Contradiction for a Gravitational Plane Wave Pulse Colliding with a Mass

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Abstract

We consider a system of a gravitational plane wave pulse colliding with a mass. We assume as size and mass go to zero that the the path of the mass approaches a geodesic of a plane gravitational wave pulse having zero Ricci tensor. Assume also that energy and momentum are conserved. We show these assumptions lead to a contradiction.

1 Plane gravitational wave pulse metric

Define $u = t - x$ and let the metric $g_{\mu\nu}(u)$ be [1]

$$ds^2 = -dt^2 + dx^2 + [L(u)]^2 e^{2\beta(u)} dy^2 + [L(u)]^2 e^{-2\beta(u)} dz^2 \tag{1}$$

having $L(u) = 1$ and $\beta(u) = 0$ for $u < 0$ hence $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$. Let $\beta \neq 0$ and let $L(u)$ satisfy the equation

$$\frac{d^2 L}{du^2}(u) + \left[\frac{d\beta}{du}(u)\right]^2 L(u) = 0 \tag{2}$$

The metric $g_{\mu\nu}(u)$ then has zero Ricci tensor hence $R_{\mu\nu} = 0$. It is then the metric of a gravitational plane wave pulse. We have by (2) as u increases from $u = 0$ that $L(u)$ decreases from $L(0) = 1$ and become zero at some point $u_0 > 0$. Consequently $g_{22}(u) > 0$ for $u < u_0$.

2 Proper Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1 + \cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation [2] such that

$$\begin{aligned} t &= t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta & (3)\\ x &= 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta & (4)\\ y &= 2t'\cot\theta - 2x'\cot\theta + y' & (5)\\ z &= z' & (6) \end{aligned}$$

By (3) and (4) we get $t - x = t' - x' = u'$. By (3)-(6) we get the metric $g'_{\mu\nu}(u')$

$$\begin{aligned} ds^2 = &\Big\{-1 - 4[1 - g_{22}(u')]\cot^2\theta\Big\}dt'^2 + 8[1 - g_{22}(u')]\cot^2\theta\, dt'dx' \\ + &\Big\{1 - 4[1 - g_{22}(u')]\cot^2\theta\Big\}dx'^2 - 4[1 - g_{22}(u')]\cot\theta\, dt'dy' \\ + &4[1 - g_{22}(u')]\cot\theta\, dx'dy' + g_{22}(u')dy'^2 + g_{33}(u')dz'^2 \end{aligned} \tag{7}$$

The metric $g'_{\mu\nu}(u')$ satisfying $R'_{\mu\nu}(u') = 0$ and $g'_{\mu\nu}(u') = \eta_{\mu\nu}$ for $u' < 0$ is then also the metric of a gravitational plane wave pulse.

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3 Geodesic curve

The curve

$$t'(\lambda) = (1+2\cot^2\theta)\lambda - 2\cot^2\theta\int_0^\lambda \frac{dw}{g_{22}(w)} \quad (8)$$

$$x'(\lambda) = 2\cot^2\theta\lambda - 2\cot^2\theta\int_0^\lambda \frac{dw}{g_{22}(w)} \quad (9)$$

$$y'(\lambda) = -2\cot\theta\lambda + 2\cot\theta\int_0^\lambda \frac{dw}{g_{22}(w)} \quad (10)$$

$$z'(\lambda) = 0 \quad (11)$$

satisfies the geodesic equation for the metric $g'_{\mu\nu}(u')$ and so is a geodesic curve. For $\lambda < 0$ we have $t'(\lambda) = \lambda, x'(\lambda) = y'(\lambda) = z'(\lambda) = 0$. Now by (8)

$$\frac{dt'}{d\lambda} = 1 + 2\cot^2\theta - \frac{2\cot^2\theta}{g_{22}(\lambda)} \quad (12)$$

Since $g_{22}(0) = 1$ and $g_{22}(u) \to 0$ as $u \to u_0$ there is then a $\lambda_0 > 0$ such that $(dt'/d\lambda)(\lambda_0) = 0$. We then have $(dt'/d\lambda)(\lambda) < 0$ for $\lambda_0 < \lambda < u_0$. Consequently for $\lambda_0 < \lambda < u_0$ the geodesic curve goes backward in t'.

4 Energy-momentum tensor

Now consider a system of gravitational plane wave pulse colliding with a mass M of finite size and finite mass density. Let $\widetilde{g}_{\mu\nu}(t,x,y,z)$ be the metric of the combined system of colliding wave and M. Require $\widetilde{g}_{\mu\nu}(t,x,y,z) \to g_{\mu\nu}(t-x)$ as size and mass density of M go to zero where $g_{\mu\nu}(t-x)$ is the metric (1). Transforming to t',x',y',z' coordinates by (3)-(6) gives the metric $\widetilde{g}'_{\mu\nu}(t',x',y',z')$. Let the mass density of M be $\rho(t',x',y',z')$ and pressure $p(t',x',y',z')$. Also there is an equation between p and ρ. The energy-momentum tensor of M is

$$T'^{\mu\nu} = p\widetilde{g}'^{\mu\nu} + (p+\rho)\frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau} \quad (13)$$

with

$$\widetilde{g}'_{\mu\nu}\frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau} = -1 \quad (14)$$

From (13) and (14) we get

$$T'^{\mu\nu} = p\widetilde{g}'^{\mu\nu} + \frac{(T'^{0\mu} - p\widetilde{g}'^{0\mu})(T'^{0\nu} - p\widetilde{g}'^{0\nu})}{T'^{00} - p\widetilde{g}'^{00}} \quad (15)$$

and

$$p + \rho = -\widetilde{g}'_{\mu\nu}\frac{(T'^{0\mu} - p\widetilde{g}'^{0\mu})(T'^{0\nu} - p\widetilde{g}'^{0\nu})}{T'^{00} - p\widetilde{g}'^{00}} \quad (16)$$

Assuming conservation of energy and momentum $T'^{\mu\nu}{}_{;\nu} = 0$ we have

$$\frac{\partial T'^{0\mu}}{\partial t'} = -\frac{\partial T'^{1\mu}}{\partial x'} - \frac{\partial T'^{2\mu}}{\partial y'} - \frac{\partial T'^{3\mu}}{\partial z'} - \Gamma^\mu{}_{\alpha\beta}T'^{\alpha\beta} - \Gamma^\alpha{}_{\alpha\beta}T'^{\beta\mu} \quad (17)$$

where $\Gamma^\alpha_{\mu\nu}$ is constructed using the metric $\widetilde{g}'_{\mu\nu}(t',x',y',z')$. From $T'^{\mu\nu}$ at t' and having $\widetilde{g}'_{\mu\nu}$ at all points we can use (17) to determine $T'^{0\mu}$ at $t' + \delta t'$. We can then use (15), (16), and equation between p and ρ to determine $T'^{\mu\nu}$ and ρ at $t' + \delta t'$. Now choose M of constant mass density. There is then a constant C such that $\rho = C$ for all points of M and $\rho = 0$ outside M.

5 Backward in time

Let M have small mass and size so that, using the assumption, the path of M is approximately the geodesic (8)-(11). Define $S(t')$ to be the set of points (x', y', z') such that $\rho(t', x', y', z') = C$. We have for large negative t' that $S(t')$ is not empty. If $S(t')$ is not empty then by (16) there are points (x', y', z') such that $T'^{0\mu}(t', x', y', z') \neq 0$. Consequently by (17) there are (x', y', z') and a small $\delta > 0$ such that $T'^{0\mu}(t' + \delta, x', y', z') \neq 0$. By (13) then $\rho(t' + \delta, x', y', z') \neq 0$. Now ρ at a point is either C or zero hence $\rho(t' + \delta, x', y', z') = C$. Consequently $S(t' + \delta)$ is not empty.

Since the paths of the different fluid elements making up M do not intersect $S(t')$ does not go to a point as t' increases. Consequently $S(t')$ is not empty for all t'. Following $S(t')$ as t' increases we then have M does not go backward in t'.

6 Contradiction

From section 3 we have for $\lambda_0 < \lambda < u_0$ that the geodesic goes backward in t'. For our example of wave colliding with M having small mass and size the path of M is approximately this geodesic. There are then points on the path of M where M goes backward in t'. From section 5 this can not happen. We have a contradiction.

References

[1] C. Misner, K. Thorne, J. Wheeler, *Gravitation*, p. 957

[2] K. De Paepe, *Physics Essays*, June 2018