Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make one identity regarding p and q from one equation that gives Riemann zeta function $\zeta(s)$ analytic continuation and 2 formulas ($1/2+a\pm bi$, $1/2-a\pm bi$) that show non-trivial zero point of $\zeta(s)$. 2. We find that the above identity holds only at $a=0$. 3. Therefore non-trivial zero points of $\zeta(s)$ must be $1/2\pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $0 < Re(s)$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+\cdots\cdots=(1-2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$. i is $\sqrt{-1}$.

$$S_0 = 1/2+a\pm bi \tag{2}$$

The following (3) also shows non-trivial zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1-S_0 = 1/2-a\mp bi \tag{3}$$

We define the range of a and b as $0 \le a < 1/2$ and $14 < b$ respectively. Then we can show all non-trivial zero points of $\zeta(s)$ by the above (2) and (3). Because non-trivial zero points of $\zeta(s)$ exist in the critical strip of $\zeta(s)$ ($0 < Re(s) < 1$) and non-trivial zero points of $\zeta(s)$ found until now exist in the range of $14 < b$.
We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero

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respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. The identity regarding p and q

We define $f(n)$ as the following (8).

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from the above (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from the above (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) regarding p and q from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. p and q are any real numbers.

$$\begin{aligned} 0 \equiv\, & p\{\text{the right side of (9)}\} + q\{\text{the right side of (10)}\} \\ =\, & p\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - \cdots\cdots\} \\ & + q\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - \cdots\cdots\} \\ =\, & f(2)\{p\cos(b\log 2) + q\sin(b\log 2)\} - f(3)\{p\cos(b\log 3) + q\sin(b\log 3)\} \\ & + f(4)\{p\cos(b\log 4) + q\sin(b\log 4)\} - \cdots \\ & + (-1)^n f(n)\{p\cos(b\log n) + q\sin(b\log n)\} + \cdots\cdots \end{aligned} \tag{11}$$

In order for the above (11) to hold for any value of p and q, the following (12) i.e. $a = 0$ must hold.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{12}$$

3. Conclusion

a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$ $(0 < Re(s) < 1)$. However, a cannot have any value but zero as shown in the above item 2. Therefore non-trivial zero point of $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$.

Appendix 1. : Equation construction

We can construct (9), (10) and (11) by applying the following Theorem 1[1].

Theorem 1

If the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) hold.

$$\begin{aligned}
&\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A \\
&\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B \\
&\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B \\
&\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B
\end{aligned}$$

1.1 We can have (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

1.2 We can have (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

1.3 We can have (11) as (Series 3) by regarding the following (13) and (14) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = p\{\text{the right side of (9)}\} \equiv 0 \tag{13}$$

$$\text{(Series 2)} = q\{\text{the right side of (10)}\} \equiv 0 \tag{14}$$

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

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