Elementary Integral

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Abstract

In this note we study a trigonometric integral

Introduction

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \tag{1}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\ ,\ 0 \le k \le n \tag{2}$$

where

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \tag{3}$$

The Gauss Hypergeometric function is defined by

$$F(a,\, b,\, c,\, x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!}\, x^n\ ,\ |x| < 1 \tag{4}$$

where $(a)_n = a\,(a+1)\,(a+2)\ldots(a+n-1)$; $(a)_0 = 1$, See Olver et al., [3]. In this note we study a trigonometric integral.

Integral

$$I = \frac{\pi}{2}\cos\left(\frac{1}{\sqrt{2}}\right) + \sqrt{2}\int_0^1 \sin\left(\frac{x}{\sqrt{2}}\right)\sin^{-1}\left(\sqrt{\frac{2x}{x+\sqrt{8+x^2}}}\right)dx \tag{5}$$

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{-n}}{(4\, n+1)\, (2\, n)!} F\left(1,\, n+1,\, 2\, n+\frac{3}{2},\, \frac{1}{2}\right) \tag{6}$$

$$I = \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{-2\, n}}{(4\, n+1)\, (2\, n)!} F\left(n+\frac{1}{2},\, 2\, n+\frac{1}{2},\, 2\, n+\frac{3}{2},\, \frac{1}{2}\right) \tag{7}$$

$$I = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k\, 2^{n-2\, k}}{(4\, k+1)\, (2\, k)!} \binom{n}{k} \binom{4\, k+2}{2\, k+1} \binom{2\, n+2\, k+2}{n+k+1}^{-1} \binom{n+k+1}{2\, k+1}^{-1} \tag{8}$$

$$I = \frac{\pi}{2} \cos\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8} \sum_{n=0}^{\infty} 2^{-3\, n} \sum_{k=0}^{[n/2]} \frac{(-1)^k\, 2^{4\, k}}{(2\, k+1)!\, (2\, n-4\, k+1)} \binom{2\, n-4\, k}{n-2\, k} G(n,\, k) \tag{9}$$

where

$$G(n,\, k) = \frac{4}{2\, n+5} F\left(k+2,\, n+\frac{5}{2},\, n+\frac{7}{2},\, \frac{1}{2}\right) - \frac{1}{2\, n+7} F\left(k+2,\, n+\frac{7}{2},\, n+\frac{9}{2},\, \frac{1}{2}\right) \tag{10}$$

$$I = \frac{\pi}{2} \cos\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8} \sum_{n=0}^{\infty} \frac{2^{-3\, n}\, (6\, n+23)}{(2\, n+5)\, (2\, n+7)} \sum_{k=0}^{[n/2]} \frac{(-1)^k\, 2^{4\, k}}{(2\, k+1)!} \sum_{m=0}^{n-2\, k} \frac{2^{2\, m}}{2\, n-4\, k-2\, m+1} \binom{2\, n-4\, k-2\, m}{n-2\, k-m} \binom{k+m+1}{m} \tag{11}$$

References

[1] Boros, G., and Moll, V., Irresistible Integrals, Cambridge University Press, 2004.

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[3] Olver, F.W.J., et al., NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

[4] Spanier, J., and Oldham, K.B., An Atlas of Functions. Hemisphere Publishing, 1987.

[5] Ramanujan, S., Notebooks. Vols. 1, 2, Tata Institute of Fundamental Research, Bombay, 1957.