \alpha(a_1), \ldots, \alpha(a_n) \rangle$$

If β preserves Φ, then $\overrightarrow{\beta \circ \alpha} = \vec{\beta} \circ \vec{\alpha}$ is a homomorphism.

Let $\dot{\mathcal{S}}$ be a subcategory of $\mathcal{S}$ defined by

Objects: the objects of $\mathcal{S}$
morphisms: the set $\{\alpha \in Hom_S(A, B) \mid \alpha$ preserves the n-ary relation $\Phi\}$

The we may define a functor.

Proposition 4.1. *Let F be a morphism from $\dot{\mathcal{S}}$ to $\mathcal{S}$ given by*

Object: $\boldsymbol{A} \mapsto \Phi_A$;
morphism: $\alpha \mapsto \vec{\alpha}$;

Then F is a functor[3].

Proof. We have that $F(Id_A) = \overrightarrow{Id_A} = Id_{\Phi_A}$. Suppose that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{S}$, $\alpha: \boldsymbol{A} \to \boldsymbol{B}$ and $\beta: \boldsymbol{B} \to \boldsymbol{C}$ are morphisms in $\dot{\mathcal{S}}$. Then we have that

$$F(\beta \circ \alpha) = \overrightarrow{\beta \circ \alpha} = \vec{\beta} \circ \vec{\alpha} = F(\beta) \circ F(\alpha)$$ □

Hence the statement is true.

5. Lattice of Languages

We say that A is a structure[2] of language $\emptyset$ if A is a set. Let $\boldsymbol{\Omega}$ be a set of some languages[2, 4].

Proposition 5.1. *Let $\tau, \sigma \in \boldsymbol{\Omega}$. Define*

$$\tau \vee \sigma := \tau \cup \sigma$$
$$\tau \wedge \sigma := \tau \cap \sigma$$

If $\tau \vee \sigma \in \Omega$ and $\tau \wedge \sigma \in \Omega$, then $\boldsymbol{\Omega}$ is a lattice.

Proof. Let $\sigma, \tau, \upsilon \in \mathbf{\Omega}$. Then we have $\tau \vee \sigma \in \mathbf{\Omega}$ and $\tau \wedge \sigma \in \mathbf{\Omega}$. And $\mathbf{\Omega}$ satisfies the following equations:

(Idempotent)	$\tau \vee \tau = \tau$
(Idempotent)	$\tau \wedge \tau = \tau$
(Commutative)	$\tau \vee \sigma = \sigma \vee \tau$
(Commutative)	$\tau \wedge \sigma = \sigma \wedge \tau$
(Associative)	$(\tau \vee \sigma) \vee \upsilon = \tau \vee (\sigma \vee \upsilon)$
(Associative)	$(\tau \wedge \sigma) \wedge \upsilon = \tau \wedge (\sigma \wedge \upsilon)$
(Absorption)	$\tau \vee (\tau \wedge \sigma) = \tau$
(Absorption)	$\tau \wedge (\tau \vee \sigma) = \tau$

□

Suppose that $\boldsymbol{A}$ is a structure of language τ. If f is an n-ary operation in τ, then let f^A[4] denote an n-ary operation of $\boldsymbol{A}$.

Definition 5.1. Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are τ-structure and σ-structure, respectively. Then a function $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism provided

$$\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n))$$

for all operations $f \in \tau \wedge \sigma$ and all $a_1, \ldots, a_n \in \boldsymbol{A}$; And if

$$\langle b_1, \ldots, b_m \rangle \in \Theta_A$$

then

$$\langle \varphi(b_1), \ldots, \varphi(b_m) \rangle \in \Theta_B$$

for all relations $\Theta \in \tau \wedge \sigma$ and all $b_1, \ldots, b_m \in \boldsymbol{A}$. If $\tau \wedge \sigma = \emptyset$, then φ is a function of the sets.

Proposition 5.2. *If φ is the homomorphism which is defined in definition 5.1, then the image of φ is a structure of language $\tau \wedge \sigma$.*

Proof. It is obvious. □

6. Algebraic Language Algebra

Suppose that σ is an algebraic language. Let $f \in \sigma$ be an n-ary operation and $g_i \in \sigma$ n_i-ary operations for $1 \le i \le n$. Then $h \coloneqq f(g_1, \ldots, g_n)$ is a $(\sum_1^n n_i)$-ary operation. And the operation h is a composition of $f, g_1, \ldots, g_n$. A compostion of the compostions of some operations in σ is a composition of the operations in σ. Let $\mathbf{\Sigma}$ be the set of the compositions of operations in σ.

Proposition 6.1. *The set $\mathbf{\Sigma}$ is an algebra of the language σ.*

Proof. It is obvious. □

We say that the algebra $\mathbf{\Sigma}$ is generated by σ.

Corollary 6.1. *The algebra $\mathbf{\Sigma}$ has the language $\mathbf{\Sigma}$.*

Proof. It is obvious. □

Proposition 6.2. *We define a binary relation Ξ in $\mathbf{\Sigma}$: $\langle f, g \rangle \in \Xi$ if f and g have same arity*[2, 4]. *Then the binary relation Ξ is a congruence relation.*

Proof. It is obvious. □

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