

Primality test and factorization using Lambert series near $q=1$

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Abstract

In this paper we will demonstrate that we can check if a number is prime, the number of factors it has and even information about these factors using the following integral:

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (1)$$

Where p is the number to be checked and $\mathcal{L}_{(e^{-j\omega})}(s, 2)$ represents the Lambert series that will be studied in the paper. All the steps to arrive to that integral will be shown although they will be very similar to the already shown in [12][13].

In this paper to calculate the Lambert series apart from the summation that comes from its definition, it will be used the paper by Banerjee-Wilkerson [14] that provides closed solutions for different cases.

In the conclusions, it will be shown if this exercise is computationally worth to make the primality test and calculate the factors of a number p .

Keywords

Primality test, factorization, pq-number, semiprime, prime, number theory, Lambert Series, Banerjee Wilkerson paper

1. Introduction

In this paper we will demonstrate that we can check if a number is prime, the number of factors it has and even information about these factors using the following integral:

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (1)$$

Where p is the number to be checked and $\mathcal{L}_{(e^{-j\omega})}(s, 2)$ represents the Lambert series that will be studied in the paper. All the steps to arrive to that integral will be shown although they will be very similar to the already shown in [12][13].

In this paper to calculate the Lambert series apart from the summation that comes from its definition, it will be used the paper by Banerjee-Wilkerson [14] that provides closed solutions for different cases.

In the conclusions, it will be shown if this exercise is computationally worth to make the primality test and calculate the factors of a number p .

2. Matrix of composite numbers

If you have already read [12] or [13], chapters 1 to 7 will be very familiar to you. Anyhow, I recommend to, at least, check them quickly as there are some differences, mainly to prepare the path to introduce the Banerjee-Wilkerson paper [14] in the calculations.

It is possible to create a matrix with the only composite numbers in certain domain. We can create it assuring that each element of the matrix is the product of every two integers in the domain. We can perform this operation one by one until having used all the integers. So, the result will be all the composite numbers in a certain domain.

You can see an example in figure 1,

$$\begin{bmatrix} 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 5 \\ 3 \cdot 2 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 5 \\ 4 \cdot 2 & 4 \cdot 3 & 4 \cdot 4 & 4 \cdot 5 \\ 5 \cdot 2 & 5 \cdot 3 & 5 \cdot 4 & 5 \cdot 5 \end{bmatrix}$$

Fig.1

And the result attached in figure 2:

$$\begin{bmatrix} 4 & 6 & 8 & 10 \\ 6 & 9 & 12 & 15 \\ 8 & 12 & 16 & 20 \\ 10 & 15 & 20 & 25 \end{bmatrix}$$

Fig.2

We can see that all the composite numbers between 4 and 10 are included in the matrix: 4,6,8,9,10. And as expected, the prime numbers 2,3,5 and 7 are not. But we can see that the number 14 that is composite is not included. Why? Because it is composed by $7 \cdot 2$ and the 7 is not included in the multiplications. This means, the domain where it is true that all the composite numbers are included (and none of the primes) finishes in $2 \cdot 5 = 10$ (being 5 the last number we are using in the multiplications).

But we can consider a general matrix with infinite rows and infinite columns. This means, we are including all the composite numbers in this matrix, meaning that the natural numbers that will not appear in this hypothetical matrix are all the prime numbers.

$$\begin{bmatrix} 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 5 & \dots \\ 3 \cdot 2 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 5 & \dots \\ 4 \cdot 2 & 4 \cdot 3 & 4 \cdot 4 & 4 \cdot 5 & \dots \\ 5 \cdot 2 & 5 \cdot 3 & 5 \cdot 4 & 5 \cdot 5 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Fig.3

$$\begin{bmatrix} 4 & 6 & 8 & 10 & \dots \\ 6 & 9 & 12 & 15 & \dots \\ 8 & 12 & 16 & 20 & \dots \\ 10 & 15 & 20 & 25 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Fig.4

3. Including the elements of the composite matrix in the discrete time domain

Now, the next movements will be quite straight forward. The first thing we will do, is to introduce these calculated composite numbers in the discrete time domain. We will put delta functions in the positions of the x axis indicated by the matrix of composite numbers (figure 5).

The difference with respect to previous papers [12][13] is that instead of locating unitary delta functions, we will have them escalated (the height of the Dirac delta) by one of the factors that composes these composite numbers. More precisely, the escalation will correspond to the factor to the power of a real parameter that we will call s. See graph below:

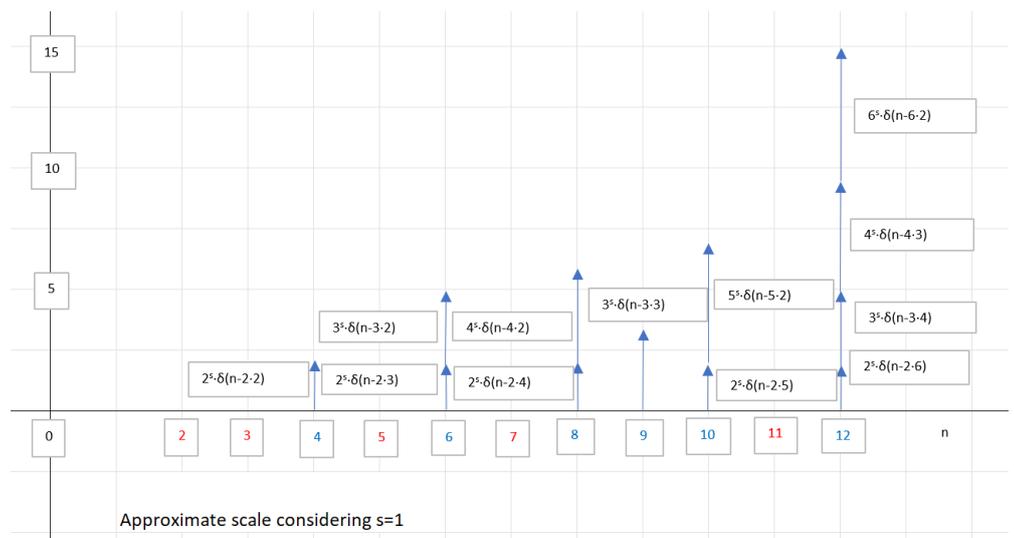


Fig.5

We can see in the above figure, that in the horizontal axis, each Dirac delta is located in the position of a composite number that is obtained as the product of two integers. This is, each Dirac delta position represents the factorization of two numbers of each composite number. And the order of the factors counts. This is, $\delta(n-2 \cdot 3)$ is one delta and $\delta(n-3 \cdot 2)$ counts as a second delta in the same position.

Therefore, if we do not have any delta in a certain position (see the red numbers), this means that the number is prime.

If we have only one Dirac Delta in one position, it means that the number is the square of a prime number (see $4=2 \cdot 2$ or $9=3 \cdot 3$ for example). The reason for this, is that if the order counts, the only way to have only one product -and not a second, exchanging the order- is that the two numbers of the product are the same (leading to a square).

If the number of Dirac deltas is odd but higher than one, the number has an integer square root (but this root is not a prime). If the number of deltas is even, it has different permutations of different products, but it does not have an integer square root.

There is also the special case when the number of deltas is two. In general, this would mean that the number is semiprime (pq-number) [3]. As generally, two deltas represent the product of two prime numbers in the two possible orders. See the number 6 for example ($2 \cdot 3$ and $3 \cdot 2$). But you can also check that the number 8 has two deltas but is not a semiprime. We will come back to this later.

Regarding the vertical axis, as commented, the height of each delta it depends in one of the factors that composes the composite number. In fact, the height of each delta is the multiplication of this factor to the power of a parameter that we have called s. We will come to this (and the reason for this) later.

4. Creating the functions f(n) and g(n)

We can create a function that is the sum of all these deltas. And this function will have the information of all these composite numbers. This function in the discrete domain is defined like this:

$$\begin{aligned}
 f(n) &= \sum_{k_2=2}^{k_2=\infty} \sum_{k_1=2}^{k_1=\infty} k_1^s \cdot \delta(n - k_1 \cdot k_2) = \sum_{k_2=2}^{k_2=\infty} (2^s \cdot \delta(n - 2 \cdot k_2) + 3^s \cdot \delta(n - 3 \cdot k_2) + 4^s \\
 &\quad \cdot \delta(n - 4 \cdot k_2) + \dots) = \\
 &\quad = 2^s \cdot \delta(n - 2 \cdot 2) + 2^s \cdot \delta(n - 2 \cdot 3) + \dots + \\
 &\quad + 3^s \cdot \delta(n - 3 \cdot 2) + 3^s \cdot \delta(n - 3 \cdot 3) \dots 4^s \cdot \delta(n - 4 \cdot 2) + \dots \quad (2)
 \end{aligned}$$

This function f(n) has the information regarding all the composite numbers. But the important information there, is what it is not included. All the integers that are not included in this function, are the prime numbers.

How do we spot them? If we want to know if the number 5 is prime or not. We have to check if there is a Dirac delta in position 5. This means, we have to check if the following function g(n) (or more precisely whatever scalation of it) is inside the function f(n):

$$g(n) = \delta(n - 5) \quad (3)$$

In general, if we want to check if whatever number p is prime, g(n) would be:

$$g(n) = \delta(n - p) \quad (4)$$

We could do this performing the multiplication one by one for all the points in the x axis between the function f(n) and the function g(n), as follows:

$$\sum_{n=-\infty}^{n=\infty} f(n) \cdot g(n) = \sum_{n=4}^{n=\infty} f(n) \cdot g(n) \quad (5)$$

As the function g(n) is zero everywhere except in the point p, if the result of the multiplication point by point between f(n) and g(n) is zero, it means that f(n) does not have a Dirac delta in p. So, in that case (no deltas in p), p would be prime.

If the result is different, it will tell us the value of f(n) at that point. And the value of f(n) at that point is the sum of the heights of all the deltas included at that point. This means, this sum will give us information about the factors that compose that number.

The problem is that to perform this operation in the time domain, does not add any value. Because to know the result, we have to perform all the multiplications between f(n) and g(n) in all the points of the domain, to see if f(n) has a value different from zero in the position where g(n) has the delta.

The only way to perform this operation efficiently is to do it in the frequency domain as we will see now.

5. The frequency domain

What we will do is to convert the functions f(n) and g(n) to the frequency domain. To do so, we use the Fourier transform.

We start with f(n). We know from signal theory that the Fourier transform of a shifted Dirac delta is:

$$\delta(n - a) \rightarrow e^{-aj\omega} \quad (6)$$

So, the function f(n), in the frequency domain has the form F(ω):

$$\begin{aligned} F(\omega) &= \sum_{k_1=2}^{k_1=\infty} \sum_{k_2=2}^{k_2=\infty} k_1^s \cdot k_2^s \cdot e^{-k_1 \cdot k_2 j\omega} = \\ &= 2^s \cdot e^{-2 \cdot 2 j\omega} + 2^s \cdot e^{-2 \cdot 3 j\omega} + 2^s \cdot e^{-2 \cdot 4 j\omega} + \dots + \\ &\quad + 3^s \cdot e^{-3 \cdot 2 j\omega} + 3^s \cdot e^{-3 \cdot 3 j\omega} + 3^s \cdot e^{-3 \cdot 4 j\omega} + \dots \\ &\quad + 4^s \cdot e^{-4 \cdot 2 j\omega} + 4^s \cdot e^{-4 \cdot 3 j\omega} + 4^s \cdot e^{-4 \cdot 4 j\omega} + \dots \quad (7) \end{aligned}$$

We can put above sum in the following form:

$$\begin{aligned} F(\omega) &= 2^s \cdot (e^{-2j\omega})^2 + 2^s \cdot (e^{-2j\omega})^3 + 2^s \cdot (e^{-2j\omega})^4 + \dots + \\ &\quad + 3^s \cdot (e^{-3j\omega})^2 + 3^s \cdot (e^{-3j\omega})^3 + 3^s \cdot (e^{-3j\omega})^4 + \dots + \\ &\quad + 4^s \cdot (e^{-4j\omega})^2 + 4^s \cdot (e^{-4j\omega})^3 + 4^s \cdot (e^{-4j\omega})^4 + \dots \quad (8) \end{aligned}$$

If we consider the sum when m tends to infinity, we see that each line is a geometric series sum to infinity [8]. Yes, we will discuss regarding convergence later (this discussion could be a thesis in itself). I know the module of the element that is being powered is exactly 1 (not less than 1 as should be in principle required). Anyhow, considering that this could be done:

$$\begin{aligned}
F(\omega) &= 2^s \cdot \frac{e^{-2 \cdot 2j\omega}}{1 - e^{-2j\omega}} + \\
&\quad + 3^s \cdot \frac{e^{-3 \cdot 2j\omega}}{1 - e^{-3j\omega}} + \\
&\quad + 4^s \cdot \frac{e^{-4 \cdot 2j\omega}}{1 - e^{-4j\omega}} + \dots = \\
&= \sum_{k=2}^{k=\infty} \frac{k^s \cdot e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \quad (9)
\end{aligned}$$

Now we change the k to start counting in 1 instead of 2. To do so we have to subtract the first term. Remember that whatever s is, 1^s will always be 1, so that factor disappears for the first term:

$$F(\omega) = -\frac{e^{-2j\omega}}{1 - e^{-j\omega}} + \sum_{k=1}^{k=\infty} \frac{k^s \cdot e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \quad (10)$$

Now yes, the summation in the right has the form of the definition of Lambert series given by Banerjee-Wilkerson [14] (1.1):

$$\mathcal{L}_q(s, x) = \sum_{k=1}^{k=\infty} \frac{k^s \cdot q^{k \cdot x}}{1 - q^k} \quad (11)$$

In fact, if we make with x=2 and q=e^{-jw}:

$$\mathcal{L}_{(e^{-j\omega})}(s, 2) = \sum_{k=1}^{k=\infty} \frac{k^s \cdot (e^{-j\omega})^{k \cdot 2}}{1 - (e^{-j\omega})^k} = \sum_{k=1}^{k=\infty} \frac{k^s \cdot e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \quad (12)$$

So, taking equations (10) and (12) and reordering we have:

$$F(\omega) = \mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (13)$$

This is key, because the Lambert series are studied for different values of s even when the module of q is near 1 in the paper Banerjee-Wilkerson [14]. And we can take advantage of that to calculate F(ω) in closed forms (not needing even sometimes to make the summation). As in our case q is an imaginary exponential e^{-jw} (its module is exactly |q|=1) we can use the properties commented in that paper [14]. We will see this in more detail.

For the function g(n), following the same process:

$$g(n) = \delta(n - p) \quad (14)$$

The Fourier transform $G(\omega)$ of $g(n)$ would be:

$$g(n) = \delta(n - p) \rightarrow G(\omega) = e^{-pj\omega} \quad (15)$$

6. Parseval's theorem

Now, we get to the point. In the chapter 4, we wanted to perform the following operation:

$$\sum_{n=-\infty}^{n=\infty} f(n) \cdot g(n) = \sum_{n=4}^{n=\infty} f(n) \cdot g(n) \quad (16)$$

If the result q is different from zero, it means that p is composite, as it means that:

$$g(n) = \delta(n - p) \quad (17)$$

$g(n)$ has found delta/s in the same position in the function $f(n)$ defined before, that has only deltas in composite number positions.

This operation in the time domain does not have added value (as you have to perform all multiplications), but fortunately a guy called Marc-Antoine Parseval [6][7] saved our day discovering the following theorem.

The general Parseval's theorem applied to discrete functions (as it is the case) says the following [7] Page 48/51:

$$\sum_{n=-\infty}^{n=\infty} f(n) \cdot g^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) \cdot G^*(\omega) d\omega \quad (18)$$

The asterisk* means the complex conjugate. All the functions we are dealing in this paper with, are real in the time domain so in the left-hand side no conjugation is necessary.

$$g^*(n) = g(n) \quad (19)$$

Remember that in chapter 3 we have commented that we will use always a real value for s . In a general study about Lambert series, s could be complex. But for our study of this paper (regarding primality and factors of a number p) we will consider s always real.

For the righthand side, yes, we have complex functions. But as you can check all of them are exponential based. So, the conjugate is just to change the sign of the imaginary part of the exponent. This is immediate from Euler's formula [9].

$$G(\omega) = e^{-pj\omega} \quad (19.1)$$

$$G^*(\omega) = e^{pj\omega} \quad (20)$$

So, coming back, this means, we can perform the operation of multiplying point by point

the two functions in the time domain, just making the definite integral form $-\pi$ to π of the multiplication of the two functions (its Fourier Transform) in the frequency domain.

So, the integral in the frequency domain (that represents the multiplication point by point of the functions $f(n)$ and $g(n)$ in the time domain) according Parseval's theorem is the following (and we will call its result q):

$$q = \sum_{n=-\infty}^{n=\infty} f(n) \cdot g^*(n) = \sum_{n=-\infty}^{n=\infty} f(n) \cdot g(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) \cdot G^*(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(-\frac{e^{-2j\omega}}{1 - e^{-j\omega}} + \mathcal{L}_{(e^{-j\omega})}(s, 2) \right) \cdot e^{pj\omega} d\omega \quad (21)$$

Reordering terms for future convenience:

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (22)$$

We will call q the result of above expression. If q is zero, p is prime, if q is different from zero, p is composite, and the number q has information regarding the factors that from the composite number. Remember that the definition of the Lambert series component we have given in (11) and (12).

One important property of above integral is that the expression:

$$\left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) \quad (23)$$

Does not depend on p . This means whatever study or pre-calculation of that expression (23) we make on advance, will be valid independently of the number p that we want to check.

And once we have that expression studied or precalculated, we only have to perform the definite integral (22) which yes it has a factor ($e^{pj\omega}$) that depends on p .

We will make some examples later.

7. Relation of q with the factors of p

As we have seen in equations (22), (16) and in the figure 5 when we calculate q , the result is the sum of the factors of all the possible factorizations (of two components) of the number. We can see it better in a example. Let's check figure 5 again:

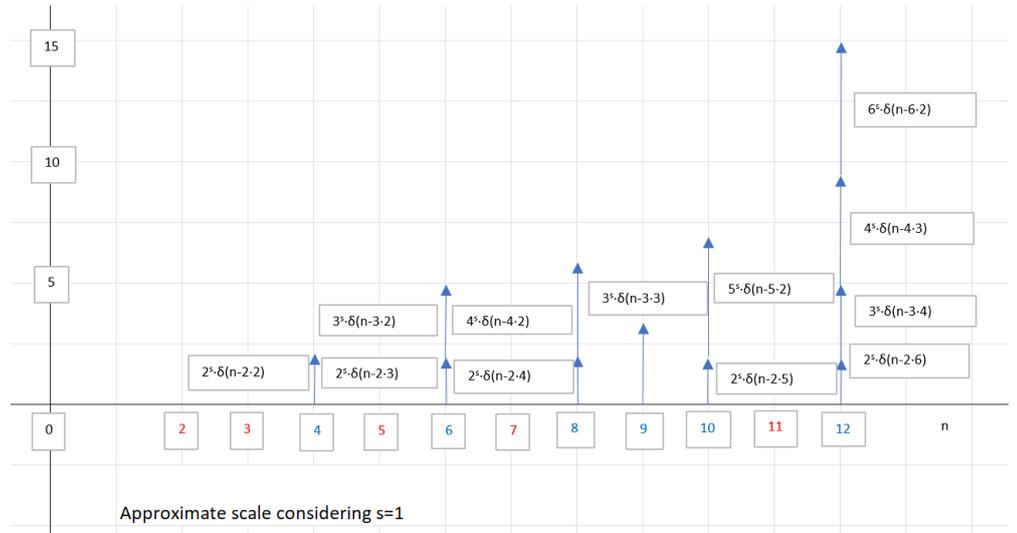


Fig.5

If we calculate q with equation (22) using $p=6$, the result will be the sum of the heights of the Dirac Deltas in 6, that is $q=2^s+3^s$. If we calculate with $p=9$, the result will be 3^s . If we calculate for $p=12$ the result will be $q=2^s+3^s+4^s+6^s$.

This has some interesting properties. First one, if use the case where $s=0$, the value of q will not depend on the value of the factors but only in the number of them (as the height of each delta will be 1, independently of the value of the factor).

When $s=0$, if $q=1$, it means that the number p is a perfect square (it only has one factor that is repeated, so only one combination of multiplication). This is the case of $p=4$ and $p=9$ in figure above.

When $s=0$ and $q=2$, p is a semi-prime number. It is only composed by two different primes. $q=2$ because there are two combinations of multiplication of these two factors (one combination and the second is reversing the order of the factors). This is the case of $p=6$ or $p=10$, see figure above.

But there is another case, when $s=0$ and $q=2$. It could be perfect cube. The reason for this, is that q can only see the perfect cube as the combination of two multiplications (the square of the number by the number itself and the reverse order). It cannot detect it is a cube, it only sees a composition of two multiplications. This is the case of $p=8$ above. It is composed by $2 \cdot 4$ and by $4 \cdot 2$ (it is treated as if it were a semiprime number).

Continuing with the case of $s=0$, the general equation to obtain the number of factors is the following:

$$q = (n_{f_1} + 1)(n_{f_2} + 1)(n_{f_3} + 1) \dots (n_{f_z} + 1) - 2 \quad (24)$$

Being nf_1 the number of times the factor f_1 is repeated, nf_2 the number of times f_2 is repeated and so on.

Putting in another way is:

$$q + 2 = (n_{f_1} + 1)(n_{f_2} + 1)(n_{f_3} + 1) \dots (n_{f_z} + 1) \quad (25)$$

So, the factors of the obtained number $q+2$ (much smaller than p) tell us how many factors the number p originally has and how many of them are repeated.

Above cases are considering always $s=0$, so the information about the value of the factors is lot. We are only left with information regarding the number of factors.

If we go where $s \neq 0$, the things get a little more complicated. For example, for the case of $p=12$ we have that $q=2^s+3^s+4^s+6^s$ (see figure (5) above). In general, this leads us to a situation where it is very difficult to obtain information about the factors. Except in one case: the semiprime numbers. The numbers that are composed only by two independent prime factors.

See the case of $p=10$ for example. Its $q=2^s+5^s$. Let's imagine that we know that $p=10$ is a semiprime number (this we can know using $s=0$, as commented above). But now we want to know which are the factors that compose them. The way to do it, would be the following:

We calculate q with an $s \neq 0$ using equation (22) with $p=10$:

$$\begin{aligned} q &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{10 \cdot j\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (26) \end{aligned}$$

And we know (being a and b the unknown factors):

$$q = a^s + b^s \quad (27)$$

Also, we know by definition that:

$$p = a \cdot b \quad (28)$$

$$a = \frac{p}{b} \quad (29)$$

So, this system is solvable, substituting a in (27):

$$q = \left(\frac{p}{b}\right)^s + b^s \quad (30)$$

$$qb^s = p^s + b^{2s} \quad (31)$$

$$b^{2s} - qb^s + p^s = 0 \quad (32)$$

$$b^s = \frac{q \pm \sqrt{q^2 - 4p^s}}{2} \quad (33)$$

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (22)$$

$$b = \sqrt{\frac{q \pm \sqrt{q^2 - 4p^s}}{2}} \quad (34)$$

$$a = \frac{p}{b} \quad (35)$$

With these three equations above we can calculate directly the factor a and b of a semiprime number p. p is the semiprime we want to factorize, and s is a real parameter that can take whatever value we want different from zero.

For example, imagine we want to factorize the number 10. We use the parameter s=1. So, we calculate (22):

$$\begin{aligned} q &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{10 \cdot j\omega} \left(\mathcal{L}_{(e^{-j\omega})}(1, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (36) \end{aligned}$$

You can use the Mathematica program in Annex A3 to calculate it, as we will explain later. We obtain that the value of q=7.

Then we calculate b:

$$b = \sqrt{\frac{7 \pm \sqrt{7^2 - 4 \cdot 10^1}}{2}} = \frac{7 \pm 3}{2} \quad (37)$$

This is, b can be 5 or 2. And depending on that, a will be 2 or 5.

We have calculated the factors of 10, and we have not made any factorization. Just a definite integral (22) and analytical calculation (34).

We will check more details later.

8. Closed solutions for F(ω)

8.1 Solution when s=0 using q-polygamma function that does not converge when |q|=1

Here, you will find the closed version of F(ω) used in the paper [12]. The problem with this solution is that it does not converge when |q|=1, this is, when e^{jw} is a pure imaginary

exponential. The only way to get convergence is to use the exponential $e^{-c-j\omega}$. Where c is a real number as near as zero as possible but not zero. You can find a study about this possibility in the paper [12]. So, we use the standard solution for the Lambert series when $s=0$ depending on the q -polygamma function. See [12]:

$$\mathcal{L}_q(0, x) = \sum_{k=1}^{k=\infty} \frac{q^{x \cdot k}}{1 - q^k} = \frac{1}{\ln(q)} (\psi_q(x) + \ln(1 - q)) \quad (38)$$

Where ψ_q is the q -polygamma function [2]. So, using equations (12) and (38):

$$\begin{aligned} \mathcal{L}_{(e^{-j\omega})}(0,2) &= \sum_{k=1}^{k=\infty} \frac{k^0 \cdot (e^{-j\omega})^{k \cdot 2}}{1 - (e^{-j\omega})^k} = \sum_{k=1}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \\ &= \frac{1}{\ln(e^{-j\omega})} (\psi_{(e^{-j\omega})}(2) + \ln(1 - e^{-j\omega})) \quad (39) \end{aligned}$$

So, using (13):

$$\begin{aligned} F(\omega)_{s=0} &= \sum_{k=2}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \\ &= \sum_{k=1}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} = \mathcal{L}_{(e^{-j\omega})}(0,2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \\ &= \frac{1}{\ln(e^{-j\omega})} (\psi_{(e^{-j\omega})}(2) + \ln(1 - e^{-j\omega})) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (40) \end{aligned}$$

When $s=0$. As commented, this solution does not converge when $e^{-j\omega}$ is a pure imaginary exponential. An exponential that includes a real part in the exponent as $e^{-c-j\omega}$ is needed to solve it.

8.2 Solution when $s=0$ using digamma function as Banerjee-Wilkerson paper that converges even when $|q|=1$

In the paper [14] a solution is found when $s=0$ and $|q|=1$. This means valid for a pure imaginary exponential of the type $e^{-j\omega}$.

$$\begin{aligned} \mathcal{L}_q(0, x) &= \sum_{k=1}^{k=\infty} \frac{q^{x \cdot k}}{1 - q^k} \\ &= \frac{1}{\ln(q)} (\psi(x) \\ &\quad + \ln(\ln(\frac{1}{q}))) - \sum_{k=1}^{k=\infty} \frac{\zeta(1-k)}{k!} B_k(x) (\ln(q))^{k-1} \quad (41) \end{aligned}$$

Where $\psi(x)$ is the digamma function (different form the previous point when it was the q -polygamma function). ζ is the Zeta function. And $B_k(x)$ are Bernoulli polynomials.

Putting $x=2$ and $q=e^{-j\omega}$:

$$\begin{aligned}
\mathcal{L}_{(e^{-j\omega})}(0,2) &= \sum_{k=2}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \\
&= \frac{1}{\ln(e^{-j\omega})} (\psi(2)) \\
&\quad + \ln\left(\ln\left(\frac{1}{e^{-j\omega}}\right)\right) - \sum_{k=1}^{k=\infty} \frac{\xi(1-k)}{k!} B_k(2) (\ln(e^{-j\omega}))^{k-1} \quad (42)
\end{aligned}$$

So, if the closed form for $F(\omega)$ for this case ($s=0$ but converging for $|q|=1$) we use:

$$\begin{aligned}
F(\omega)_{s=0} &= \sum_{k=2}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \\
&= \sum_{k=1}^{k=\infty} \frac{e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} = \mathcal{L}_{(e^{-j\omega})}(0,2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (43)
\end{aligned}$$

$$\begin{aligned}
F(\omega)_{s=0} &= \frac{1}{\ln(e^{-j\omega})} (\psi(2) + \ln\left(\ln\left(\frac{1}{e^{-j\omega}}\right)\right)) - \sum_{k=1}^{k=\infty} \frac{\xi(1-k)}{k!} B_k(2) (\ln(e^{-j\omega}))^{k-1} \\
&\quad - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (44)
\end{aligned}$$

8.3 Solution for $s \neq 0, -1, -2, \dots$ as per Banerjee-Wilkerson paper that converges even when $|q|=1$

In the paper [14] a solution is found for every $s \neq 0, -1, -2, \dots$ And valid when $|q|=1$. This means is valid for a pure imaginary exponential of the type $e^{j\omega}$.

$$\begin{aligned}
\mathcal{L}_q(s, x) &= \sum_{k=1}^{k=\infty} \frac{k^s q^{x \cdot k}}{1 - q^k} \\
&= \frac{1}{\left(\ln\left(\frac{1}{q}\right)\right)^{1+s}} \Gamma(1+s) \\
&\quad \cdot \xi(1+s, x) - \sum_{k=0}^{k=\infty} \frac{\xi(1-s-k)}{k!} B_k(x) (\ln(q))^{k-1} \quad (45)
\end{aligned}$$

Where Γ is gamma function, $\xi(x, y)$ is the Hurwitz Zeta function [17], $\xi(x)$ is the Riemann Zeta function and $B_k(x)$ are the Bernoulli polynomials.

$$\begin{aligned}
\mathcal{L}_{(e^{-j\omega})}(s, 2) &= \sum_{k=1}^{k=\infty} \frac{k^s e^{-2j\omega k}}{1 - e^{-j\omega k}} \\
&= \frac{1}{\left(\ln\left(\frac{1}{e^{-j\omega}}\right)\right)^{1+s}} \Gamma(1+s) \\
&\quad \cdot \xi(1+s, 2) - \sum_{k=0}^{k=\infty} \frac{\xi(1-s-k)}{k!} B_k(2) (\ln(e^{-j\omega}))^{k-1} \quad (46)
\end{aligned}$$

$$\begin{aligned}
F(\omega)_{s \neq 0, -1, -2, \dots} &= \sum_{k=2}^{k=\infty} \frac{k^s e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} \\
&= \sum_{k=1}^{k=\infty} \frac{k^s e^{-k \cdot 2j\omega}}{1 - e^{-kj\omega}} - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} = \mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (47)
\end{aligned}$$

$$\begin{aligned}
&F(\omega)_{s \neq 0, -1, -2, \dots} \\
&= \frac{1}{\left(\ln\left(\frac{1}{e^{-2j\omega k}}\right)\right)^{1+s}} \Gamma(1 + s) \\
&\cdot \xi(1 + s, 2) - \sum_{k=0}^{k=\infty} \frac{\xi(1 - s - k)}{k!} B_k(2) (\ln(e^{-2j\omega k}))^{k-1} - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (48)
\end{aligned}$$

8.4 Solution for s=-1,-2... as per Banerjee-Wilkerson paper that converges even when |q|=1

In the paper [14] they use the parameter -m=s. This means, it applies when -m=-1,-2,-3...

$$\begin{aligned}
&\mathcal{L}_q(-m, x) \\
&= \left[m \cdot \xi'(1 - m, x) \right. \\
&+ \left(\ln\left(\ln\frac{1}{q}\right) \right. \\
&\left. \left. - H_{m-1} \right) B_m(x) \right] \frac{(\ln(q))^{m-1}}{m!} - \sum_{\substack{k=0 \\ k \neq m}}^{k=\infty} \frac{\xi(1 + m - k)}{k!} B_k(x) (\ln(q))^{k-1} \quad (49)
\end{aligned}$$

Where $\xi'(x, y)$ is the derivative of the Hurwitz Zeta function with respect to m, $\xi(x)$ is the Riemann Zeta function, $B_k(x)$ are the Bernoulli polynomials and H_m are the Harmonic numbers.

$$\begin{aligned}
\mathcal{L}_{(e^{-j\omega})}(-m, 2) &= \left[m \cdot \xi'(1 - m, 2) \right. \\
&+ \left(\ln\left(\ln\frac{1}{e^{-j\omega}}\right) \right. \\
&\left. \left. - H_{m-1} \right) B_m(x) \right] \frac{(\ln(e^{-j\omega}))^{m-1}}{m!} - \sum_{\substack{k=0 \\ k \neq m}}^{k=\infty} \frac{\xi(1 + m - k)}{k!} B_k(2) (\ln(e^{-j\omega}))^{k-1} \quad (50)
\end{aligned}$$

$$\begin{aligned}
F(\omega)_{s=-m=-1,-2\dots} &= \left[m \cdot \xi'(1-m, 2) \right. \\
&+ \left(\ln \left(\ln \frac{1}{e^{-j\omega}} \right) \right. \\
&\left. \left. - H_{m-1} \right) B_m(x) \right] \frac{(\ln(e^{-j\omega}))^{m-1}}{m!} - \sum_{\substack{k=0 \\ k \neq m}}^{k=\infty} \frac{\xi(1+m-k)}{k!} B_k(2) (\ln(e^{-j\omega}))^{k-1} - \frac{e^{-2j\omega}}{1-e^{-j\omega}} \quad (51)
\end{aligned}$$

9. Testing different equations to calculate the Lambert series

You can find a program in Mathematica in Annex A3. It is a very raw program but prepared to calculate q according to the equation (22) using different ways to calculate F(ω).

When a solution for |q|=1 does not converge, it is added a real c to the exponential $e^{-c-j\omega}$ near zero but not being zero.

When the solution does not apply for a negative integer for s for example, it is calculated for s being a real number very near to that integer. So, we can compare to the solution that cannot apply to an integer with the solution that applies for it.

And the results are stunning. The equations from the paper Banerjee-Wilkerson really works.

First comment is that you check Annex A1 to see the two types of summation for calculating the Lambert series in the “traditional” way (Type 1 and 2 as explained in the Annex A1). So, what we will do is to compare the values obtained with the “traditional” way (the summation definition of the Lambert series) to the values obtained with the equations from the Banerjee-Wilkerson.

Copy the program in Annex A3 in Mathematica application [15]. If you do not have a license, you can try with trial or online versions to run the program.

And just click “Evaluate Notebook”, you will see the following as first part of the output:

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```
q=Exp(-0.5 - 0.5 i)
```

```
x=2.
```

```
For s=0
```

```
Using q-polygamma
```

```
-0.15708792501811 - 0.75397788086692 i
```

```
Using Type 1 of summation of the Lambert Series
```

```
-0.15708792430093 - 0.75397788035223 i
```

```
Using Type 2 of summation of the Lambert Series
```

```
-0.15710397656587 - 0.75398686251663 i
```

```
For s=0 and valid for |q|=1 using Banerjee-Wilkerson for s=0
```

```
-0.15708792501811 - 0.75397788086692 i
```

```
s=2
```

```
If s<>0
```

```
Using Type 1 of summation of the Lambert Series
```

```
-0.88688721218109 - 0.72054840294982 i
```

```
Using Type 2 of summation of the Lambert Series
```

```
-0.8869035917059 - 0.72055760061645 i
```

```
s=2 s<>0, <>-1,-2,-3 and valid for |q|=1 using Banerjee-Wilkerson for s<>0, <>-1,-2,-3
```

```
-0.88688754022698 - 0.720548618615 i
```

```
s=-2.001 s<>0, <>-1,-2,-3 and valid for |q|=1 using Banerjee-Wilkerson for s<>0, <>-1,-2,-3
```

```
-0.024354959967214 - 0.6938113059879 i
```

```
s=-2
```

```
If s=-1,-2,-3... and valid for |q|=1 using Banerjee-Wilkerson for s=-1,-2,-3
```

```
-0.024381068625833 - 0.69382826109088 i
```

This means, it has been calculated the Lambert series of the number $q=e^{-0.5-0.5j}$ and $x=2$ for different values of s and with different methods.

$$\mathcal{L}_{(e^{-0.5-0.5j\omega})}(s, 2) = \sum_{k=2}^{k=\infty} \frac{k^s e^{k \cdot 2(-0.5-0.5j\omega)}}{1 - e^{k(-0.5-0.5j\omega)}} \quad (52)$$

You can check above that independently of the method used (if you want details of each method, you can see it in the program with explanations) you get the same result for every different value of s .

You can play with the program changing the values of s , x , c and w (the real and imaginary part of the exponential in q) in the first part of the program:

```
nmax = 20;  
p2 = 15.;  
s2 = 2;  
x2 = 2.;  
c2 = 0.5;  
w2 = 0.5;
```

N_{max} is the number of iterations used when summations are needed. You can check the rest of the program for other different changes of s depending on the method.

You will see that when there is no error or lack of convergence the results are coherent

with the different methods including Banerjee-Wilkerson even for pure imaginary exponentials (c=0), results that you cannot obtain if you use “traditional” summations (Type 1 or 2 for Lambert series).

10. Testing different integrals to calculate q

The idea of calculating the Lambert series even with values of c=0 (q being a pure imaginary exponential) was that the Lambert series appear on the function F(ω), this way:

$$F(\omega) = \mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \quad (13)$$

And we use the function F(ω) to calculate the number of factors of a number p when s=0 or even these factors when s≠0 and p is a semiprime number (it contains only two factors). See chapters 6 and 7 above.

So run the program again (“Evaluate Notebook”) but now check the second part of the output:

```
Copyright 2022 Jesus Sanchez
If you consider this helpful, do not hesitate to drop your BTC here
bc1q0qce9tqykrm6gzzhemn836cnkp6hmel51mz36f
p=15.
Calculating integrals using Exp(-pwj) and F(w) from -pi to +pi.
The functions are 'antisymmetrized' so the integral is converted to an integral from 0 to +pi
Integrals with diffent values of s=2, 2, 2, -2
Using type 1 of Lambert series summation and s=2 q=34. +0. i
Using type 2 of Lambert series summation and s=2 q=34. +0. i
Banerjee-Wilkerson for s<>0,-1,-2 s=2 q=40.652909637072 + 7.1994859909328 x 10-16 i
Banerjee-Wilkerson for s=-1,-2,-3 s=-2 q=-1.6817160605379 +0. i
Integrals with s=0
Using type 1 of summation for Lambert series s=0 q=2. +0. i
Using type 2 of summation for Lambert series s=0 q=2. +0. i
Using solution for s=0 using q-polygamma function and c=0.5 q=2.0000000000001 +0. i
Using Banerjee-Wilkerson for s=0 q=-1.8637208454154 +0. i
Calculating the factors using Type 1 of summation of Lambert series s=2 p=15. q=34. +0. i
5. +0. i
3. +0. i
```

What we are doing here is to calculate the integral:

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{p(c+j\omega)} \left(\mathcal{L}_{(e^{-c-j\omega})}(s, 2) - \frac{e^{2(-c-j\omega)}}{1 - e^{(-c-j\omega)}} \right) d\omega \quad (53)$$

Similar to (22) but with the real factor c to assure convergence whatever method we use to calculate it. In fact, the objective is to obtain expressions valid for c=0 and different values of s using Banerjee-Wilkerson equations [14].

To improve the numerical calculation, instead of using the integral form -π to +π we use the most numerical efficient from 0 to +π:

$$q = \frac{1}{2\pi} \int_0^\pi \left[e^{p(c+j\omega)} \left(\mathcal{L}_{(e^{-c-j\omega})}(s, 2) - \frac{e^{2(-c-j\omega)}}{1 - e^{(-c-j\omega)}} \right) + e^{p(c-j\omega)} \left(\mathcal{L}_{(e^{-c+j\omega})}(s, 2) - \frac{e^{2(-c+j\omega)}}{1 - e^{(-c+j\omega)}} \right) \right] d\omega \quad (54)$$

You can see that this change is correct in Annex A2.

Now, let's check the output of the program.

What we are doing is to try to know the nature of the number p=15 (is it prime, semiprime, general composite?). And in case that it is semiprime, to calculate the factors.

Obviously, we know that 15 is a semiprime with factors 3 and 5. But will the program calculate it, using the value of q obtained in the integral?

If p=15 was a prime, we know that q should be zero (see chapters 6 and 7). We see that in all the calculations with different values of s, q is never zero. So, this is ok.

```
Integrals with diffent values of s=2, 2, 2, -2
Using type 1 of Lambert series summation and s=2 q=34. +0. i
Using type 2 of Lambert series summation and s=2 q=34. +0. i
Banerjee-Wilkerson for s<>0,-1,-2 s=2 q=40.652909637072 + 7.1994859909328 x 10-16 i
Banerjee-Wilkerson for s=-1,-2,-3 s=-2 q=-1.6817160605379 +0. i
Integrals with s=0
Using type 1 of summation for Lambert series s=0 q=2. +0. i
Using type 2 of summation for Lambert series s=0 q=2. +0. i
Using solution for s=0 using q-polygamma function and c=0.5 q=2.0000000000001 +0. i
Using Banerjee-Wilkerson for s=0 q=-1.8637208454154 +0. i
Calculating the factors using Type 1 of summation of Lambert series s=2 p=15. q=34. +0. i
```

The next issue, we have seen in chapter 7 is that when s=0, q gives us information of the number of factors of the number p. If q=0, the number p should be or semi-prime or a perfect cube. We see:

```
Integrals with s=0
Using type 1 of summation for Lambert series s=0 q=2. +0. i
Using type 2 of summation for Lambert series s=0 q=2. +0. i
Using solution for s=0 using q-polygamma function and c=0.5 q=2.0000000000001 +0. i
Using Banerjee-Wilkerson for s=0 q=-1.8637208454154 +0. i
```

When we use the summations (Type 1 or Type 2) of the Lambert series we obtain the value q=2. This is correct, as 15 is a semiprime number.

Also, using the q-polygamma function we see that q=2 as expected.

But when we go to Banerjee-Wilkerson [14] equations we do not obtain the desired number.

We will see that this happens always when we use the Banerjee-Wilkerson equations to calculate the integral (q). Later I will comment about this.

Now, let's go with a value of $s \neq 0$. Using the Lambert series summation (Type 1 and 2) for $s=2$ we get the value $q=34$.

```
Integrals with different values of s=2, 2, 2, -2
Using type 1 of Lambert series summation and s=2 q=34. + 0. i
Using type 2 of Lambert series summation and s=2 q=34. + 0. i
```

We see that for $s=2$, this is correct as $3^s+5^s=3^2+5^2=9+25=34$.

And the program is able to calculate these factors 3 and 5 using the equations in chapter 7 as you can see in:

```
Calculating the factors using Type 1 of summation of Lambert series s=2 p=15. q=34. + 0. i
5. + 0. i
3. + 0. i
```

To calculate these factors, it has not used a brute-force checking by divisions. It has used the value of the integral q and the equations in chapter 7.

You can change the value of the number p in the beginning of the program ($p2$):

```
nmax = 20;
p2 = 15.;
s2 = 2;
x2 = 2.;
c2 = 0.5;
w2 = 0.5;
```

You will see that for low values of p everything works. When you start increasing p the program starts "collapsing". The only way that this could have an added value is pre-calculating values of $F(\omega)$, storing in tables and then using this integral to make the calculation for different values of p .

Regarding the Banerjee-Wilkerson equations we see that they do not work also for the integral in the case $s \neq 0$.

```
Banerjee-Wilkerson for s<>0,-1,-2    s=2 q=40.652909637072 + 7.1994859909328 x 10^-16 i
Banerjee-Wilkerson for s=-1,-2,-3    s=-2 q=-1.6817160605379 + 0. i
```

For $s=2$, the result should be 34 and it is 40.652.

For $s=-2$, the result should be $3^s+5^s=3^{-2}+5^{-2}=0.1511$ but it is -1.681

The curious thing is that we have seen in the chapter 9 that these Banerjee-Wilkerson equations work for specific values. But they do not work to calculate the complete integral.

The reasons for this could be:

- a) The program I made is wrong or has some bugs (perfectly possible).
- b) The interval of convergence (or validity) for these equations does not fulfill the complete integral (from 0 to π). This means, for example, they are valid only from 0 to 1. Or

depending on s , they have different intervals of validity. So, the result of the integral is not valid.

c) The result they give is correct in “the world of mathematics”. But they are equations to transform what are by nature divergent results to convergent results, so although the result is theoretically correct, it cannot be used in numerical applications. With this, I mean, for example, the Riemman Zeta(-1)=1+2+3+4... =-1/12 and this is correct and has a mathematical meaning. But there is no numerical way to find that -1/12 is composed by 1+2+3+4+... (to start, numerically you would need at least one negative number to be able to sum -1/12). So, something similar could happen with the Banerjee-Wilkerson equations, they force a result to get convergence and it is mathematically correct, but it cannot be used to obtain numerical information.

d) Other possible reasons I did not think about.

So, summing up, all the methods commented in the paper to obtain information regarding the primality or the number of factors (or even their value) of a number, in general work. The problem is that to be valid you need to make the summations of the Lambert series that computationally are very expensive and slow.

The idea of trying to use the Banerjee-Wilkerson equations for this goal (and reducing the computational effort and time) is still not working and more studies are needed to know the reason. If this reason is found and corrected, it could be that in the future the primality and factorization using the tools commented in the paper (the integral q) could be computationally feasible to be used and even win over the brute-force checking by trying to find divisors.

Anyhow, as commented in previous papers [12] the possibility of creating tables making a pre-calculation of the Lambert series so you reduce the total amount of time once you know which number p you have to test, is always there. Again, it will require time for this prework and very probably storage space for all the results. But once done, it will be valid to check any p (until certain predefined value for which the precision of the calculations has been chosen in a first place).

11. Conclusions

It has been shown how it is possible to check if a number is prime and the number of factors it is composed by and even the value of these factors, calculating the following integral:

$$q = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{pj\omega} \left(\mathcal{L}_{(e^{-j\omega})}(s, 2) - \frac{e^{-2j\omega}}{1 - e^{-j\omega}} \right) d\omega \quad (22)$$

The problem is that the Lambert series are computationally very expensive. That is the reason it has been tried to use the Banerjee-Wilkerson paper to try to find closed solutions for these Lambert series.

The result is that the equations proposed on Banerjee-Wilkerson paper work and give correct results as checked by a Mathematica program.

But when they are used to calculate the above integral, the results are not the ones expected. Different possible reasons for this issue have been proposed in chapter 10. If the reason for this problem is found and corrected, possibly a cheap computationally solution for primality test and factorization could be used using above integral.

Bilbao, 18th June (viXra-v1).

16. Acknowledgements

AAAACDEEILNÑOPRSTUV

To my family and friends.

If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

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A1. Annex A1. Type 1 and Type 2 summation of Lambert series

$$\begin{aligned} & 1^s q^{0+x} + 2^s q^{0+2x} + 3^s q^{0+3x} + 4^s q^{0+4x} + \dots \\ & 1^s q^{1+x} + 2^s q^{2+2x} + 3^s q^{3+3x} + 4^s q^{4+4x} + \dots \\ & 1^s q^{2+x} + 2^s q^{4+2x} + 3^s q^{6+3x} + 4^s q^{8+4x} + \dots \\ & 1^s q^{3+x} + 2^s q^{6+2x} + 3^s q^{9+3x} + 4^s q^{12+4x} + \dots \end{aligned}$$

Fig. A1.0

Summing in columns and using geometric sums [8]:

$$\frac{1^s q^x}{1-q^1} + \frac{2^s q^{2x}}{1-q^2} + \frac{3^s q^{3x}}{1-q^3} + \frac{4^s q^{4x}}{1-q^4} + \dots \quad (A1.1)$$

So, in the form of summation, we have the typical definition of Lambert series:

$$\mathcal{L}_q(s, x) = \sum_{k=1}^{k=\infty} \frac{k^s \cdot q^{k \cdot x}}{1-q^k} \quad (A1.2)$$

This type of summation is called summation Type 1 in the program in the Annex A3.

For the special case when $s=0$, this is

$$\mathcal{L}_q(0, x) = \sum_{k=1}^{k=\infty} \frac{q^{k \cdot x}}{1-q^k} \quad (A1.3)$$

But if instead of summing in columns, we sum in arrows the fig A1.0 we get:

$$\begin{aligned} & \sum_{k=1}^{\infty} k^s (q^x)^k + \sum_{k=1}^{\infty} k^s (q^{1+x})^k \\ & + \sum_{k=1}^{\infty} k^s (q^{2+x})^k + \sum_{k=1}^{\infty} k^s (q^{3+x})^k + \dots \end{aligned} \quad (A1.4)$$

Passing the k^s to the denominator, we have:

$$\sum_{k=1}^{\infty} \frac{(q^x)^k}{k^{-s}} + \sum_{k=1}^{\infty} \frac{(q^{1+x})^k}{k^{-s}} + \sum_{k=1}^{\infty} \frac{(q^{2+x})^k}{k^{-s}} + \sum_{k=1}^{\infty} \frac{(q^{3+x})^k}{k^{-s}} + \dots \quad (A1.5)$$

If we compare with the definition of the Polylogarithm [16]:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \quad (A1.6)$$

We have:

$$Li_{-s}(q^x) + Li_{-s}(q^{1+x}) + Li_{-s}(q^{2+x}) + Li_{-s}(q^{3+x}) + \dots \quad (A1.7)$$

Putting it in the form of summation, we have:

$$\mathcal{L}_q(s, x) = \sum_{k=0}^{k=\infty} Li_{-s}(q^{k+x}) \quad (A1.8)$$

This is what we call summation Type 2 in the Mathematica program in Annex A3.

For the specific case when s=0, the Polylogarithm gets the form of the geometric sum:

$$\mathcal{L}_q(0, x) = \sum_{k=0}^{k=\infty} \frac{q^{k+x}}{1 - q^{k+x}} \quad (A1.9)$$

You can check this easily as for s=0 the equation (A1.5) gets to the following and each element is a geometric sum:

$$\sum_{k=1}^{\infty} (q^x)^k + \sum_{k=1}^{\infty} (q^{1+x})^k + \sum_{k=1}^{\infty} (q^{2+x})^k + \sum_{k=1}^{\infty} (q^{3+x})^k + \dots \quad (A1.10)$$

$$\frac{q^x}{1 - q^x} + \frac{q^{1+x}}{1 - q^{1+x}} + \frac{q^{2+x}}{1 - q^{2+x}} + \frac{q^{3+x}}{1 - q^{3+x}} + \dots = \sum_{k=0}^{k=\infty} \frac{q^{k+x}}{1 - q^{k+x}} \quad (A1.11)$$

So, believe or not, the next two following summations gives the same result (result of Lambert series when s=0):

$$\sum_{k=1}^{k=\infty} \frac{q^{k \cdot x}}{1 - q^k} = \sum_{k=0}^{k=\infty} \frac{q^{k+x}}{1 - q^{k+x}} \quad (A1.12)$$

A2. Annex A2. Transformation of an integral originally from -a to a to 0 to a

For any integral of the type:

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad (A2.1)$$

If in the first integral, we make the substitution x=-y that implies dx=-dy (and also knowing that what was -a for x, will be +a for y) we get:

$$\int_{-a}^a f(x) dx = \int_a^0 f(-y)(-dy) + \int_0^a f(x) dx \quad (A2.2)$$

Now we “reverse” the first integral. Instead of going from a to 0, we will go from 0 to a. This implies adding a negative sign that will be cancelled with the (-dy) sign.

$$\int_{-a}^a f(x)dx = \int_0^a f(-y)(dy) + \int_0^a f(x)dx \quad (A2.3)$$

As the variables are “dummies” inside of each definite integral, we can rename them as we want. To avoid confusion, with the previous substitution, I will rename all the variables as t. But we could have used x again with no problem:

$$\int_{-a}^a f(t)dt = \int_0^a f(-t)(dt) + \int_0^a f(t)dt \quad (A2.4)$$

Reordering and putting in the same integral:

$$\int_{-a}^a f(t)dt = \int_0^a (f(t) + f(-t)) dt \quad (A2.5)$$

And this is the one we use in the program in Annex A3.

A3. Annex A3. Mathematica program

```
(* Copyright 2022 by Jesus Sanchez*)
(* If you consider this helpful, do not hesitate to drop your BTC here:
bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f
*)

pi2=N[Pi];
Remove[func];
Remove[func2];

Remove[y2];

(*
nmax: number of terms when there are summations
p2: number to be prime tested
s2 the s in the Lambert series in the k^s term
x2 the x in the Lambert series. It is always 2 when you want to calculate F(w)
c2 the c in the exponential q=Exp(-c-wj)
w2 the c in the exponential q=Exp(-c-wj)
*)

nmax=20;
p2=15.;
s2=2;
x2=2.;
c2=0.5;
w2=0.5;

(*c3 changes slightly c2 when c2=0 to be able to converge some series*)
c3=If[c2==0.,c2+0.01,c2];

(* fors=0 but not valid for |q|=1 *)
(* using q-polygamma*)
funconpoly[x_,c_,w_]:= (QPolyGamma[x,Exp[-c-I*w]]+Log[1-Exp[-c-I*w]])/Log[Exp[-c-I*w]];

(* using directly 1st type of summation of Lambert Series*)
funcsuma[x_,c_,w_,nmax_]:= (a=0;Do[a=a+Exp[-c-I*w]^(n*x)/(1-Exp[-c-I*w]^(n)),{n,1,nmax}];a);

(* using directly 2nd type of summation of Lambert Series*)
funcsuma2[x_,c_,w_,nmax_]:= (a=0;Do[a=a+Exp[-c-I*w]^(n+x)/(1-Exp[-c-I*w]^(n+x)),{n,0,nmax}];a);

(* for s=0 and valid for |q|=1 *)
(* using Banarjee-Wilkerson solution for Lambert Series when s=0 and |q|=1 *)
funcbanerwilkers0[x_,c_,w_,nmax_]:= (a=(PolyGamma[x]+Log[Log[1/Exp[-c-I*w]]])/Log[Exp[-c-I*w]]);Do[a=a-Zeta[1-k]*BernoulliB[k,x]/k!*(Log[Exp[-c-I*w]])^(k-1),{k,1,nmax}];a);
(* for s<>0,<>-1,-2,-3... *)
```

```
(* using directly 1nd type of summation of Lambert Series*)
funcsumas[x_,c_,w_,nmax_,s_]:=:(a=0;Do[a=a+n^s*Exp[-c-I*w]^(n*x)/(1-Exp[-c-I*w]^(n)),{n,1,nmax}];a);
```

```
(* using directly 2nd type of summation of Lambert Series*)
funcsuma2s[x_,c_,w_,nmax_,s_]:=:(a=0;Do[a=a+PolyLog[-s,Exp[-c-I*w]^(n+x)],{n,0,nmax}];a);
```

```
(* for s<>0,<-1,-2,-3... and valid for |q|=1 *)
(* using Banarjee-Wilkerson solution for Lambert Series when for s<>0,<-1,-2,-3... and |q|=1 *)
funcbanerwilkers[x_,c_,w_,nmax_,s_]:=:(a=Gamma[1+s]*HurwitzZeta[1+s,x]/(Log[1/Exp[-c-I*w]]^(1+s));Do[a=a-Zeta[1-s-k]*BernoulliB[k,x]/k!*Log[Exp[-c-I*w]]^(k-1),{k,0,nmax}];a);
(* for s=-m=-1,-2,-3... and valid for |q|=1 *)
(* using Banarjee-Wilkerson solution for Lambert Series when for s=-m=-1,-2,-3... and |q|=1 *)
Remove[m];Remove[y];hzprime[m_,y_]:=D[HurwitzZeta[m,y],m];
```

```
funcbanerwilkersni[x_,c_,w_,nmax_,s_]:=:(m=-s;a=(m*hzprime[1-m,x]+(Log[Log[1/Exp[-c-I*w]]]-HarmonicNumber[m-1])*BernoulliB[m,x])*(1/m!)*Log[Exp[-c-I*w]]^(m-1));Do[a=a-Zeta[1+m-k]*BernoulliB[k,x]/k!*Log[Exp[-c-I*w]]^(k-1),{k,0,m-1}];Do[a=a-Zeta[1+m-k]*BernoulliB[k,x]/k!*Log[Exp[-c-I*w]]^(k-1),{k,m+1,nmax}];a);
```

```
(*Starting calculations*)
```

```
Print["Copyright 2022 Jesus Sanchez"]
```

```
Print["q=Exp(", -c-2*w2*I, ")"]
Print["x=", x2]
Print["For s=0"]
Print["Using q-polygamma"]
Print[funconpoly[x2,c3,w2]];
```

```
Print["Using Type 1 of summation of the Lambert Series"]
```

```
Print[funcsuma[x2,c2,w2,nmax]];
Print["Using Type 2 of summation of the Lambert Series"]
Print[funcsuma2[x2,c2,w2,nmax]];
```

```
Print["For s=0 and valid for |q|=1 using Banerjee-Wilkerson for s=0"]
Print[funcbanerwilkers0[x2,c2,w2,nmax]];
```

```
Print["s=", s2]
```

```
Print["If s<>0"]
Print["Using Type 1 of summation of the Lambert Series"]
Print[funcsumas[x2,c2,w2,nmax,s2]];
Print["Using Type 2 of summation of the Lambert Series"]
Print[funcsuma2s[x2,c2,w2,nmax, s2]];
```

```
s4=-Abs[Round[s2]];
s3=If[s2==s4,s2+0.001,s2];
```

```
Print["s=", s3, " s<>0,<-1,-2,-3 and valid for |q|=1 using Banerjee-Wilkerson for s<>0,<-1,-2,-3"];
```

```
Print[funcbanerwilkers[x2,c2,w2,nmax,s3]];
Print["s=", s4-0.001, " s<>0,<-1,-2,-3 and valid for |q|=1 using Banerjee-Wilkerson for s<>0,<-1,-2,-3"];
```

```
Print[funcbanerwilkers[x2,c2,w2,nmax,s4-0.001]];
```

```
Print["s=", s4]
Print["If s=-1,-2,-3... and valid for |q|=1 using Banerjee-Wilkerson for s=-1,-2,-3"]
```

```
Print[funcbanerwilkersni[x2,c2,w2,nmax,s4]];
```

```
Print[""];
```

```
(* Copyright 2022 by Jesus Sanchez*)
(* If you consider this helpful, do not hesitate to drop your BTC here
bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f
*)
```

```
Print["Copyright 2022 Jesus Sanchez"]
Print["If you consider this helpful, do not hesitate to drop your BTC here:"]
Print["bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f"]
Print["p=", p2];
Print["Calculating integrals using Exp(-pwj) and F(w) from -pi to +pi."]
```

```

Print["The functions are 'antisymmetrized' so the integral is converted to an integral from 0 to +pi"]
Print["Integrals with diffent values of s=",s2," ",s2," ",s3," ",s4]

q=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcsuma2s[x2,c2,w,nmax,s2]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcsuma2s[x2,c2,-w,nmax,s2]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};

q1=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcsumas[x2,c2,w,nmax,s2]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcsumas[x2,c2,-w,nmax,s2]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};

q2=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcbanerwilkers[x2,c2,w,nmax,s3]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcbanerwilkers[x2,c2,-w,nmax,s3]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};

q3=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcbanerwilkersni[x2,c2,w,nmax,s4]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcbanerwilkersni[x2,c2,-w,nmax,s4]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};
Print["Using type 1 of Lambert series summation and s=",s2," q=","q];

Print["Using type 2 of Lambert series summation and s=",s2," q=","q1];

Print["Banerjee-Wilkerson for s<0,-1,-2 s=",s3," q=","q2];

Print["Banerjee-Wilkerson for s=-1,-2,-3 s=",s4," q=","q3];

(*See below in the Prints the meaning of each integral*)

q4=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcsuma2[x2,c2,w,nmax]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcsuma2[x2,c2,-w,nmax]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};
q45=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcsumaf[x2,c2,w,nmax]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcsumaf[x2,c2,-w,nmax]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};

q5=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c3+w*I)]*(funconpoly[x2,c3,w]-Exp[2*(-c3-w*I)]/(1-Exp[-c3-w*I]))+Exp[p2*(c3-w*I)]*(funconpoly[x2,c3,-w]-Exp[2*(-c3+w*I)]/(1-Exp[-c3+w*I]))],{w,0,N[Pi]};

q6=(1/(2*N[Pi]))*NIntegrate[Exp[p2*(c2+w*I)]*(funcbanerwilkers0[x2,c2,w,nmax]-Exp[2*(-c2-w*I)]/(1-Exp[-c2-w*I]))+Exp[p2*(c2-w*I)]*(funcbanerwilkers0[x2,c2,-w,nmax]-Exp[2*(-c2+w*I)]/(1-Exp[-c2+w*I]))],{w,0,N[Pi]};

Print["Integrals with s=0"]

Print["Using type 1 of summation for Lambert series s=0 q=","q45];

Print["Using type 2 of summation for Lambert series s=0 q=","q4];

Print["Using solution for s=0 using q-polygamma function and c=","c3," q=","q5];

Print["Using Banerjee-Wilkerson for s=0 q=","q6];

(* Functions that solve factors a and b in a semiprime number when you know p(the semiprime) and q (the result of the integral when s<0*)

funcfactora[q_,p_,s_]:=N[(((1/2)*(q+Sqrt[q^2-4*p^s]))^(1/s));
funcfactorb[q_,p_,s_]:=N[(((1/2)*(q-Sqrt[q^2-4*p^s]))^(1/s));
Print["Calculating the factors using Type 1 of summation of Lambert series s=",s2," p=","p2," q=","q];
Print[funcfactora[q,p2,s2]];

Print[funcfactorb[q,p2,s2]];

(*AAAACDEEILNÑOPRSTUV*)

```