Proof that the Einstein's field equations are invalid: exposition of the unimodular defect

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Abstract

Albert Einstein first presented his gravitational field equations in unimodular coordinates. In these coordinates, the field equations can be written explicitly in terms of the Einstein pseudotensor for the energy-momentum of the gravitational field. Since this pseudotensor produces, by contraction, a first-order intrinsic differential invariant, it violates the laws of pure mathematics. This is sufficient to prove that Einstein's unimodular field equations are invalid. Since the unimodular form must hold in the General Theory of Relativity, it follows that the latter is also physically and mathematically unsound, lacking a proper mathematical foundation.[a]

1. INTRODUCTION

In the original derivation of his field equations for the gravitational field, Albert Einstein invoked unimodular coordinates[1,2], which are characterised mathematically by the condition $\sqrt{-g} = 1$, where g is the determinate of the metric tensor, "*whereby the equations of the theory experience an eminent simplification.*"[1] Although the unimodular condition is not mandatory, the related field equations must still hold in the General Theory of Relativity. Demonstration that the unimodular field equations are invalid is sufficient to prove the invalidity of the General Theory of Relativity.

Using the unimodular coordinates, Einstein's field equations can be written explicitly in terms of his pseudotensor for the energy-momentum of his gravitational field alone. Einstein introduced his pseudotensor in order to make his theory conform to the usual conservation of energy and momentum for a closed system: "*It must be remembered that besides the energy density of the matter there must also be given an energy density of the gravitational field, so that there can be no talk of principles of conservation of energy and momentum for matter alone.*"[3] Although his pseudotensor is not a tensor, its use is justified by Einstein on the grounds that it acts 'like a tensor' under linear transformations of coordinates. His pseudotensor however, produces, upon contraction, a first-order intrinsic differential invariant, i.e. a differential invariant that depends solely upon the components of the metric tensor and their first derivatives.[4] But G. Ricci-Curbastro and T. Levi-Civita proved, in 1901, that first-order intrinsic differential invariants do not exist.[5] His pseudotensor is therefore invalid. Consequently, Einstein's unimodular field equations are invalid. Therefore, his General Theory of Relativity is invalid.

The proof herein proceeds in the following fashion:

1. Write Einstein's field equations in his original form using his unimodular coordinates;

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[a] **To cite this paper:** Crothers, S.J. Proof that Einstein's field equations are invalid: Exposition of the unimodular defect, *Physics Essays*, Volume 34: Pages 420-428, 2021. (Published Online: 17 September, 2021): http://www.physicsessays.org/browse-journal-2/product/1896-2-stephen-j-crothers-proof-that-einstein-s-field-equations.html

2. Convert the field equations into an equivalent unimodular form containing the Einstein pseudotensor explicitly;
3. Since the pseudotensor is invalid, the field equations containing it are invalid;
4. Therefore the original form of Einstein's field equations in unimodular coordinates is also invalid.

II. THE UNIMODULAR FIELD EQUATIONS

In modern notation, the Riemann-Christoffel symbol of the second kind, denoted $\Gamma^{\alpha}_{\beta\gamma}$, is defined by,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\omega}\left(\frac{\partial g_{\omega\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\gamma\beta}}{\partial x^{\omega}} + \frac{\partial g_{\beta\omega}}{\partial x^{\gamma}}\right). \tag{1}$$

In older notations now out of favour, but used by Einstein[1,2], the Riemann-Christoffel symbol of the second kind is written in a form of the 'three index symbols',

$$\{\beta\lambda, \alpha\} \equiv \Gamma^{\alpha}_{\beta\lambda}. \tag{2}$$

Einstein introduced an unconventional negative sign, in this way,

$$\Gamma^{\alpha}_{\beta\lambda} = -\{\beta\lambda, \alpha\}. \tag{3}$$

With these notational features in mind, the unimodular generalised field equations can be written as[1,2],

$$\begin{gathered}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right), \\ \sqrt{-g} = 1\end{gathered} \tag{4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor for the material sources of the gravitational field and $T = T^{\mu}_{\mu}$ is Laue's scalar.[b] Only the energy-momentum tensor for the material sources appears in this form of the unimodular field equations. The pseudotensor for the energy-momentum of the gravitational field is latent. The unimodular field equations explicit in the Einstein pseudotensor are[2],

$$\begin{gathered}\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) = -\kappa\left[\left(t^{\sigma}_{\mu} + T^{\sigma}_{\mu}\right) - \frac{1}{2}\delta^{\sigma}_{\mu}\left(t + T\right)\right]. \\ \sqrt{-g} = 1\end{gathered} \tag{5}$$

where the pseudotensor is defined by[2]

$$t^{\sigma}_{\mu} = \frac{1}{\kappa}\left(\frac{1}{2}\delta^{\sigma}_{\mu} g^{\alpha\nu}\Gamma^{\lambda}_{\alpha\beta}\Gamma^{\beta}_{\nu\lambda} - g^{\alpha\nu}\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu}\right) \tag{6}$$

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[b] Einstein used the obsolete notation $\partial/\partial x_{\alpha}$ in place of $\partial/\partial x^{\alpha}$.

The invariant $t = t_\sigma^\sigma$ is the contraction of the pseudotensor t_μ^σ

$$t = t_\sigma^\sigma = \frac{1}{\kappa} g^{\mu\nu} \Gamma_{\mu\beta}^{\lambda} \Gamma_{\nu\lambda}^{\beta} \,. \tag{7}$$

By Eq. (1), it is immediately evident that the invariant Eq. (7) depends solely upon the components of the metric tensor and their first-derivatives, i.e., it is a first-order intrinsic differential invariant.[4] But first-order intrinsic differential invariants do not exist.[5] One can write a mathematical expression for an invariant that is constructed solely from the components of the metric tensor and their first derivatives, but it has no mathematical validity. Therefore, the Einstein pseudotensor is mathematically invalid and so too all equations that contain it.

Note that when $T_\mu^\sigma = 0$ the field equations in the absence of matter can be written[2],

$$\begin{gathered} \frac{\partial}{\partial x^\alpha}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) = -\kappa\left(t_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma t\right), \\ \sqrt{-g} = 1 \end{gathered} \tag{8}$$

which can also be written[1,2],

$$\begin{gathered} \frac{\partial \Gamma_{\mu\nu}^{\alpha}}{\partial x^\alpha} + \Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} = 0, \\ \sqrt{-g} = 1 \end{gathered} \tag{9}$$

precisely how Einstein originally presented his field equations 'in the absence of matter'[1,2]. Since the pseudotensor t_μ^σ has no mathematical existence, all of these unimodular field equations are invalid. Consequently, the General Theory of Relativity is unsound. The detailed derivation of these unimodular field equations and their related invalid conservation laws are provided in the appendix.

III. CONCLUSIONS

The unimodular form of Einstein's field equations for the gravitational field rarely appears in the contemporary literature despite them being the original form, in which he presented his general theory of relativity. Examination of this form of his field equations reveals that they are invalid, because his pseudotensor does not exist. It follows from this that the general theory of relativity is without merit since the unimodular form must hold in the theory. From the very outset, Einstein's formulation of his general theory of relativity was invalid. Consequently, everything that has been derived from it is invalid.

Any attempt to formulate a set of field equations in terms of the alternative Landau-Lifshitz pseudotensor cannot surmount this outcome because it too, upon contraction, produces a first-order intrinsic differential invariant.[4]

APPENDIX: DERIVATION OF THE UNIMODULAR FIELD EQUATIONS

Einstein's field equations relate his gravitational field to the material sources thereof,

$$R^{\sigma}_{\mu} - \frac{1}{2} R g^{\sigma}_{\mu} = -\kappa T^{\sigma}_{\mu} \, , \tag{A1}$$

where R^{σ}_{μ} is the Ricci tensor and R the scalar curvature obtained by contraction of R^{σ}_{μ}. Equation (A1) is often written in terms of the Einstein tensor, $G^{\sigma}_{\mu} = R^{\sigma}_{\mu} - \frac{1}{2} R g^{\sigma}_{\mu}$, thus,

$$G^{\sigma}_{\mu} = -\kappa T^{\sigma}_{\mu} \, . \tag{A2}$$

Contracting Eq.(A1) gives, $R = \kappa T$. Substituting this into Eq.(A1) Einstein's field equations have the alternative form,

$$R^{\sigma}_{\mu} = -\kappa \left(T^{\sigma}_{\mu} - \frac{1}{2} T g^{\sigma}_{\mu} \right) . \tag{A3}$$

If material sources are absent $T^{\sigma}_{\mu} = 0$ and Eq.(A3) reduces to,

$$R^{\sigma}_{\mu} = 0 \, , \tag{A4}$$

which Einstein calls "*The Field Equations of Gravitation in the Absence of Matter*"[2]. Einstein maintains that a material source is nevertheless present in relation to Eq.(A4) by asserting that it holds 'outside' that material source because Eq.(A4) is, he says, the general relativistic analogue of the Laplace equation[2]. The solution to Eq.(A4) is the so-called 'Schwarzschild solution', which allegedly contains a point-mass within a spherical surface centred on that point-mass.

The foregoing forms of Einstein's field equations (Eqs.(A1) – (A4)), material sources present or not, do not explicitly exhibit his pseudotensor t^{σ}_{μ} for the energy-momentum of his gravitational field simply because they are not in the form containing it. In the foregoing forms it is latent. His field equations explicitly in terms of his pseudotensor and his energy-momentum tensor for material sources are[2],

$$\begin{gathered} \frac{\partial}{\partial x^{\alpha}} \left(g^{\sigma\beta} \Gamma^{\alpha}_{\mu\beta} \right) = -\kappa \left[\left(t^{\sigma}_{\mu} + T^{\sigma}_{\mu} \right) - \frac{1}{2} \delta^{\sigma}_{\mu} \left(t + T \right) \right] \\ \sqrt{-g} = 1 \end{gathered} \, . \tag{A5}$$

If $T^{\sigma}_{\mu} = 0$ the field equations are,

$$\begin{gathered} \frac{\partial}{\partial x^{\alpha}} \left(g^{\sigma\beta} \Gamma^{\alpha}_{\mu\beta} \right) = -\kappa \left(t^{\sigma}_{\mu} - \frac{1}{2} \delta^{\sigma}_{\mu} t \right) \\ \sqrt{-g} = 1 \end{gathered} \, . \tag{A6}$$

Equations (A5) and (A6) require a full explanation. The covariant form of the Ricci tensor is,

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\mu\alpha} - \frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}. \tag{A7}$$

Since $\Gamma^{\mu}_{\nu\mu} = \frac{\partial}{\partial x^{\nu}}\ln\sqrt{-g}$ where $g = \det(g_{\mu\nu})$, Eq.(A7) can be written,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} + \frac{\partial^2 \ln\sqrt{-g}}{\partial x^{\nu}\partial x^{\mu}} - \Gamma^{\alpha}_{\mu\nu}\frac{\partial \ln\sqrt{-g}}{\partial x^{\alpha}}. \tag{A8}$$

Einstein[1] constrained Eq.(A8) by setting the condition $\sqrt{-g} = 1$. Coordinates constrained by the condition $\sqrt{-g} = 1$ are called 'unimodular'. Hence Eq.(A8) reduces to,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}. \tag{A9}$$

For reasons arcanum, Einstein arbitrarily changed the sign of the Riemann-Christoffel symbol of the second-kind (see Eq.(3) in the text body), thus,

$$\Gamma^{\alpha}_{\beta\gamma} = -\frac{1}{2}g^{\alpha\omega}\left(\frac{\partial g_{\omega\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\gamma\beta}}{\partial x^{\omega}} + \frac{\partial g_{\beta\omega}}{\partial x^{\gamma}}\right). \tag{A10}$$

Hence, Eq.(A9) becomes,

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}. \tag{A11}$$

His field equations in the absence of matter therefore become[2],

$$\begin{aligned} &\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0 \\ &\sqrt{-g} = 1. \end{aligned} \tag{A12}$$

According to Einstein the field equations Eq.(A12) "*correspond to the laws of momentum and energy*".[2] He writes the Hamiltonian form[2],

$$\begin{aligned} \delta\int H\,d\tau = 0 \\ H = g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}, \\ \sqrt{-g} = 1 \end{aligned} \tag{A13}$$

where H is regarded[2] as a function of $g^{\mu\nu}$ and $\partial g^{\mu\nu}/\partial x^{\sigma} \equiv g^{\mu\nu}_{\sigma}$. Note that H is just the second term on the left side of Eq.(A12) multiplied by $g^{\mu\nu}$.

From Eq.(A13),

$$
\begin{aligned}
\delta H &= \delta g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\delta\Gamma^{\beta}_{\nu\alpha} \\
&= \delta g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + g^{\mu\nu}\delta\Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\mu\alpha} + g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\delta\Gamma^{\beta}_{\nu\alpha} \\
&= \delta g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + g^{\mu\nu}\delta\Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\mu\beta} + g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\delta\Gamma^{\beta}_{\nu\alpha} \\
&= \delta g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + 2g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\delta\Gamma^{\beta}_{\nu\alpha} \quad .
\end{aligned} \tag{A14}
$$

Now,

$$\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) = \delta g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha} + g^{\mu\nu}\delta\Gamma^{\beta}_{\nu\alpha}, \tag{A15}$$

so

$$g^{\mu\nu}\delta\Gamma^{\beta}_{\nu\alpha} = \delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) - \delta g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}. \tag{A16}$$

Therefore,

$$2g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\delta\Gamma^{\beta}_{\nu\alpha} = 2\Gamma^{\alpha}_{\mu\beta}\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) - 2\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\delta g^{\mu\nu}. \tag{A17}$$

Substituting Eq.(A17) into Eq.(A14) gives,

$$\delta H = -\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\delta g^{\mu\nu} + 2\Gamma^{\alpha}_{\mu\beta}\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right). \tag{A18}$$

Now, with Einstein's sign change in Eq.(A10),

$$\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) = -\frac{1}{2}\delta\left[g^{\mu\nu}g^{\beta\lambda}\left(\frac{\partial g_{\nu\lambda}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\nu}}\right)\right]. \tag{A19}$$

The last two terms in the parentheses cancel out, thus,

$$-g^{\mu\nu}g^{\beta\lambda}\frac{\partial g_{\alpha\nu}}{\partial x^{\lambda}} + g^{\mu\nu}g^{\beta\lambda}\frac{\partial g_{\alpha\lambda}}{\partial x^{\nu}} = -g^{\mu\lambda}g^{\beta\nu}\frac{\partial g_{\alpha\lambda}}{\partial x^{\nu}} + g^{\beta\nu}g^{\mu\lambda}\frac{\partial g_{\alpha\lambda}}{\partial x^{\nu}} = 0, \tag{A20}$$

so Eq.(A19) reduces to,

$$\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) = -\frac{1}{2}\delta\left(g^{\mu\nu}g^{\beta\lambda}\frac{\partial g_{\nu\lambda}}{\partial x^{\alpha}}\right). \tag{A21}$$

Owing to the general relation,

$$-g^{\mu\nu}g^{\beta\lambda}\frac{\partial g_{\nu\lambda}}{\partial x^{\alpha}} = \frac{\partial g^{\mu\beta}}{\partial x^{\alpha}} = g^{\mu\beta}_{\alpha}, \tag{A22}$$

Eq.(A21) reduces further to,

$$\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) = \frac{1}{2}\delta\frac{\partial g^{\mu\beta}}{\partial x^{\alpha}} = \frac{1}{2}\delta g^{\mu\beta}_{\alpha}. \tag{A23}$$

Substituting Eq.(A23) into Eq.(A18),

$$\delta H = -\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\delta g^{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\delta g^{\mu\beta}_{\alpha}. \tag{A24}$$

Hence,

$$\begin{aligned} \frac{\partial H}{\partial g^{\mu\nu}} &= -\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\ \frac{\partial H}{\partial g^{\mu\nu}_{\alpha}} &= \Gamma^{\alpha}_{\mu\nu} \end{aligned} \quad . \tag{A25}$$

The variation of Eq.(A13) with respect to $g^{\mu\nu}$ and $g^{\mu\nu}_{\alpha}$ yields,

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial H}{\partial g^{\mu\nu}} = 0\,. \tag{A26}$$

Substituting Eqs.(A25) into Eq.(A26) gives Eq.(A12), seemingly to satisfy Einstein's assertion of correspondence of the laws of energy and momentum with his field equations (A12). The result however is misleading. Having no clear definition of energy and momentum for this gravitational field Einstein constructed a Hamiltonian using components of his field equations in the absence of matter. He effectively goes on to define energy and momentum by means of his Hamiltonian. But a Hamiltonian does not define energy or momentum; it can only process energy and momentum terms that exist independently of it in order to produce equations of motion. The claimed correspondence is a vicious circle. The self-serving nature of Einstein's Hamiltonian manifests further in the meaningless expression Einstein thereafter obtained for the energy-momentum of his gravitational field alone; his pseudotensor, which he manufactured as follows.

Multiplying Eq.(A26) by $g^{\mu\nu}_{\sigma}$ gives,

$$g^{\mu\nu}_{\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) - g^{\mu\nu}_{\sigma}\frac{\partial H}{\partial g^{\mu\nu}} = 0\,. \tag{A27}$$

Recall that according to Einstein, "*we regard H as a function of the $g^{\mu\nu}$ and the $g^{\mu\nu}_{\alpha}$*." Hence the first term on the left side of Eq.(A27) is zero. Therefore,

$$g^{\mu\nu}_{\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) = 0\,. \tag{A28}$$

Now,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) = \frac{\partial g^{\mu\nu}_{\sigma}}{\partial x^{\alpha}}\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}} + g^{\mu\nu}_{\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right), \tag{A29}$$

hence,

$$g^{\mu\nu}_{\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) = \frac{\partial}{\partial x^{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial g^{\mu\nu}_{\sigma}}{\partial x^{\alpha}}\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\,. \tag{A30}$$

Using the relation,

$$\frac{\partial g^{\mu\nu}_{\sigma}}{\partial x^{\alpha}} = \frac{\partial g^{\mu\nu}_{\alpha}}{\partial x^{\sigma}}\,, \tag{A31}$$

Eq.(A30) becomes,

$$g_{\sigma}^{\mu\nu}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}\right)=\frac{\partial}{\partial x^{\alpha}}\left(g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}\right)-\frac{\partial H}{\partial x^{\sigma}}. \tag{A32}$$

Substituting Eq.(A32) into Eq.(A28) gives,

$$\frac{\partial}{\partial x^{\alpha}}\left(g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}\right)-\frac{\partial H}{\partial x^{\sigma}}=0. \tag{A33}$$

Using the Kronecker-delta, Eq.(A33) can be written,

$$\frac{\partial}{\partial x^{\alpha}}\left[g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}-\delta_{\sigma}^{\alpha}H\right]=0. \tag{A34}$$

Einstein then sets,

$$-2\kappa t_{\sigma}^{\alpha}=g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}-\delta_{\sigma}^{\alpha}H. \tag{A35}$$

This is his pseudotensor for the energy-momentum of his gravitational field alone. From Eq.(A34),

$$\frac{\partial t_{\sigma}^{\alpha}}{\partial x^{\alpha}}=0, \tag{A36}$$

for which Einstein asserts, "*This equation expresses the law of conservation of momentum and of energy for the gravitational field.*"[2]

Equation (A35) can be written,

$$\kappa t_{\sigma}^{\alpha}=\frac{1}{2}\delta_{\sigma}^{\alpha}H-\frac{1}{2}g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}. \tag{A37}$$

Using H from Eq.(A13) this becomes,

$$\kappa t_{\sigma}^{\alpha}=\frac{1}{2}\delta_{\sigma}^{\alpha}g^{\mu\nu}\Gamma_{\mu\beta}^{\lambda}\Gamma_{\nu\lambda}^{\beta}-\frac{1}{2}g_{\sigma}^{\mu\nu}\frac{\partial H}{\partial g_{\alpha}^{\mu\nu}}. \tag{A38}$$

Note the following general relation,

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}=-g^{\mu\beta}\Gamma_{\beta\sigma}^{\nu}-g^{\nu\beta}\Gamma_{\beta\sigma}^{\mu}. \tag{A39}$$

Recall that Einstein changed the sign of the Riemann-Christoffel symbol of the second kind (see Eq.(3) in the paper body), so Eq.(A39) in his context must read,

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}=g^{\mu\beta}\Gamma_{\beta\sigma}^{\nu}+g^{\nu\beta}\Gamma_{\beta\sigma}^{\mu}. \tag{A40}$$

Using Eqs.(A25) and (A40), after some tedious manipulation of suffixes,

$$
\begin{aligned}
g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}} &= \left(g^{\mu\beta}\Gamma_{\beta\sigma}^{\nu} + g^{\nu\beta}\Gamma_{\beta\sigma}^{\mu}\right)\Gamma_{\mu\nu}^{\alpha} \\
&= g^{\mu\beta}\Gamma_{\beta\sigma}^{\nu}\Gamma_{\mu\nu}^{\alpha} + g^{\nu\beta}\Gamma_{\beta\sigma}^{\mu}\Gamma_{\mu\nu}^{\alpha} \\
&= g^{\mu\nu}\Gamma_{\nu\sigma}^{\beta}\Gamma_{\mu\beta}^{\alpha} + g^{\nu\mu}\Gamma_{\mu\sigma}^{\beta}\Gamma_{\beta\nu}^{\alpha} \\
&= g^{\mu\nu}\left(\Gamma_{\nu\sigma}^{\beta}\Gamma_{\mu\beta}^{\alpha} + \Gamma_{\mu\sigma}^{\beta}\Gamma_{\beta\nu}^{\alpha}\right) \\
&= g^{\mu\nu}\left(\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta} + \Gamma_{\nu\beta}^{\alpha}\Gamma_{\mu\sigma}^{\beta}\right) \\
&= g^{\mu\nu}\left(\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta} + \Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta}\right) \\
&= 2g^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta} \quad .
\end{aligned} \qquad \text{(A41)}
$$

Putting Eq.(A41) into Eq.(A38) finally gives the explicit form of Einstein's pseudotensor for his unimodular field equations,

$$
\kappa t_\sigma^\alpha = \frac{1}{2}\delta_\sigma^\alpha g^{\mu\nu}\Gamma_{\mu\beta}^{\lambda}\Gamma_{\nu\lambda}^{\beta} - g^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta}\,. \qquad \text{(A42)}
$$

Equation (A36) is not yet Einstein's conservation law because it relates only to his gravitational field. It does however reveal how Einstein conjured an energy-momentum pseudotensor for his gravitational field from his unimodular field equations. His reasoning is fallacious because he cannot in fact define energy-momentum by a Hamiltonian and his resultant pseudotensor is invalid. This is actually an insurmountable problem for General Relativity: it is not possible for General Relativity to satisfy the usual conservation laws; which invalidates the theory on physical grounds.

To incorporate his pseudotensor into his field equations in the presence of matter and derive his 'conservation law', Einstein proceeded to a reformulation of his field equations. Multiplying Eq.(A12) by $g^{\nu\sigma}$ gives,

$$
g^{\nu\sigma}\frac{\partial\Gamma_{\mu\nu}^{\alpha}}{\partial x^\alpha} + g^{\nu\sigma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} = 0\,. \qquad \text{(A43)}
$$

Now,

$$
\frac{\partial}{\partial x^\alpha}\left(g^{\nu\sigma}\Gamma_{\mu\nu}^{\alpha}\right) = \frac{\partial g^{\nu\sigma}}{\partial x^\alpha}\Gamma_{\mu\nu}^{\alpha} + g^{\nu\sigma}\frac{\partial\Gamma_{\mu\nu}^{\alpha}}{\partial x^\alpha}\,. \qquad \text{(A44)}
$$

Hence

$$
g^{\nu\sigma}\frac{\partial\Gamma_{\mu\nu}^{\alpha}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\left(g^{\nu\sigma}\Gamma_{\mu\nu}^{\alpha}\right) - \frac{\partial g^{\nu\sigma}}{\partial x^\alpha}\Gamma_{\mu\nu}^{\alpha}\,. \qquad \text{(A45)}
$$

Using the general relation Eq.(A40), and mindful of Einstein's change of the sign of the Riemann-Christoffel symbol of the second kind (see Eq.(A39)),

$$
\begin{aligned}
g^{\nu\sigma}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} &= \frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu}\right) - g^{\nu\beta}\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu} - g^{\sigma\beta}\Gamma^{\nu}_{\beta\alpha}\Gamma^{\alpha}_{\mu\nu} \\
&= \frac{\partial}{\partial x^{\alpha}}\left(g^{\beta\sigma}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\nu\delta}\Gamma^{\sigma}_{\alpha\delta}\Gamma^{\alpha}_{\mu\nu} - g^{\sigma\nu}\Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\mu\beta} \\
&= \frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^{\sigma}_{\alpha\delta}\Gamma^{\alpha}_{\mu\gamma} - g^{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\
&= \frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^{\sigma}_{\beta\delta}\Gamma^{\beta}_{\mu\gamma} - g^{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\
&= \frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^{\sigma}_{\beta\gamma}\Gamma^{\beta}_{\mu\delta} - g^{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}.
\end{aligned}
\tag{A46}
$$

Putting Eq.(A46) into Eq.(A43), the field equations become,

$$
\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^{\sigma}_{\beta\gamma}\Gamma^{\beta}_{\mu\delta} - g^{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + g^{\nu\sigma}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0, \tag{A47}
$$

that is,

$$
\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^{\sigma}_{\gamma\beta}\Gamma^{\beta}_{\delta\mu} = 0. \tag{A48}
$$

Now, from Eq.(A42),

$$
-g^{\gamma\delta}\Gamma^{\sigma}_{\gamma\beta}\Gamma^{\beta}_{\delta\mu} = \kappa t^{\sigma}_{\mu} - \frac{1}{2}\delta^{\sigma}_{\mu}g^{\gamma\delta}\Gamma^{\lambda}_{\gamma\beta}\Gamma^{\beta}_{\delta\lambda}. \tag{A49}
$$

Contracting Eq.(A42),

$$
\begin{aligned}
\kappa t = \kappa t^{\sigma}_{\sigma} &= \frac{1}{2}\delta^{\sigma}_{\sigma}g^{\gamma\delta}\Gamma^{\lambda}_{\gamma\beta}\Gamma^{\beta}_{\delta\lambda} - g^{\gamma\delta}\Gamma^{\sigma}_{\gamma\beta}\Gamma^{\beta}_{\delta\sigma} \\
&= 2g^{\gamma\delta}\Gamma^{\lambda}_{\gamma\beta}\Gamma^{\beta}_{\delta\lambda} - g^{\lambda\delta}\Gamma^{\lambda}_{\gamma\beta}\Gamma^{\beta}_{\delta\lambda} \\
&= g^{\gamma\delta}\Gamma^{\lambda}_{\gamma\beta}\Gamma^{\beta}_{\delta\lambda}.
\end{aligned}
\tag{A50}
$$

Substituting Eq.(A50) into Eq.(A49) gives,

$$
-g^{\gamma\delta}\Gamma^{\sigma}_{\gamma\beta}\Gamma^{\beta}_{\delta\mu} = \kappa t^{\sigma}_{\mu} - \frac{1}{2}\delta^{\sigma}_{\mu}\kappa t. \tag{A51}
$$

Putting Eq.(A51) into Eq.(A48) yields the field equations as,

$$
\begin{aligned}
\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right) &= -\kappa\left(t^{\sigma}_{\mu} - \frac{1}{2}\delta^{\sigma}_{\mu}t\right) \\
\sqrt{-g} &= 1.
\end{aligned}
\tag{A52}
$$

It is now immediately clear that Einstein's field equations (A52) in the absence of matter are unsound due to violation of the rules of pure mathematics (i.e. the presence of his invalid pseudotensor). Consequently his field equations (A12) are meaningless. Nevertheless, Einstein maintained that Eq.(A52) is an analogue of the Laplace equation for his gravitational field. Recall the Laplace and Poisson equations are respectively,

$$\nabla^2\phi = 0$$

$$\nabla^2\phi = 4\pi\kappa\rho \qquad \text{(A53)}$$

where ρ denotes matter density.

Einstein[2] then remarks "*The system of equations (51) shows how this energy-tensor (corresponding to the density ρ in Poisson's equation) is to be introduced into the field equations of gravitation. For if we consider a complete system (e.g. the solar system), the total mass of the system, and therefore its total gravitating action as well, will depend on the total energy of the system, and therefore on the ponderable energy together with the gravitational energy. This will allow itself to be expressed by introducing into (51), in place of the energy-components of the gravitational field alone, the sum* $t_\mu^\sigma + T_\mu^\sigma$ *of the energy-components of matter and of gravitational field. Thus instead of (51) we obtain the tensor equation*

$$\frac{\partial}{\partial x^\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) = -\kappa\left[\left(t_\mu^\sigma + T_\mu^\sigma\right) - \frac{1}{2}\delta_\mu^\sigma\left(t+T\right)\right]$$

$$\sqrt{-g} = 1$$

where we have set $T = T_\mu^\mu$ *(Laue's scalar). These are the required field equations of gravitation in mixed form. Working back from these, we get in place of (47)*

$$\frac{\partial}{\partial x_\alpha}\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

$$\sqrt{-g} = 1 \quad ."$$

Again, it is immediately clear that both forms of Einstein's field equations for the presence of material sources are also false, since working backwards from the first set of equations in the quotation above contains his invalid pseudotensor. Consequently, the usual presentations of Einstein's field equations, Eqs.(A1) - (A4), are also false. Note that in the these forms of Einstein's field equations the pseudotensor for the energy-momentum of the gravitational field disappears into the depths of the field equations where it lies hidden, giving no hint of its presence even though it appears explicitly in the penultimate form in the quotation above, from which Einstein extracts his conservation law. To obtain Einstein's generalised field equations in the presence of matter in unimodular coordinates and thereby discern the latent presence of his invalid pseudotensor, write Eq.(A5) as:

$$\frac{\partial}{\partial x^\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) = -\kappa\left(t_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma t\right) - \kappa\left(T_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma T\right). \qquad \text{(A54)}$$

Hence,

$$\frac{\partial}{\partial x^\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) + \kappa\left(t_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma t\right) = -\kappa\left(T_\mu^\sigma - \frac{1}{2}\delta_\mu^\sigma T\right). \qquad \text{(A55)}$$

Recall that,

$$\delta_{\mu}^{\sigma} = g_{\mu\nu} g^{\nu\sigma} . \tag{A56}$$

So Eq.(A55) can be written as,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) + \kappa\left(t_{\mu}^{\sigma} - \frac{1}{2}\delta_{\mu}^{\sigma} t\right) = -g^{\nu\sigma}\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \tag{A57}$$

But,

$$- g^{\gamma\delta}\Gamma_{\gamma\beta}^{\sigma}\Gamma_{\delta\mu}^{\beta} = \kappa\left(t_{\mu}^{\sigma} - \frac{1}{2}\delta_{\mu}^{\sigma} t\right), \tag{A58}$$

wherein Einstein's pseudotensor is given by,

$$\kappa t_{\sigma}^{\alpha} = \frac{1}{2}\delta_{\sigma}^{\alpha} g^{\mu\nu}\Gamma_{\mu\beta}^{\lambda}\Gamma_{\nu\lambda}^{\beta} - g^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\sigma}^{\beta} . \tag{A59}$$

Using Eq.(A58), Eq.(A57) becomes,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) - g^{\gamma\delta}\Gamma_{\gamma\beta}^{\sigma}\Gamma_{\delta\mu}^{\beta} = -g^{\nu\sigma}\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \tag{A60}$$

Therefore,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma_{\mu\beta}^{\alpha}\right) - g^{\gamma\delta}\Gamma_{\gamma\beta}^{\sigma}\Gamma_{\delta\mu}^{\beta} - g^{\nu\sigma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} = -g^{\nu\sigma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} - g^{\nu\sigma}\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \tag{A61}$$

Changing the dummy summation indices on the left side this can be written as,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\sigma}\Gamma_{\mu\nu}^{\alpha}\right) - g^{\nu\beta}\Gamma_{\alpha\beta}^{\sigma}\Gamma_{\mu\nu}^{\alpha} - g^{\sigma\beta}\Gamma_{\alpha\beta}^{\nu}\Gamma_{\mu\nu}^{\alpha} = -g^{\nu\sigma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} - g^{\nu\sigma}\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \tag{A62}$$

Recall the mathematical relation,

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}} = -g^{\mu\tau}\{\tau\sigma,\nu\} - g^{\nu\tau}\{\tau\sigma,\mu\} . \tag{A63}$$

Also recall that Einstein used,

$$\{\tau\sigma,\nu\} = -\Gamma_{\tau\sigma}^{\nu} . \tag{A64}$$

Therefore Eq.(A63) becomes,

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}} = g^{\mu\tau}\Gamma_{\tau\sigma}^{\nu} + g^{\nu\tau}\Gamma_{\tau\sigma}^{\mu} . \tag{A65}$$

Using the relation Eq.(A65), Eq.(A62) can be written,

$$\frac{\partial}{\partial x^\alpha}\left(g^{\nu\sigma}\Gamma^\alpha_{\mu\nu}\right)-\frac{\partial g^{\nu\sigma}}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu}=-g^{\nu\sigma}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}-g^{\nu\sigma}\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\ . \qquad \text{(A66)}$$

Now

$$\frac{\partial}{\partial x^\alpha}\left(g^{\nu\sigma}\Gamma^\alpha_{\mu\nu}\right)=\frac{\partial g^{\nu\sigma}}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu}+g^{\nu\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}\ , \qquad \text{(A67)}$$

so,

$$\frac{\partial}{\partial x^\alpha}\left(g^{\nu\sigma}\Gamma^\alpha_{\mu\nu}\right)-\frac{\partial g^{\nu\sigma}}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu}=g^{\nu\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}\ . \qquad \text{(A68)}$$

Using Eq.(A68), Eq.(A66) becomes,

$$g^{\nu\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}=-g^{\nu\sigma}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}-g^{\nu\sigma}\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\ . \qquad \text{(A69)}$$

Therefore,

$$g^{\nu\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}+g^{\nu\sigma}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}=-g^{\nu\sigma}\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\ . \qquad \text{(A70)}$$

Dividing through by $g^{\nu\sigma}$ gives,

$$\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha}+\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}=-\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right). \qquad \text{(A71)}$$

To reach Einstein's conservation law, the first step is to obtain yet another form of his field equations by contracting the field equations containing the pseudotensor explicitly, on the suffixes μ and σ, thus,

$$\frac{\partial}{\partial x^\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\sigma\beta}\right)=-\kappa\left[\left(t^\sigma_\sigma+T^\sigma_\sigma\right)-\frac{1}{2}\delta^\sigma_\sigma\left(t+T\right)\right]=\kappa\left(t+T\right). \qquad \text{(A72)}$$

Since σ herein is a dummy suffice it can be replaced by λ to give,

$$\frac{\partial}{\partial x^\alpha}\left(g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}\right)=\kappa\left(t+T\right). \qquad \text{(A73)}$$

Multiplying Eq.(A73) by ½ δ^σ_μ gives,

$$\frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\delta^\sigma_\mu g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}\right)=\kappa\frac{1}{2}\delta^\sigma_\mu\left(t+T\right). \qquad \text{(A74)}$$

The uncontracted equation from which Eq.(A73) is obtained is,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right)=-\kappa\left[\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right)-\frac{1}{2}\delta^{\sigma}_{\mu}\left(t+T\right)\right]. \tag{A75}$$

Subtracting Eq.(A74) from Eq.(A75) gives,

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}-\frac{1}{2}\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}\right)=-\kappa\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right). \tag{A76}$$

Take now the derivative of the Eq.(A76) with respect to x^{σ},

$$\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}-\frac{1}{2}\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}\right)=-\kappa\frac{\partial}{\partial x^{\sigma}}\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right). \tag{A77}$$

Expanding gives,

$$\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right)-\frac{1}{2}\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}\right)=-\kappa\frac{\partial}{\partial x^{\sigma}}\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right). \tag{A78}$$

The first term on the left side of Eq.(A78) is, again bearing in mind Einstein's change of sign for $\Gamma^{\alpha}_{\mu\nu}$,

$$\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\mu\beta}\right)=-\frac{1}{2}\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left[g^{\sigma\beta}g^{\alpha\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\beta}}+\frac{\partial g_{\beta\lambda}}{\partial x^{\mu}}-\frac{\partial g_{\mu\beta}}{\partial x^{\lambda}}\right)\right]. \tag{A79}$$

The first and third terms in the parentheses cancel, as follows,

$$g^{\sigma\beta}g^{\alpha\lambda}\frac{\partial g_{\mu\lambda}}{\partial x^{\beta}}-g^{\sigma\beta}g^{\alpha\lambda}\frac{\partial g_{\mu\beta}}{\partial x^{\lambda}}=g^{\alpha\beta}g^{\sigma\lambda}\frac{\partial g_{\mu\lambda}}{\partial x^{\beta}}-g^{\sigma\lambda}g^{\alpha\beta}\frac{\partial g_{\mu\lambda}}{\partial x^{\beta}}=0\,. \tag{A80}$$

Hence Eq.(A79) reduces to,

$$\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\sigma\beta}\Gamma^{\alpha}_{\beta\mu}\right)=-\frac{1}{2}\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\sigma\beta}g^{\alpha\lambda}\frac{\partial g_{\beta\lambda}}{\partial x^{\mu}}\right). \tag{A81}$$

Using the general relation Eq.(A22), Eq.(A81) becomes,

$$\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\alpha\beta}\Gamma^{\alpha}_{\beta\mu}\right)=\frac{1}{2}\frac{\partial^{2}g^{\sigma\alpha}_{\mu}}{\partial x^{\sigma}\partial x^{\alpha}}=\frac{1}{2}\frac{\partial^{2}g^{\alpha\beta}_{\mu}}{\partial x^{\alpha}\partial x^{\beta}}\,. \tag{A82}$$

Putting Eq.(A82) into Eq.(A78) gives,

$$\frac{1}{2}\frac{\partial^{2}g^{\alpha\beta}_{\mu}}{\partial x^{\alpha}\partial x^{\beta}}-\frac{1}{2}\frac{\partial^{2}}{\partial x^{\sigma}\partial x^{\alpha}}\left(\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}\right)=-\kappa\frac{\partial}{\partial x^{\sigma}}\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right). \tag{A83}$$

The second term on the left side of Eq.(A83) is,

$$-\frac{1}{2}\frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(\delta_\mu^\sigma g^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha\right)=-\frac{1}{2}\frac{\partial^2}{\partial x^\mu \partial x^\alpha}\left(g^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha\right). \tag{A84}$$

Bearing in mind once again Einstein's change of sign for $\Gamma_{\lambda\beta}^\alpha$, Eq.(A84) is expanded to,

$$-\frac{1}{2}\frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(\delta_\mu^\sigma g^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha\right)=\frac{1}{4}\frac{\partial^2}{\partial x^\mu \partial x^\alpha}\left[g^{\lambda\beta}g^{\alpha\delta}\left(\frac{\partial g_{\delta\lambda}}{\partial x^\beta}+\frac{\partial g_{\delta\beta}}{\partial x^\lambda}-\frac{\partial g_{\lambda\beta}}{\partial x^\delta}\right)\right]. \tag{A85}$$

The last term on the right side of Eq.(A85) is,

$$-\frac{1}{4}\frac{\partial^2}{\partial x^\mu \partial x^\alpha}\left(g^{\alpha\delta}g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\delta}\right). \tag{A86}$$

It is zero,

$$\frac{1}{2}g^{\lambda\beta}\frac{\partial g_{\lambda\beta}}{\partial x^\delta}=\frac{\partial\sqrt{-g}}{\partial x^\delta}=0, \tag{A87}$$

because in unimodular coordinates $\sqrt{-g}=1$. Therefore Eq.(A85) reduces to,

$$-\frac{1}{2}\frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(\delta_\mu^\sigma g^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha\right)=\frac{1}{4}\frac{\partial^2}{\partial x^\mu \partial x^\alpha}\left[g^{\lambda\beta}g^{\alpha\delta}\left(\frac{\partial g_{\delta\lambda}}{\partial x^\beta}+\frac{\partial g_{\delta\beta}}{\partial x^\lambda}\right)\right]. \tag{A88}$$

Furthermore,

$$\begin{aligned}\left[g^{\lambda\beta}g^{\alpha\delta}\left(\frac{\partial g_{\delta\lambda}}{\partial x^\beta}+\frac{\partial g_{\delta\beta}}{\partial x^\lambda}\right)\right]&=g^{\lambda\beta}g^{\alpha\delta}\frac{\partial g_{\delta\lambda}}{\partial x^\beta}+g^{\lambda\beta}g^{\alpha\delta}\frac{\partial g_{\delta\beta}}{\partial x^\lambda}\\&=g^{\lambda\beta}g^{\alpha\delta}\frac{\partial g_{\delta\lambda}}{\partial x^\beta}+g^{\beta\lambda}g^{\alpha\delta}\frac{\partial g_{\delta\lambda}}{\partial x^\beta}\\&=2g^{\lambda\beta}g^{\alpha\delta}\frac{\partial g_{\delta\lambda}}{\partial x^\beta}.\end{aligned} \tag{A89}$$

Then by the general relation Eq.(A22),

$$2g^{\lambda\beta}g^{\alpha\delta}\frac{\partial g_{\delta\lambda}}{\partial x^\beta}=-2\frac{\partial g^{\alpha\beta}}{\partial x^\beta}. \tag{A90}$$

Putting Eq.(A90) into Eq.(A88) gives,

$$-\frac{1}{2}\frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(\delta_\mu^\sigma g^{\lambda\beta}\Gamma_{\lambda\beta}^\alpha\right)=-\frac{1}{2}\frac{\partial^2 g_\beta^{\alpha\beta}}{\partial x^\mu \partial x^\alpha}. \tag{A91}$$

Putting Eq.(A91) into Eq.(A83) gives,

$$\frac{1}{2}\frac{\partial^2 g_\mu^{\alpha\beta}}{\partial x^\alpha \partial x^\beta}-\frac{1}{2}\frac{\partial^2 g_\beta^{\alpha\beta}}{\partial x^\mu \partial x^\alpha}=-\kappa\frac{\partial}{\partial x^\sigma}\left(t_\mu^\sigma+T_\mu^\sigma\right), \tag{A92}$$

that is,

$$\frac{1}{2}\frac{\partial^3 g^{\alpha\beta}}{\partial x^\alpha \partial x^\beta \partial x^\mu} - \frac{1}{2}\frac{\partial^3 g^{\alpha\beta}}{\partial x^\alpha \partial x^\beta \partial x^\mu} = -\kappa \frac{\partial}{\partial x^\sigma}\left(t_\mu^\sigma + T_\mu^\sigma\right), \qquad \text{(A93)}$$

or,

$$\frac{\partial\left(t_\mu^\sigma + T_\mu^\sigma\right)}{\partial x^\sigma} = 0. \qquad \text{(A94)}$$

Concerning Eq.(A94) Einstein[2] states, "*Thus it results from our field equations of gravitation that the laws of conservation of momentum and energy are satisfied. This may be seen most easily from the consideration which leads to equation (49a); except that here, instead of the energy components* t_μ^σ *of the gravitational field, we have to introduce the totality of the energy components of matter and gravitational field.*"

Eq.(A94) is invalid owing to the pseudotensor. However, is in fact impossible to write a non-tensorial conservation law in General Relativity. Furthermore, there is only one way to write a tensorial conservation law in General Relativity. But it is a conservation law only as a matter of mathematical formality as it violates the usual conservation laws determined by a vast array of experiments. This means that Einstein's field equations are also fallacious on physical grounds. Before considering the tensorial form of a conservation law from Einstein's field equations, consider his now usual field equations in the two equivalent forms of Eq.(A1) and Eq.(A3),

$$R_\mu^\sigma - \frac{1}{2} R g_\mu^\sigma = -\kappa T_\mu^\sigma,$$

$$R_\mu^\sigma = -\kappa\left(T_\mu^\sigma - \frac{1}{2} T g_\mu^\sigma\right).$$

Compare them as follows,

$$\begin{aligned} R_\mu^\sigma &= -\kappa\left(T_\mu^\sigma - \frac{1}{2} T g_\mu^\sigma\right), \\ T_\mu^\sigma &= -\frac{1}{\kappa}\left(R_\mu^\sigma - \frac{1}{2} R g_\mu^\sigma\right). \end{aligned} \qquad \text{(A95)}$$

By the first equation if $T_\mu^\sigma = 0$ then $R_\mu^\sigma = 0$, and by the second equation, if $R_\mu^\sigma = 0$ then $T_\mu^\sigma = 0$. In other words, T_μ^σ and R_μ^σ must vanish identically: 0 = 0. The expression $R_\mu^\sigma = 0$ has no meaning. Now consider the field equations with cosmological constant λ,

$$R_\mu^\sigma - \frac{1}{2}(R - 2\lambda) g_\mu^\sigma = -\kappa T_\mu^\sigma. \qquad \text{(A96)}$$

Contracting Eq.(A96) gives,

$$\begin{aligned} &R - 2(R - 2\lambda) = -\kappa T, \\ &so \\ &R = \kappa T + 4\lambda. \end{aligned} \qquad \text{(A97)}$$

Hence Eq.(A96) can be written in the equivalent form,

$$R^{\sigma}_{\mu} = -\kappa\left(T^{\sigma}_{\mu} - \frac{1}{2}Tg^{\sigma}_{\mu}\right) + \lambda g^{\sigma}_{\mu}. \tag{A98}$$

If $T^{\sigma}_{\mu} = 0$ then Eq.(A98) reduces to,

$$R^{\sigma}_{\mu} = \lambda g^{\sigma}_{\mu}. \tag{A99}$$

If also $\lambda = 0$,

$$R^{\sigma}_{\mu} = 0. \tag{A100}$$

The solution to Eq.(A99) is de Sitter's empty universe, which is empty precisely because it contains no material sources, i.e. $T^{\sigma}_{\mu} = 0$. On the other hand, Einstein maintains that a material source is present in relation to Eq.(A100) even though $T^{\sigma}_{\mu} = 0$ there too, (his gravitational field is generated by the presence of matter). Indeed, the solution for Eq.(A100) is the so-called 'Schwarzschild solution' which allegedly contains a finite mass M to generate the gravitational field associated with Eq.(A100) as an analogue of the Laplace equation. Thus, according to Einstein, material sources are both absent and present by the very same mathematical constraint, $T^{\sigma}_{\mu} = 0$, which is impossible: compare Eq.(A99) with Eq.(A100). In other words, there is matter nowhere in the universe described by Eq.(A99), because $T^{\sigma}_{\mu} = 0$, yet there is matter at the centre of a spherical surface in the universe described by Eq.(A100), for which $T^{\sigma}_{\mu} = 0$ also. Indeed, setting $\lambda = 0$ in Eq.(A99) produces Eq.(A100). Thus, there is no matter anywhere in Eq.(A99) but upon setting $\lambda = 0$ therein, matter suddenly exists in Eq.(A100).

Einstein's field equations in terms of the Einstein tensor are, in mixed form,

$$G^{\sigma}_{\mu} = -\kappa T^{\sigma}_{\mu}. \tag{A101}$$

This can be written,

$$\left(\frac{G^{\sigma}_{\mu}}{\kappa} + T^{\sigma}_{\mu}\right) = 0. \tag{A102}$$

Equation (A102) is a tensor equation. Compare this to the total energy and momentum E^{σ}_{μ} given by Einstein,

$$E^{\sigma}_{\mu} = \left(t^{\sigma}_{\mu} + T^{\sigma}_{\mu}\right). \tag{A103}$$

Thus the $G^{\sigma}_{\mu} / \kappa$ constitute the energy-momentum of his gravitational field[6] and the T^{σ}_{μ} that of the material sources thereof. Denote the covariant derivative by a subscripted semicolon. The tensor divergence of Eq.(A102) is then,

$$\left(\frac{G^{\sigma}_{\mu}}{\kappa} + T^{\sigma}_{\mu}\right)_{;} = 0. \tag{111}$$

Since the tensor divergence is zero Eq.(A102) implies a conservation law. However, by Eq.(A102) the total energy-momentum of Einstein's gravitational field and its material sources is always zero[6]. Consequently Einstein's theory violates the experimentally established conservation laws for a closed system. So, once again, his theory is false.

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