ON RIEMANN HYPOTHESIS

LUCIAN M. IONESCU

Abstract. A line of study of the Riemann Hypothesis is proposed, based on a comparison with Weil zeros and a categorification of the duality between Riemann zeros and prime numbers. The three case of coefficients, complex, p-adic and finite fields are also related.

Contents

1. Introduction

The Riemann Hypothesis (RH) in finite characteristic is proved part of the Weil Conjectures", now Theorems; see bibliography.

A natural idea is to understand the relation between Riemann zeros and Weil zeros, as well as to look for the objects behind them. Specifically, R-zeros are in duality with prime powers in the sense of distributions [7, 5], which are just sizes (integrals, hence "periods") of finite Abelian groups (Category Abf). It is expected that R-zeros, which are better understood as poles of $1/\zeta = DT(\mu)$ via the convolution identity involving the inverse of the Moebus function $\mu \star 1 = \delta$ and Dirichlet Transform (DT), may also have a similar interpretation as "counting something", i.e. periods.

To give meaning to less obvious numbers and relations, a powerful tool is Categorification (see Khovanov's work related to homology). By categorification one should look for a duality between the category Abf and a "natural category" which realize R-zeros as periods. If Tannaka-Krein duality associates to a Tannakian category a Hopf algebra, here we would look at group rings of finite Abelian groups as Hopf algebras. Their representation category is thus TK-dual to the original finite Abelian group.

In what follows we only try to expand the above and point to some related work.

2. Primes Numbers and Riemann Zeros

Riemann zeros correspond to prime powers, and not just to prime numbers, under a Fourier transform (duality) at the level of distributions.

2.1. The Duality.

This duality is well documented in [7]; see also [5] for the Fourier transform interpretation of the duality, extended to the context of distributions (loc.. cit. §3.2). An important point is that all the prime powers are involved as dual to R-zeros and also the appearance of $\sqrt{p}$ (quadratic extension to adjoin i; see below) and $logp$ ("Lie algebra" generator of multiplicative rational numbers group: see quasi-crystals [5] §3).

The bicharacter that needs to be studied is $z^s : (C^*, \cdot) \times (C, +) \to C^*$. It extends n^s with its generating function the R-zeta function and has a natural embeding into p-adic numbers of all primes p. Then complex numbers analytic continuation can be related to p-adic analytic continuation via theory of harmonic forms (p-adic de Rham - Hodge Theory and Jacobian matrix). How does the duality translate to the p-adic realm? (see Conclusions regarding de Rham isomorphism / cohomological duality and RH within Weil Conjectures, proved by B. Dwork using p-adic techniques).

Recall that Fourier isomorphism, from an algebraic point of view, is the iso between the group ring and convolution algebra. This can be applied to finite abelian groups, including Z/p^k as truncations of p-adic numbers, which are Frobenius related to finite fields as algebraic extensions.

Now the group ring elements define the algebraic varieties to be studied cohomologically. Their seizes determine the corresponding generating functions: zeta functions. Their zeros have a cohomological interpretation, possibly related to de Rham isomorphism periods.

How do all these dualities (isomorphisms) relate to one another?

Finally, the Fourier duality for the primes and R-zeros, at the level of distributions, needs only Dirac delta functions, as distributions; this suggests that there maybe a pro-algebraic version of the duality at the level of pro-algebraic group rings and convolution functions, within the realm of periods extending the algebraic numbers within the quadratic extension of the reals.

2.2. Categorifying Prime Numbers.

Prime powers are sizes of primary abelian groups Z/p^k [5] (categorification), the irreducibles of the category Ab_f. Note also that these are n-th order deformations of Z/p, hence part of p-adic numbers, within the theory in characteristic zero!

When considering the symmetries of $(Z/p, +)$ symmetries we get the finite fields F_p (see refs.) and derive the POSet structure of the prime numbers [17]. So, finite fields F_{p^k} are tightly related to p-adic numbers, the later being deformations of the former [39, 38]. Lattice models of finite fields also alows to see the role of $\sqrt{p}$ and its relation with Weil Conjectures.

The higher powers of primes are categorified as truncations of p-adic groups or as sizes of finite fields. These two types of objects are related by Frobenius map [39]. The relation between p-adic numbers to finite fields F_{p^k} is very rich: Deformation Theory (p-adic numbers) meets Galois Theory (algebraic extensions theory). Hence prime powers are but shadows of a very rich theory ... What about R-zeros!? One should look first at Weil zeros (and poles of the corresponding zeta function).

2.3. **Lessons from Weil Conjectures.** Here *Weil zeros* refers to zeros (and poles) of the generating functions of number of points of an algebraic equation over finite fields F_{p^n} in characteristic p (corresponding to prime powers), as part of Weil Conjectures (see Modern Introduction to Number Theory; [23, 24]). It is a rational function having a cohomological interpretation [25].

This zeta function is related to a path integral (loc. cit. §2.4), hence has the flavor of both counting points (sizes) and of being related to periods. The associated RH is more amenable to interpretation, meaning that we expect the real part to be 1/2 since we expect $\sqrt{p}$ to enter into the picture (finite fields "complex plane" allowing for finite rotations; see Gauss periods and cyclotomic numbers).

Since finite fields correspond to p-adic numbers via Frobenius, what would be the theory of algebraic varieties and corresponding Weil zeros over p-adic numbers, hence passing from characteristic p and algebraic extensions to characteristic 0 and Deformation Theory.

It is natural to look for a meaning for the R-zeros, in char zero. The two cases are of course related within the theory of adeles. One deforms Z to the "right" (reals[1]) or "left" (p-adic numbers[2]). How are these two cases related?

3. Now what!?

The relation between field extensions of F_p in characteristic p $(Aut(Z/p, +))$ and deformations of the same object as a tangent space in the sense of Deformation

[1] Decimal representation with carryover 2-cocycle going against the topological 10-adic grading.
[2] 2-cocycle carryover compatible with the topological grading.

Theory, yielding the p-adic numbers form a 2-dim categorical diagram:

$$
\begin{array}{ccccccc}
 & & \dots & & & & \\
 & & \uparrow & & & & \\
 & & F_{p^3} & \longrightarrow & Z/p^2[\alpha_2] & & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & F_{p^2} & \longrightarrow & Z/p^2[\alpha_1] & & \dots \\
 & & \uparrow & & \uparrow & & \\
 F_p = Z/p & & & \longrightarrow & Z/p^2 & \longrightarrow & Z/p^3 \quad \dots
\end{array}
$$

The relation to Witt vectors and the corresponding convolution algebra is essential to be understood.

The "p-adic version" of Weil Conjectures should be the missing link between RH in characteristic p and RH for reals (complex numbers), both in characteristic zero as part of Theory of Adeles! The Frobenius map should provide the required tool relating Weil zeros "algebraically" (over finite fields) to Weil zeros over p-adic numbers (p-adic Complex Analysis: zeta function, RH, path integrals etc.).

Other pieces of the puzzle in the algebraic realm are: the group ring of Z/p, cyclotomic units and Gauss periods; quadratic reciprocity exhibits $\sqrt{p}$ as involved for possibly common reasons. How do these translate in the analytic realm, of p-adic Complex Analysis!?

4. Conclusions and Further Developments

The Polya-Hilbert suggestion regarding the interpretation of R-zeros is too vague / generic to be of direct use; a more structured program was presented above, based on the duality between the prime numbers powers and R-zeros, to be categorified to a TK-duality. The comparison with Weil zeros and their role in Weil Conjectures is essential; perhaps one should look there for a categorification that is general enough to be transferred from local-to-global cases. Since the overall impression is that this is a "quadratic case" (quadratic reciprocity and RH) one may hope the transfer is possible, the abstract setting controlling both cases. The adeles containing these both cases play the role of a multiplicative duality at the number level (the product 1 of components of an adele behaves like the augmentation morphism).

But the main point remaining to be studied is the analog of Weil Conjectures over p-adic numbers! The algebraic realm linked by Frobenius map to analytic realm provides the bridge to RH char 0: for p-adic numbers and for real numbers. The two cases should be unified within the Theory of Adeles.

A cohomological interpretation is clear for Weil zeros of zeta function of algebraic varieties: l-adic cohomology; what is the zero char homological interpretation (think of Galois extensions and groups as algebraic fundamental group)!? One should first examine the p-adic numbers version of l-adic cohomology interpretation, namely the Monski-Washnitzer cohomology where the theory of periods is well studied (see [40]): Grothendieck-de Rham period isomorphisms for p-adic algebraic varieties. What are the p-adic Weil zeros here, as zeros of the generating function of sizes of p-adic algebraic varieties? Comparing with the Riemann surfaces case of periods of de Rham isomorphism, what is the period matrix (Jacobian etc.) now over p-adic numbers and is it related to the R-zeros over p-adic numbers? (path integrals of basic harmonic forms over a homological basis). Do these periods generalize the number of points game? (see the Elliptic Curve case). Will their generating function generalize the Weil zeta function? One should look at the Weil cohomological picture and Lipschitz theory for p-adic case (see Bernard Dwork proof using p-adic methods [25] etc.). The link to Betti numbers (de Rham iso) in etale cohomology can be then used to deal with the p-adic case of Weil Conjectures via Frobenius.

So, are R-zeros adeles dual to the whole family of p-adic Weil zeros (how are the period matrices related?)? Are p-adic Weil zeros (over p-adic numbers) Frobenius related / "deformations" of Weil zeros (over finite fields)?

$char/duality$	0 $(Global)$	p $(Local)$	Both: Adeles
$Categorification:$	Ab_f vs. H-Mod (TK)	l-adic cohomology (Poincare)	?
$Numbers$	$Primes\ vs.\ R-zeros$	$@p\ Weil\ zeros$	?

Note that "finite characteristic" is misleading: primary finite fields are just infinitesimal tangent spaces for p-adic numbers as deformations. Hence *"everything" happens in characteristic zero*: p-adic numbers as algebraic deformations Z_p have real numbers "Z_0" as a projective completion in some sense, captured by the adeles.

So the RH has several frameworks, related by Adeles, Theory of Algebraic Extensions (Galois Theory / algebraic fundamental Group) and Deformation Theory (p-adic Analysis) as part of corresponding analogues of Weil Conjectures (WC).

$$\begin{array}{cccc}
 & Analytic \xleftrightarrow{Adeles} & Analytic & \xleftarrow{Frobenius} Algebraic \\
 & & \searrow^{Dwork} & \\
RH & Complex\ Numbers & p-adic\ RH & \xleftarrow{Hasse}\ RH@p \\
WC & Riemann\ Surfaces & p-adic\ WC & WC@p.
\end{array}$$

Note that a form of Hasse Principle together with the Frobenius map, relate the 3rd and 2nd column of the above diagram: p-adic solutions of equations can be obtained by extending initial solutions over F_{p^k}. The full version, also called Local-to-Global Principle, involves now the 1st column (solutions for real numbers): integral solutions (common to all p-adic numbers) exist if they have initial conditions at p and real solutions. For quadratic extensions this is a correspondence. So this principle should help relate these various RH.

Perhaps de Rham-Grothendieck period isomorphism (duality) "is" a categorification of (related to / generalizes) the prime powers - Riemann zeros Fourier duality ... Recall that Galois Theory (the Algebraic picture for field extensions) is an instance of the algebraic fundamental group, part of the de Rham Period isomorphism.

Since real numbers are in some sense 0-adic numbers[3] ("projective closure" of the rational part of the adeles) the way to prove RH is via p-adic numbers and adeles, at the "categorical" level of fundamental group / cohomology level: Grothendieck-de Rham Period Isomorphism. Is, perhaps, Hurewicz map / Abelianization, of interest here, relating algebraic varieties over algebraic extensions realm (Galois Theory) and analytic realm (Deformation Theory)?

Recall again that the 2-dim lattice of coefficients of algebraic varieties) has an entry the k-th extension of p-adic numbers as a deformation of F_{p^k}. Hence Galois Theory (fundamental Group) meets Deformation Theory (p-adic) in an essential way. How does the RH behave under this two theories related by Frobenius map?

Then the adeles join p-adic numbers and reals in a duality that is reminiscent of a "point at infinity" completion. Again, how then, within the adeles, does RH for $C = R[i]$ (R as a 2-cocycle deformation of polynomials, part of a group ring, against the l-adic completion) relates to $Z_p[i]$ the one for a quadratic extension of p-adic numbers, via WC at p?

Finally, the reals or p-adic numbers are too many (e.g. Liouville number etc.); can we restrict the framework to periods (algebraic integrals / de Rham Period Iso)?? Even better, can we obtain the adeles, targeting the reals, another way, as a projective completion of the p-adic sector of the adeles? (p-adic numbers are genuine deformations) and then restrict to periods only (including algebraic extensions)? Getting rid of pesky real numbers is an umbrella goal of modern Math and Physics (Quantum Everything!).

The bibliography contains many more references then cited, as stepping stones that helped the author reach the above considerations, and are included as a "menu" for the reader's choice. Additional references can be found in author's "old" drafts on RZ and Weil Conjectures available at https://vixra.org/author/lucian_m_ionescu

[3] $1/p \to 0$ or $p \to \infty$-numbers ...

We apologise for the scattered ideas presented, but they were encountered while studying the subject by the author. Hopefully this draft will be expanded if time will allow it ...

Appendix

4.1. **Real vs. p-adic numbers.** p-adic numbers are a deformation of F_p with a 2-cocycle corresponding to the carryover unit for addition: $c : Z/p \times Z/p \to Z/p$. The trivial deformation $(Z/p[x], +)$ turns into $Z_p = (F_p[h], \star)$. This carryover is compatible with the grading that gives the h-adic topology ("down" to the right).

The reals under the decimal numbers representation, or better p-base representation have the carryover unit (2-cocycle) going "up" to the left, w.r.t. to grading; l-adic topology goes "down", in the *opposite direction of the h-adic / topological grading.*

Hence each p-adic numbers can be paired with a copy of the reals represented in base p. All these copies of the reals are isomorphic. Compare this with just one copy of R in the adeles.

The incompatibility of addition with the grading / h-adic topology yields continuum's lack of structure, otherwise present in the p-adic numbers. Yet, there are many formal similarities that can be used to compare the two theories (Analysis).

4.2. **Primes vs. irreducible polynomials.** Cohn's Th. [41] gives an interesting relation between primes p and irreducible polynomials obtained by representing the prime in a basis (deformation) and then reinterpreting as a polynomial (trivial deformation).

Weil zeros (and poles) of the zeta function of a variety, are subject to the Galois group. The Weil zeta function $\zeta(T)$ has a homological interpretation and in each degree it is the determinant of $I - TF$ (F the Frobenius automorphism) on the l-adic cohomology group H^i [25]. The roots / poles are determined modulo Galois group; so instead one may want to study the corresponding groups.

The transition from WC in char p and its p-adic analog is done via Hasse Principle (extending initial solutions at F_p to a p-adic solution). So one may wish to study the Galois Theory aspects of the Weil p-adic zeta function in trying to understand the Riemann zeta function via adeles.

Recall that Weil zeros are related by Viete relations (symmetric polynomials) to its zeta function coefficients (number of solutions of an algebraic equation over finite fields). Its rationality stems from the cohomological interpretation, hence it captures the cohomology of the algebraic variety.

4.3. **Transfer p-adic proof of RH to reals.** A final thought regarding RH in relation with the p-representation of the reals R_p and p-adic numbers is to view $Z[i] \to R_p[i] \times Z_p[i] \to Ad@p$ as a completion, in analogy with the Riemann sphere for example, and consider the inversion $1/z$ playing the role of a conformal transformation. Alternatively just use two copies of $Z_p[i]$; the inversion $w = 1/z : Z_p[i]^* \to$

$R_p[i] = C_p * : 1/w$ (on the image of the first inversion) provides an analog of the two charts of a projective space as a "bifield", with certain similarities to the case of the Riemann sphere. Then see how WC (the projective case) over p-adic numbers relates to this object.

References

[1] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, GTM Series #84, Springer New York, 2010.

[2] O. Shanker, "Entropy of Riemann zeta zero sequence", AMO - Advanced Modeling and Optimization, Volume 15, Number 2, 2013, https://camo.ici.ro/journal/vol15/v15b18.pdf

[3] K. Ford and A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function, J. reine angew. Math. **579** (2005), 145-158. www.researchgate.net, 2005.

[4] K. Ford, K. Soundararajan, A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function, II, 0805.2745, 2009.

[5] L. M. Ionescu, "On prime numbers and Riemann zeros", 2017, https://arxiv.org/abs/2204.00899.

[6] L. M. Ionescu, "A statistics study of Riemann zeros", 2014, to appear.

[7] B. Mazur, W. Stein, *Primes: What is Riemann's Hypothesis?*, Cambridge University Press, 2016; http://modular.math.washington.edu/rh/rh.pdf; new title: "Prime Numbers and the Riemann Hypothesis"

[8] P. Garrett, "Riemann's explicit/exact formula", 2015, http://www-users.math.umn.edu/g̃arrett/m/mfms/notes_2015-16/03_Riemann_and_zeta.pdf

[9] V. Munoz and R. P. Marco, Unified treatment of explicit and trace formulas via Poisson-Newton formula, https://arxiv.org/abs/1309.1449

[10] A. Connes, "An essay on the Riemann Hypothesis", http://www.alainconnes.org/docs/rhfinal.pdf

[11] I. Volovich, Number Theory as the Ultimate Physics Theory.

[12] L.M. Ionescu, "Remarks on physics as number theory", *Proceedings of the 19th National Philosophy Alliance* Vol. 9, pp. 232-244, http://www.gsjournal.net/old/files/4606_Ionescu2.pdf.

[13] Wikipedia, Riemann-Roch duality.

[14] Wikipedia, Jacobian Variety.

[15] H. Rademacher, *Fourier analysis in number theory*, Collected Works, pp.434-458.

[16] M. Kontsevich, D. Zagier, Periods, IHES 2001, http://www.maths.ed.ac.uk/ãar/papers/kontzagi.pdf

[17] L. M. Ionescu, A natural partial order on the prime numbers, Notes on Number Theory and Discrete Mathematics, Volume 21, 2015, Number 1, Pages 1?9; arxiv.org/abs/1407.6659, 2014.

[18] V. R. Pratt, Every prime has a succinct certificate, SIAM J. Comput. Vol.4, No.3, Sept. 1975, 214-220.

[19] L. M. Ionescu, On Prime Numbers and Riemann zeros, https://vixra.org/abs/2204.0105

[20] Wen Wang, Notes on character sums, http://wstein.org/edu/2010/414/projects/wen_wang.pdf

[21] L. M. Ionescu, Topics in Number Theory MAT 410, Fall 2015, Fall 2016, Google presentation.

[22] B. Ossermann, Weil conjectures, https://www.math.ucdavis.edu/ osserman/math/pcm.pdf

[23] L. M. Ionescu, On Weil Conjectures, https://vixra.org/abs/2204.0124

[24] L. M. Ionescu, On Weil zeros, https://vixra.org/abs/2204.0125

[25] Wikipedia, Weil Conjectures, https://en.wikipedia.org/wiki/Weil_conjectures

[26] L. M. Ionescu, "A natural partial order on the prime numbers", *Notes on Number Theory and Discrete Mathematics*, Volume 21, 2015, Number 1, Pages 1?9; arxiv.org/abs/1407.6659, 2014.

[27] S. S. Kudla, "Tate's thesis", *An Introduction to the Langlands Program*, pp 109-131, Editors: Joseph Bernstein, Stephen Gelbart, Springer, 2004.

[28] K. Conrad, "The character group of Q", http://www.math.uconn.edu/˜kconrad/blurbs/gradnumthy/characterQ.pdf
[29] I. M. Gel'fand M. I. Graev, I. I. Pyateskii-Shapiro, *Representation theory and automorphic forms*, Academic Press, 1990.
[30] F. Gouvea, *p-adic Numbers: An Introduction*, Springer-Verlag, 1993.
[31] L. M. Ionescu, Recent presentations, http://my.ilstu.edu/ lmiones/presentations_drafts.htm
[32] R. Meyer, "A spectral interpretation of the zeros of the Riemann zeta function", math/0412277
[33] J. F. Burnol, "Spectral analysis of the local conductor operator", math/9809119.pdf, 1998.
[34] John Baez, "Quasicrystals and the Riemann Hypothesis", golem.ph.utexas.edu/category/2013/06/quasicrystals_and_the_riemann.html
[35] Freeman Dyson, "Frogs and Birds", AMS 56 (2009), 212-223.
[36] A. M. Odlyzko, "Primes, quantum chaos and computers", *Proc. Symp.*, May 1989, Washington DC, pp.35-46.
[37] Wikipedia: "Riemann Zeta Function".
[38] L. M. Ionescu and Mina Zarrici, Lattice models of finite fields, https://arxiv.org/abs/1708.09302
[39] L. M. Ionescu, On p-adic Frobenius lifts and p-adic periods, from a Deformation Theory viewpoint, https://arxiv.org/abs/1801.07570
[40] L. M. Ionescu, On Periods: from Local to Global, https://arxiv.org/abs/1806.08726
[41] M. R. Murty, Prime numbers and irreducible polynomials, https://mast.queensu.ca/ murty/monthly.pdf
[42] Y. Tian, Weil conjecture I.
[43] M. Mustaza, Lecture 1: The Hasse-Weil Zeta Functions.
[44] Colin Hayman, The Weil conjectures, Master Thesis 2008.
[45] Eyal Z. Goren, Gauss and Jacobi sums, Weil conjectures, http://www.math.mcgill.ca/goren/SeminarOnCohomology/mycohomologytalk.pdf
[46] F. Thaine, On Gaussian periods that are rational integers.
[47] Wikipedia: 1) Gaussian period; 2) Gauss sum; 3) Jacobian variety.
[48] Paramand Singh, Gauss and regular polygons: Gaussian periods.
[49] W. Duke, S.R. Garcia, B. Lutz, The graphic nature of Gaussian periods.

Department of Mathematics, Illinois State University, IL 61790-4520
Email address: lmiones@ilstu.edu