Hom-Sets Category

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Abstract Let $\mathcal{C}$ be a category. Suppose that the hom-sets of $\mathcal{C}$ is small. Let $\mathcal{C}_{\mathcal{H}}$ be a category consist of the hom-sets of $\mathcal{C}$. Then we define a morphism of $\mathcal{C}_{\mathcal{H}}$ by a morphisms pair $\langle \nu, \mu \rangle$. Hence the morphism is monic if and only if ν is epi and μ is monic. An object $Hom_{\mathcal{C}}(P,E) \in \mathcal{C}_{\mathcal{H}}$ is an injective object if and only if P is a projective object and E is an injective object. There exists a bifunctor $T : (\mathcal{C} \downarrow A)^{op} \times (B \downarrow \mathcal{C}) \to (Hom(A,B) \downarrow \mathcal{C}_{\mathcal{H}})$. And the bifunctor T is bijective. There exist the products in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the products and coproducts in $\mathcal{C}$. There exist the pullback in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the pushout and pullback in $\mathcal{C}$.

1. Introduction

In this paper, $\mathcal{C}$ is a category. Then we define a category $\mathcal{C}_{\mathcal{H}}$:

Objects: Hom-sets of $\mathcal{C}$. If $A, B \in \mathcal{C}$, then $Hom_{\mathcal{C}}(A,B)$ is an object of $\mathcal{C}_{\mathcal{H}}$.

Morphisms: Pairs of morphisms of $\mathcal{C}$. Let $\nu : C \to A$, $\mu : B \to D$. Then $\langle \nu, \mu \rangle$ is a morphism $Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{C}}(C,D)$ given by $f \mapsto \mu \circ f \circ \nu$ for all $f \in Hom_{\mathcal{C}}(A,B)$.

The hom-set $Hom_{\mathcal{C}_{\mathcal{H}}}(Hom_{\mathcal{C}}(A,B), Hom_{\mathcal{C}}(C,D))$ is a quotient set. And a hom-set of two objects in $\mathcal{C}_{\mathcal{H}}$ is determined by other objects of $\mathcal{C}_{\mathcal{H}}$. To avoid trouble, we suppose that $\mu \circ f \circ \nu = \mu' \circ f \circ \nu'$ for all $f \in Hom(A,B)$ if and only if $\nu = \nu'$ and $\mu = \mu'$. In subsection 3.1, we discuss morphisms of $\mathcal{C}_{\mathcal{H}}$ in more detail.

The category $\mathcal{C}_{\mathcal{H}}$ is a subcategory of **Sets**[1], *not* full. Hence for an object $A \in \mathcal{C}$, $Hom(A, -)$ is a functor from $\mathcal{C}$ to $\mathcal{C}_{\mathcal{H}}$.

Proposition (proposition 3.3)**.** *The functor $Hom(A,-)$ preserves*[1] *all monic*[1] *morphisms and limits*[1] *in $\mathcal{C}_{\mathcal{H}}$.*

It is determined by morphisms of $\mathcal{C}$ that a morphism in $\mathcal{C}_{\mathcal{H}}$ is monic(epi)[1].

Proposition (propositions 3.1 and 3.2)**.** *The morphism $\langle \nu, \mu \rangle$ in $\mathcal{C}_{\mathcal{H}}$ is monic if and only if μ is monic and ν is epi.*

Hence every monic of $\mathcal{C}_{\mathcal{H}}$ consists of an epi and a monic of $\mathcal{C}$. And an object of $\mathcal{C}_{\mathcal{H}}$ is projective(injective)[1] is determined by the objects of $\mathcal{C}$.

Proposition (propositions 3.4 and 3.5)**.** *An object $Hom(P,E) \in \mathcal{C}_{\mathcal{H}}$ is a projective object if and only if $P \in \mathcal{C}$ is a projective object and $E \in \mathcal{C}$ is an injective object.*

For an object $Hom(A,B) \in \mathcal{C}_{\mathcal{H}}$, if $\langle \langle \nu, \mu \rangle, Hom(C,D) \rangle$ is an object of comma category[1] $(Hom(A,B) \downarrow \mathcal{C}_{\mathcal{H}})$, then $\langle \nu, C \rangle \in (\mathcal{C} \downarrow A)$ and $\langle u, D \rangle \in (B \downarrow \mathcal{C})$. See subsection 3.5 for detail.

The situation of (co)products[1] and pullback(pushout)[1] are similar.

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Proposition (propositions 3.7, 3.8, 3.10 and 3.11)**.** *There exist the (co)products in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the products and coproducts in $\mathcal{C}$.*

Suppose that $\mathcal{J}$ be a category. Let F be a functor from $\mathcal{J}$ to $\mathcal{C}_{\mathcal{H}}$. We have that

Proposition (propositions 3.14 and 3.15)**.** *There exists the (co)limit of F if and only if there exist the (co)limit of $T \circ F$ and the (co)limit of $S \circ F$. The funtors T and S are defined in subsection 3.8.*

Contents

2. Preliminaries

2.1. Monic and Epi.

Definition 2.1 (Monic[1])**.** A morphism μ is monic when it is left cancellable, $\mu \circ f = \mu \circ f'$ implies $f = f'$.

Definition 2.2 (Epi[1])**.** A morphism ν is epi when it is right cancellable, $f \circ \nu = f' \circ \nu$ implies $f = f'$.

2.2. Limit and Colimit.

Definition 2.3 (natural transformation[1])**.** Let $\mathcal{D}$ be a category, T and S be two functors from $\mathcal{D}$ to $\mathcal{C}$. Then a natural transformation $\tau\colon T \xrightarrow{\bullet} S$ is a function which send every object $D \in \mathcal{D}$ to a morphism $\tau_D\colon T(D) \to S(D)$ of $\mathcal{C}$ in such a way that every morphism $f\colon D \to D'$ of $\mathcal{D}$ makes the diagram(2.3) commute.

$$
\begin{array}{ccc}
T(D) & \xrightarrow{\tau_D} & S(D) \\
{\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle S(f)} \\
T(D') & \xrightarrow[\tau D']{} & S(D')
\end{array}
\tag{2.1}
$$

Let $\mathcal{J}$ be a category, F a functor from $\mathcal{J}$ to $\mathcal{C}$. Suppose that Δ is a diagnal functor[1]: $\mathcal{C} \to \mathcal{C}^{\mathcal{J}}$.

Definition 2.4 (Limit[1])**.** An object $\varprojlim F$ of $\mathcal{C}_{\mathcal{H}}$ is the limit of F provided that for all $X \in \mathcal{C}$ with a natural transformation $\tau \colon \Delta(X) \xrightarrow{\bullet} F$, there exists unique natural transformation $\pi \colon \Delta(X) \xrightarrow{\bullet} \Delta(\varprojlim F)$ such that the diagram(2.2) is commutative. The natural transformation $\omega \colon \Delta(\varprojlim F) \xrightarrow{\bullet} F$ is called limit cone[1].

$$
\begin{array}{ccc}
\Delta(\varprojlim F) & \dashleftarrow & \Delta(X) \\
& \searrow \quad \swarrow & \\
& F &
\end{array}
\tag{2.2}
$$

Definition 2.5 (Colimit[1])**.** An object $\varinjlim F$ of $\mathcal{C}_{\mathcal{H}}$ is the colimit of F provided that for all $X \in \mathcal{C}$ with a natrual transformation $\tau \colon F \to \Delta(X)$, there exists unique natural transformation $\pi \colon \Delta(\varinjlim F) \xrightarrow{\bullet} \Delta(X)$ such that the diagram(2.3) is commutative. The natural transformation $\omega \colon F \xrightarrow{\bullet} \Delta(\varinjlim F)$ is called colimit cone[1].

$$
\begin{array}{ccc}
\Delta(\varinjlim F) & \overset{\pi}{\dashrightarrow} & \Delta(X) \\
& \nwarrow \quad \nearrow & \\
& F &
\end{array}
\tag{2.3}
$$

2.3. **Functor** $Hom(A, -)$**.** If $A \in \mathcal{C}$, then $Hom_{\mathcal{C}}(A, -)$ is a functor from $\mathcal{C}$ to **Sets**[1].

Theorem 2.1 (Preserves monic[1])**.** *The functor $Hom_{\mathcal{C}}(A, -)$ preserves monic for all $A \in \mathcal{C}$.*

Proof. Let $B, C \in \mathcal{C}$, f a monic morphism from B to C. Then f induces a morphism $f^* : Hom_{\mathcal{C}}(A, B) \to Hom(A, C)$ given by $f^* : u \mapsto f \circ u$. Hence for all $u, v \in Hom_{\mathcal{C}}(A, B)$, $f \circ u = f \circ v$ implies $u = v$. Therefore, f^* is monic. □

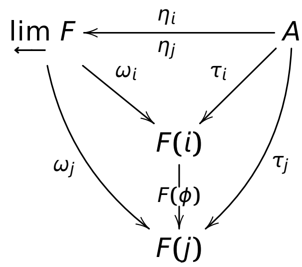
Theorem 2.2 (Preserves limits[1])**.** *The functor $Hom_{\mathcal{C}}(A, -)$ preserves the limits for all $A \in \mathcal{C}$.*

Proof. Let $\mathcal{J}$ be a category, F a functor from $\mathcal{J}$ to $\mathcal{C}$. Then for every $A \in \mathcal{C}$ we have a functor $Hom(A, F-) \colon \mathcal{J} \to$ **Sets** what is composition of F and $Hom(A, -)$.

$$
\mathcal{J} \xrightarrow{F} \mathcal{C} \xrightarrow{Hom(A,-)} \textbf{Sets}
$$

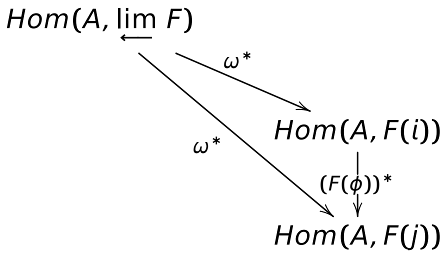
Suppose that the limit of F exists in $\mathcal{C}$ with limit cone $\omega \colon \Delta(\varprojlim F) \xrightarrow{\bullet} F$ where the functor Δ is a diagonal functor. Hence for all $A \in \mathcal{C}$ with a natural transformation $\tau \colon \Delta(A) \xrightarrow{\bullet} F$, there exists unique natural transformation $\eta \colon \Delta(A) \xrightarrow{\bullet} \Delta(\varprojlim F)$ factors through τ. Hence for all $i, j \in \mathcal{J}$ and every morphism $\phi \colon i \to j$, the diagram(2.4) is commutative in $\mathcal{C}$.

(2.4)



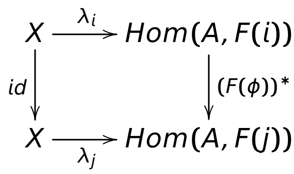
Hence the diagram(2.5) is commutative in **Sets**.

(2.5)



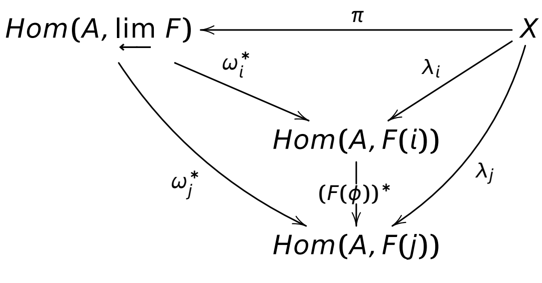
Suppose that X is a set and that $\lambda\colon \Delta(X) \xrightarrow{\bullet} Hom(A, F-)$ is a natural transformation. Then the diagram(2.6) is commutative for all $i, j \in \mathcal{J}$ and every $\phi\colon i \to j$.

(2.6)



Hence for every $x \in X$, we have $F(\phi) \circ \tau_i = \tau_j$ where $\tau_i := \lambda_i(x)$, $\tau_j := \lambda_j(x)$. Since the diagram(2.4) is commutative, there exists unique $\eta \in Hom(A, \varprojlim F)$ such that $\omega_i \circ \eta = \tau_i$, $\omega_j \circ \eta = \tau_j$ and $F(\phi) \circ \omega_i \circ \eta = \omega_j \circ \eta = \tau_j$. Hence we may define a morphism $\pi\colon X \to Hom(A, \varprojlim F)$ given by $\pi\colon x \mapsto \eta$ for every $i \in \mathcal{J}$. The morphism π makes the diagram(2.7) commutative. It is obvious that π is unique such that the diagram(2.7) is commutative.

(2.7)



Hence for all $X \in$ **Sets** with a natural transformation $\lambda\colon \Delta(X) \xrightarrow{\bullet} Hom(A, F-)$, there exists unique natural transformation $\overset{\bullet}{\pi}\colon \Delta(X) \xrightarrow{\bullet} \Delta(Hom(A, \varprojlim F))$ given by

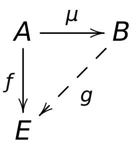
$(\pi\colon X \to Hom(A, \varprojlim F))_{i\in\mathcal{J}}$ such that $\dot{\pi}$ factors through λ. Therefore,

$$Hom(A, \varprojlim F) \cong \varprojlim Hom(A, F-) \qquad \square$$

2.4. **Projective and Injective.**

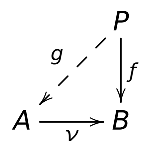
Definition 2.6 (Injective objective[1])**.** If $E \in \mathcal{C}$ is an injective object, then for every morphism $f\colon A \to E$ and every monic $\mu\colon A \to B$ there exists $g\colon B \to E$ such that the diagram(2.8) is commutative.

(2.8)



Definition 2.7 (Projective object[1])**.** If $P \in \mathcal{C}$ is a projective object, then for every morphism $f\colon P \to A$ and every epi $\nu\colon A \to B$ there exists $g\colon P \to B$ such that the diagram(2.9) is commutative.

(2.9)



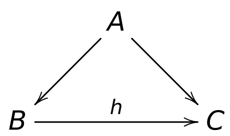
2.5. **Comma category.**

Definition 2.8 (Comma category[1])**.** Let $A, B, C \in \mathcal{C}$. Then $(A \downarrow \mathcal{C})$ is a comma category if

Object: $\langle f, B\rangle$, where $f\colon A \to B$.

Morphism: $h\colon \langle f, B\rangle \to \langle g, C\rangle$ makes the diagram(2.10) commute in $\mathcal{C}$.

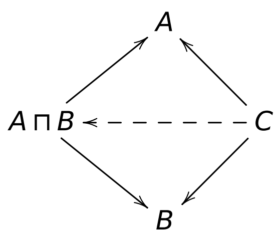
(2.10)



2.6. **Product and Coproduct.**

Definition 2.9 (Product[1])**.** Let $A, B \in \mathcal{C}$. Then for all $C \in \mathcal{C}$ and for every morphism $C \to A$, $C \to B$ there exists an object $A \sqcap B \in \mathcal{C}$ and unique morphism $C \to A \sqcap B$ such that the diagram (2.11) is commutative. We call $A \sqcap B$ product.

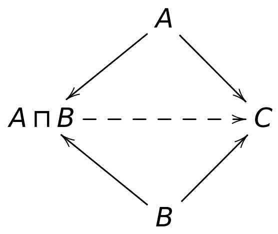
(2.11)



Definition 2.10 (Coproduct[1])**.** Let $A, B \in \mathcal{C}$. Then for all $C \in \mathcal{C}$ and for every morphism $A \to C$, $B \to C$ there exists an object $A \sqcup B \in \mathcal{C}$ and unique morphism

$A \sqcap B \to C$ such that the diagram (2.12) is commutative. We call $A \sqcup B$ coproduct.

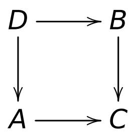
(2.12)



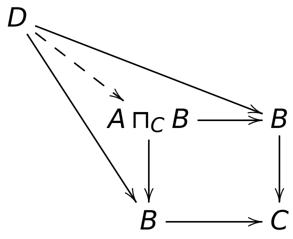
2.7. **Pullback and Pushout.**

Definition 2.11 (Pullback[1]). Let $A \to C \leftarrow B$ be morphisms in $\mathcal{C}$. For all $D \in \mathcal{C}$, if the diagram(2.13) is commutative, then there exists an object $A \sqcap_C B \in \mathcal{C}$ and unique morphism $D \to A \sqcap_C B$ such that the diagram(2.14) is commutative. That $A \sqcap_C B$ is the pullback.

(2.13)

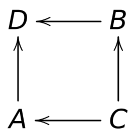


(2.14)

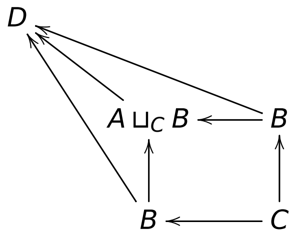


Definition 2.12 (Pushout[1]). Let $A \leftarrow C \to B$ be morphisms in $\mathcal{C}$. For all $D \in \mathcal{C}$, if the diagram(2.15) is commutative, then there exists an object $A \sqcup_C B \in \mathcal{C}$ and unique morphism $A \sqcup_C B \to D$ such that the diagram(2.16) is commutative. That $A \sqcup_C B$ is the pushout.

(2.15)



(2.16)



3. Hom-Set Category

We defined the category $\mathcal{C}_{\mathcal{H}}$. Now, we prove that the definition of $\mathcal{C}_{\mathcal{H}}$ is well-defined.

Proof. There exists identity morphism $id\colon Hom_{\mathcal{C}}(A,B) \to Hom_{\mathcal{C}}(A,B)$ where $id := \langle id_A, id_B \rangle$. Suppose that $Hom(A,B), Hom(C,D), Hom(E,F), Hom(G,H)$ are objects in $\mathcal{C}_{\mathcal{H}}$. Let $\nu_0 : C \to A$, $\mu_0 : B \to D$, $\nu_1 : E \to C$, $\mu_1 : D \to F$, $\nu_2 : G \to E$, $\mu_2 : F \to H$ be morphisms in $\mathcal{C}$. Then we have

$$Hom(A,B) \xrightarrow{\langle \nu_0, \mu_0 \rangle} Hom(C,D) \xrightarrow{\langle \nu_1, \mu_1 \rangle} Hom(E,F) \xrightarrow{\langle \nu_2, \mu_2 \rangle} Hom(G,H)$$

For every $f \in Hom(A,B)$, suppose that the diagram(3.1) is commutative.

(3.1)
$$\begin{array}{ccccccc} A & \xleftarrow{\nu_0} & C & \xleftarrow{\nu_1} & E & \xleftarrow{\nu_2} & G \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow[\mu_0]{} & D & \xrightarrow[\mu_1]{} & F & \xrightarrow[\mu_2]{} & H \end{array}$$

We define the composition

(3.2)
$$\langle \nu_1, \mu_1 \rangle \circ \langle \nu_0, \mu_0 \rangle := \langle \nu_0 \circ \nu_1, \mu_1 \circ \mu_0 \rangle$$

Hence we have $\big(\langle \nu_2, \mu_2 \rangle \circ \langle \nu_1, \mu_1 \rangle\big) \circ \langle \nu_0, \mu_0 \rangle = \langle \nu_2, \mu_2 \rangle \circ \big(\langle \nu_1, \mu_1 \rangle \circ \langle \nu_0, \mu_0 \rangle\big)$ Therefore, $\mathcal{C}_{\mathcal{H}}$ is a category. □

3.1. **Hom-Set of $\mathcal{C}_{\mathcal{H}}$.** Let $Hom_{\mathcal{C}}(A,B), Hom_{\mathcal{C}}(C,D) \in \mathcal{C}_{\mathcal{H}}$. Suppose that $\langle \nu, \mu \rangle$ and that $\langle \nu', \mu' \rangle$ are morphisms of $\mathcal{C}_{\mathcal{H}}$:

$$Hom_{\mathcal{C}}(A,B) \xrightarrow[\langle \nu, \mu \rangle]{\langle \nu', \mu' \rangle} Hom_{\mathcal{C}}(C,D)$$

We define a binary relation: $\langle \nu, \mu \rangle \sim \langle \nu', \mu' \rangle$ when $\mu \circ f \circ \nu = \mu' \circ f \circ \nu'$ for all $f \in Hom_{\mathcal{C}}(A,B)$. It is obvious that '$\sim$' is an equivalence relation. Then $Hom_{\mathcal{C}_{\mathcal{H}}}(Hom_{\mathcal{C}}(A,B), Hom_{\mathcal{C}}(C,D))$ is a quotient set of $Hom(C,A) \times Hom(B,D)$ by $\sim$. These may arise some trouble, hence we suppose that $\mu \circ f \circ \nu = \mu' \circ f \circ \nu'$ for all $f \in Hom(A,B)$ if and only if $\nu = \nu'$ and $\mu = \mu'$ in $\mathcal{C}$. Hence we have the two hypotheses about the categroy $\mathcal{C}$, in this paper:

- The hom-sets is small.
- $\nu \circ f \circ \mu = \nu' \circ f \circ \mu'$ for all $f \in Hom(A,B)$ if and only if $\nu = \nu'$ and $\mu = \mu'$ in $\mathcal{C}$.

3.2. **Monic and Epi in $\mathcal{C}_{\mathcal{H}}$.**

Proposition 3.1. *A morphim $\langle \nu, \mu \rangle : Hom(A,B) \to Hom(C,D)$ is monic if and only if $\nu : C \to A$ is epi and $\mu : B \to D$ is monic.*

Proof. Let $f, f' \in Hom(A,B)$ with $f \neq f'$. Suppose that $\nu : C \to A$ is epi and that $\mu : B \to D$ is monic. Then $f \neq f'$ implies $f \circ \nu \neq f' \circ \nu$ and $\mu \circ f \neq \mu \circ f'$. It follows that $\mu \circ f \circ \nu \neq \mu \circ f' \circ \nu$. On the other hand, Suppose that a morphim $\langle \nu, \mu \rangle : Hom(A,B) \to Hom(C,D)$ is monic. Hence we have that $f \neq f'$ implies $\mu \circ f \circ \nu \neq \mu \circ f' \circ \nu$. Hence $f \circ \nu \neq f' \circ \nu$ and $\mu \circ f \neq \mu \circ f'$. And the morphisms f and f' are not fixed. Therefore, that $f \neq f'$ implies $f \circ \nu \neq f' \circ \nu$ and $\mu \circ f \neq \mu \circ f'$. □

Proposition 3.2. *A morphism $\langle \nu, \mu \rangle : Hom(A,B) \to Hom(C,D)$ is epi if and only if $\nu : C \to A$ is monic, $\mu : B \to D$ is epi.*

Proof. Let $Hom(E,F)$ be a object of $\mathcal{C}_{\mathcal{H}}$, $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ morphisms from $Hom(C,D)$ to $Hom(E,F)$. Then we have

$$Hom(A,B) \xrightarrow{\langle \nu, \mu \rangle} Hom(C,D) \xrightarrow[\langle \alpha', \beta' \rangle]{\langle \alpha, \beta \rangle} Hom(E,F)$$

Suppose that ν is monic and that μ is epi. Then $\nu \circ \alpha = \nu \circ \alpha'$ and $\beta \circ \mu = \beta' \circ \mu$ implies $\alpha = \alpha'$ and $\beta = \beta'$, respectively. Hence $\beta \circ \mu \circ f \circ \nu \circ \alpha = \beta' \circ \mu \circ f \circ \nu \circ \alpha'$ for all $f \in Hom(A,B)$ implies $\alpha = \alpha'$ and $\beta = \beta'$. Hence if $\langle \alpha, \beta \rangle \circ \langle \nu, \mu \rangle = \langle \alpha', \beta' \rangle \circ \langle \nu, \mu \rangle$ then $\langle \alpha, \beta \rangle = \langle \alpha', \beta' \rangle$. Hence $\langle \nu, \mu \rangle$ is epi. On the other hand, Suppose that $\langle \nu, \mu \rangle$ is epi. Then $\langle \alpha, \beta \rangle \circ \langle \nu, \mu \rangle = \langle \alpha', \beta' \rangle \circ \langle \nu, \mu \rangle$ implies $\langle \alpha, \beta \rangle = \langle \alpha', \beta' \rangle$. Hence if $\beta \circ \mu \circ f \circ \nu \circ \alpha = \beta' \circ \mu \circ f \circ \nu \circ \alpha'$ for all $f \in Hom(A,B)$ then $\alpha = \alpha'$ and $\beta = \beta'$. And $\beta \circ \mu \circ f \circ \nu \circ \alpha = \beta' \circ \mu \circ f \circ \nu \circ \alpha'$

for all $f \in Hom(A, B)$ if and only if $\beta \circ \mu = \beta' \circ \mu$ and $\alpha \circ \nu = \alpha' \circ \mu$. Therefore, that ν is monic and μ is epi. □

3.3. **Functor** $Hom(A, -)$ **from** $\mathcal{C}$ **to** $\mathcal{C}_{\mathcal{H}}$**.** Let $C, D \in \mathcal{C}$, μ a monic from C to D. Then the functor $Hom(A, -)$ sends C, D to $Hom(A, C), Hom(A, D)$, respectively. And it sends μ to $\langle id, \mu \rangle$. Hence we have,

Proposition 3.3. *The functor* $Hom(A, -)$ *preserves all monic morphisms and limits in* $\mathcal{C}_{\mathcal{H}}$.

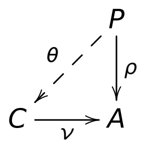
Proof. Immediate from theorems 2.1 and 2.2. □

3.4. **Injective and Projective Objects In** $\mathcal{C}_{\mathcal{H}}$**.**

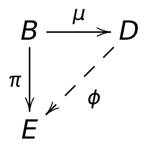
Proposition 3.4. *An object* $Hom(P, E)$ *is an injective object in* $\mathcal{C}_{\mathcal{H}}$ *if and only if* P *is a projective object and* E *is an injective object in* $\mathcal{C}$*,*

Proof. Let $\langle \nu, \mu \rangle : Hom(A, B) \to Hom(C, D)$ be a monic in $\mathcal{C}_{\mathcal{H}}$. By proposition 3.1, we have that $\nu : C \to A$ is epi and $\mu : B \to D$ is monic. Suppose that P is a projective object and that E is an injective object. Then we have P is a projective object if and only if for every morphim $\rho : P \to A$, there exists a morphism $\theta : P \to C$ such that the diagram(3.3) is commutative, and E is an injective object if and only if for every morphism $\pi : B \to E$, there exists a morphism $\phi : C \to E$ such that the diagram(3.4) is commutative.

(3.3)

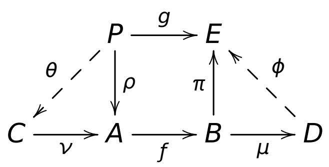


(3.4)



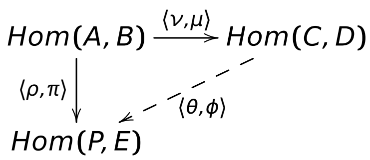
Then for all $f \in Hom(A, B)$ there exists a morphism $g : P \to E$ such that the diagram(3.5) is commutative.

(3.5)



It follows that the diagram(3.6) is commutative.

(3.6)

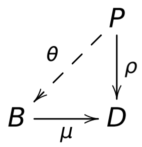


Hence $Hom(P, E)$ is an injective object in $\mathcal{C}_{\mathcal{H}}$. On the other hand, suppose that $Hom(P, E)$ is an injective object in $\mathcal{C}_{\mathcal{H}}$. Then the diagram(3.6) is commutative. Hence there exists a pair $\langle \theta, \phi \rangle$ such that the diagram(3.5) is commutative for every pair $\langle \rho, \pi \rangle$. It implies the diagrams (3.3) and (3.4) are commutative. Hence P is a projective object and E is an injective object. □

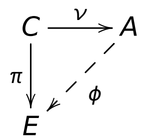
Proposition 3.5. *An object* $Hom(E, P)$ *is projective object in* $\mathcal{C}_{\mathcal{H}}$ *if and only if* E *is an injective object and* P *is a projective object in* $\mathcal{C}$.

Proof. Let morphism $\langle \nu, \mu \rangle : Hom(A,B) \to Hom(C,D)$ be an epi. By proposition 3.2, a morphism $\langle \nu, \mu \rangle$ is epi if and only if $\nu : C \to A$ is monic and $\mu : B \to D$ is epi. Suppose that P is a projective object and that E is an injective object in $\mathcal{C}$. Then for every morphism $\rho : P \to D$ there exists a morphism $\theta : P \to B$ such that the diagram(3.7) is commutative. And for every morphism $\pi : C \to E$ there exists a morphism $\phi : A \to E$ such that the diagram(3.8) is commutative.

(3.7)

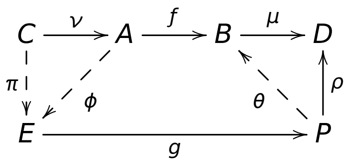


(3.8)



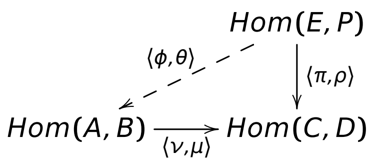
Hence for all $g \in Hom(E,P)$, there exists a morphism $f \in Hom(A,B)$ such that the diagram(3.9) is commutative.

(3.9)



It follows that for every morphism $\langle \pi, \rho \rangle$ there exist a morphism $\langle \theta, \phi \rangle$ such that the diagram(3.10) is commutative.

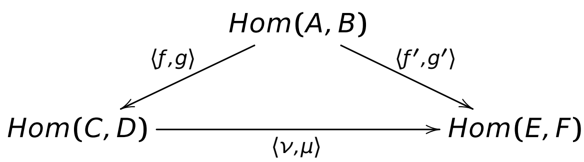
(3.10)



Hence $Hom(E,P)$ is a projective object in $\mathcal{C}_{\mathcal{H}}$. On the other hand, suppose that $Hom(E,P)$ is a projective object. Then for every morphism $\langle \pi, \rho \rangle$ there exists a morphism $\langle \phi, \theta \rangle$ such that the diagram(3.10) is commutative. Hence the diagram(3.9) is commutative. It follows that the diagrams (3.7) and (3.8) are commutative. Hence P is a projective object and E is an injective object. □

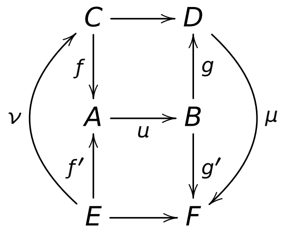
3.5. **Comma Category** $(Hom(A,B) \downarrow \mathcal{C}_{\mathcal{H}})$**.** Suppose that $\langle \langle f,g \rangle, Hom(C,D) \rangle$ and $\langle \langle f',g' \rangle, Hom(E,F) \rangle$ are objects of comma category $(Hom(A,B) \downarrow \mathcal{C}_{\mathcal{H}})$. Let $\langle \nu, \mu \rangle$ be a morphism from $\langle \langle f,g \rangle, Hom(C,D) \rangle$ to $\langle \langle f',g' \rangle, Hom(E,F) \rangle$. Then the morphism $\langle \nu, \mu \rangle$ makes the diagram(3.11) commute.

(3.11)



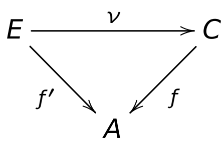
Hence for all $u \in Hom(A,B)$, the diagram(3.12) is commutative.

(3.12)

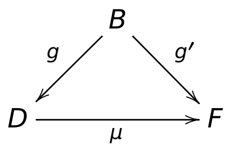


Hence the digrams (3.13) and (3.14) are commutative.

(3.13)



(3.14)



Hence we have that $\nu : \langle f, E\rangle \to \langle f', C\rangle$ in comma category $(\mathcal{C} \downarrow A)$ and that $\mu : \langle g, D\rangle \to \langle g', F\rangle$ in comma category $(B \downarrow \mathcal{C})$.

Proposition 3.6. *There exists a bifunctor $T : (\mathcal{C} \downarrow A)^{op} \times (B \downarrow \mathcal{C}) \to (Hom(A,B) \downarrow \mathcal{C}_{\mathcal{H}})$. And the bifunctor is bijective.*

Proof. We define a bifunctor T given by

$$\begin{aligned} \text{objects:} \quad & (\langle f, C\rangle, \langle g, D\rangle) \mapsto \langle\langle f, g\rangle, Hom(C,D)\rangle \\ \text{morphisms:} \quad & (\nu, \mu) \mapsto \langle \nu, \mu\rangle \end{aligned}$$

It is obvious that $T(id, id) = \langle id, id\rangle$. Suppose that $\nu' : \langle f'', G\rangle \to \langle f', E\rangle$ in $(\mathcal{C} \downarrow A)$ and that $\mu' : \langle g', F\rangle \to \langle g'', H\rangle$ in $(B \downarrow \mathcal{C})$. By Equation 3.2, we have that

$$\begin{aligned} T(\nu \circ \nu', \mu' \circ \mu) &= \langle \nu \circ \nu', \mu' \circ \mu\rangle \\ &= \langle \nu', \mu'\rangle \circ \langle \nu, \mu\rangle \\ &= T(\nu', \mu') \circ T(\nu, \mu) \end{aligned}$$

Hence T is a bifunctor. And it is obvious that T is bijective. □

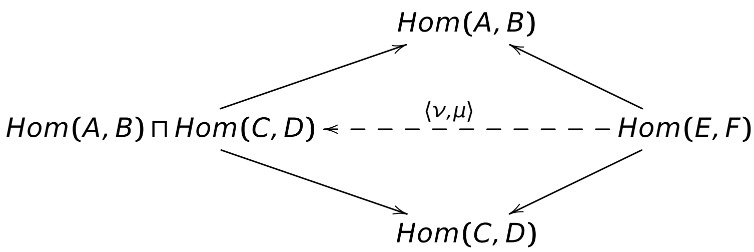
3.6. **Product and Coproduct in $\mathcal{C}_{\mathcal{H}}$.**

Proposition 3.7. *There exist the products in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the products and coproducts in $\mathcal{C}$. And*

$$Hom(A,B) \sqcap Hom(C,D) \cong Hom(A \sqcup C, B \sqcap D) \tag{3.15}$$

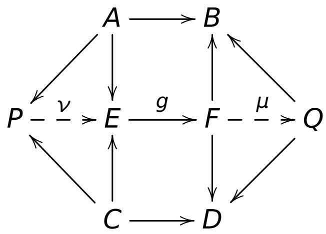
Proof. Suppose that there exist the products in $\mathcal{C}_{\mathcal{H}}$. Let $Hom(A,B), Hom(C,D) \in \mathcal{C}_{\mathcal{H}}$. Then for all $Hom(E,F) \in \mathcal{C}_{\mathcal{H}}$ and for every morphism $\langle v, u\rangle : Hom(E,F) \to Hom(A,B)$, $\langle v', u'\rangle : Hom(E,F) \to Hom(C,D)$ there exists a morphism $\langle \nu, \mu\rangle$ such that the diagram(3.16) is commutative.

(3.16)



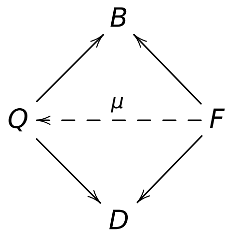
Then there exists $Hom(P,Q) \in \mathcal{C}_{\mathcal{H}}$ such that $Hom(A,B) \sqcap Hom(C,D) \cong Hom(P,Q)$. Hence for all $g \in Hom(E,F)$, the diagram(3.17) is commutative.

(3.17)

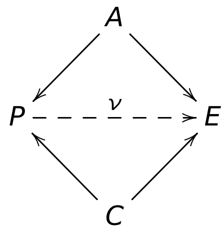


It follows that the diagrams (3.18) and (3.19) are commutative.

(3.18)

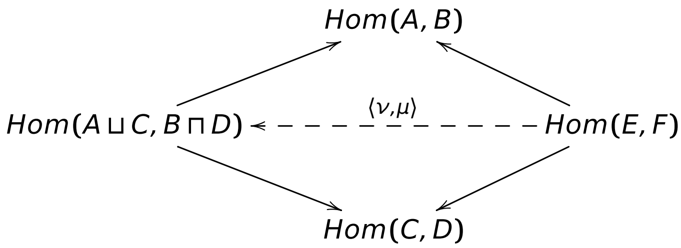


(3.19)



Hence we have that $P \cong A \sqcup C$, $Q \cong B \sqcap D$. On the other hand, suppose that there exist the products and coproducts in $\mathcal{C}$. Then the diagrams (3.18) and (3.19) are commutatives with $P \cong A \sqcup C$, $Q \cong B \sqcap D$. Hence for all $g \in Hom(E,F)$, the diagram(3.17) is commutative. Hence the diagram(3.20) is commutative.

(3.20)



Hence we have that there exist the product in $\mathcal{C}_{\mathcal{H}}$ and

$$Hom(A,B) \sqcap Hom(C,D) \cong Hom(A \sqcup C, B \sqcap D)$$ □

Proposition 3.8. *There exist the coproducts in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the products and coproducts in $\mathcal{C}$. And*

$$Hom(A,B) \sqcup Hom(C,D) \cong Hom(A \sqcap C, B \sqcup D) \tag{3.21}$$

Proof. The proof of the proposition 3.8 is similar to the proof of proposition 3.7. □

Proposition 3.9. *The products and coproducts exist in $\mathcal{C}_{\mathcal{H}}$ simultaneously.*

Proof. Immediate from propositions 3.7 and 3.8. □

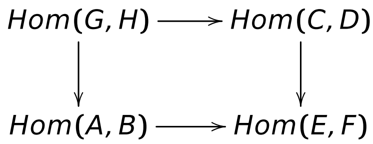
3.7. **Pushout and Pullback in $\mathcal{C}_{\mathcal{H}}$.**

Proposition 3.10. *There exist the pullback in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the pushout and pullback in $\mathcal{C}$. And*

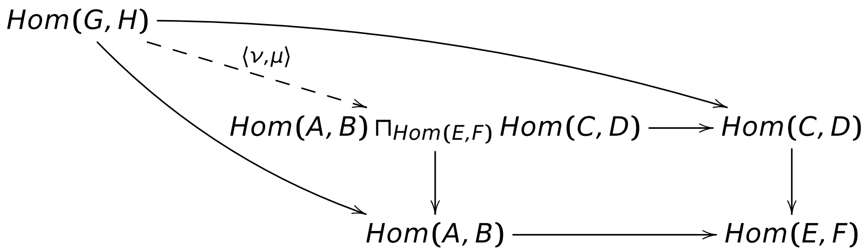
$$Hom(A,B) \sqcap_{Hom(E,F)} Hom(C,D) \cong Hom(A \sqcup_E C, B \sqcap_F D) \tag{3.22}$$

Proof. Let $Hom(A,B) \to Hom(E,F) \leftarrow Hom(C,D)$ be morphisms in $\mathcal{C}_{\mathcal{H}}$. Suppose that there exist the pullback in $\mathcal{C}_{\mathcal{H}}$. Then for all $Hom(G,H) \in \mathcal{C}_{\mathcal{H}}$, if the diagram(3.23) is commutative then there exists a morphism $\langle \nu, \mu \rangle$ such that the diagram(3.24) is commutative.

(3.23)

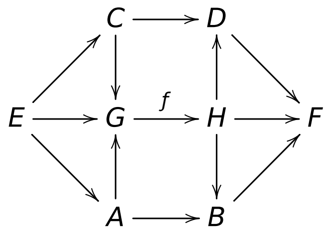


(3.24)

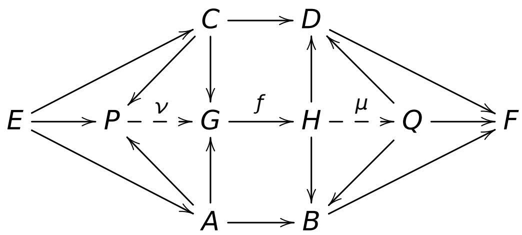


And there exists $Hom(P,Q) \in \mathcal{C}_{\mathcal{H}}$ such that $Hom(P,Q) \cong Hom(A,B) \sqcap_{Hom(E,F)} Hom(C,D)$. Hence for all $f \in Hom(G,H)$, if the diagram(3.25) is commutative then the morphisms ν, μ make the diagram(3.26) commute.

(3.25)

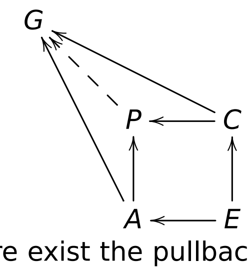


(3.26)

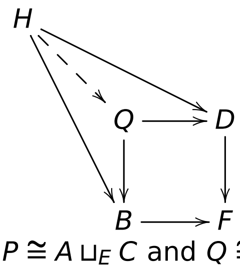


It follows that the diagrams (3.27) and (3.28) are commutative.

(3.27)



(3.28)



Hence there exist the pullback and pushout in $\mathcal{C}$. Hence $P \cong A \sqcup_E C$ and $Q \cong B \sqcap_F D$. Hence we have that

$$Hom(A,B) \sqcap_{Hom(E,F)} Hom(C,D) \cong Hom(A \sqcup_E C, B \sqcap_F D)$$

On the other hand, suppose that there exist the pullback and pushout in $\mathcal{C}_{\mathcal{H}}$. Then the diagrams (3.27) and (3.28) are commutative. Hence for all $f \in Hom(G,H)$, the diagrams (3.26) and (3.25) are commutative. Hence the diagram(3.24) is commutative. It follows that there exist pullback in $\mathcal{C}_{\mathcal{H}}$. □

Proposition 3.11. *There exist the pushout in $\mathcal{C}_{\mathcal{H}}$ if and only if there exist the pullback and pushout in $\mathcal{C}$. And*

$$Hom(A,B) \sqcup_{Hom(E,F)} Hom(C,D) \cong Hom(A \sqcap_E C, B \sqcup_F D) \tag{3.29}$$

Proof. The proof of proposition 3.11 is similar to proposition 3.10. □

Proposition 3.12. *There exist the pushout and pullback in $\mathcal{C}_{\mathcal{H}}$ simultaneously.*

3.8. **(Co)Limit In $\mathcal{C}_{\mathcal{H}}$.**

Definition 3.1. Let $T\colon \mathcal{C}_{\mathcal{H}} \to \mathcal{C}$, $S\colon \mathcal{C}_{\mathcal{H}} \to \mathcal{C}^{op}$ given by

Object:

$$T: Hom(P,Q) \mapsto Q$$
$$S: Hom(P,Q) \mapsto P$$

Morphism:

$$T: \langle \nu, \mu \rangle \mapsto \mu$$
$$S: \langle \nu, \mu \rangle \mapsto \nu$$

Let $\mathcal{D}$ be a category, G and G′ two functors from $\mathcal{D}$ to $\mathcal{C}_{\mathcal{H}}$. Suppose that τ is a natural transformation $\tau\colon G \overset{\bullet}{\to} G'$. Then we have that

Proposition 3.13. *That $T(\tau)$ is a natural transformation $T \circ G \overset{\bullet}{\to} T \circ G'$ and that $S(\tau)$ is a natural transformation $S \circ G \overset{\bullet}{\to} S \circ G'$. Hence*

$$Nat(G,G') \cong Nat(T \circ G, T \circ G') \times Nat(S \circ G, S \circ G') \tag{3.30}$$

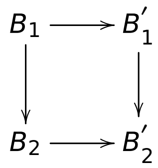
Proof. Let $D_1, D_2 \in \mathcal{D}$. Then there exist $Hom(A_1,B_1), Hom(B_2,B_2), Hom(A'_1,B'_1), Hom(A'_2,B'_2)$ in $\mathcal{C}_{\mathcal{H}}$ such that $G(D_1) = Hom(A_1,B_1)$, $G(D_2) = Hom(B_2,B_2)$, $G'(D_1) = Hom(A'_1,B'_1)$, $G'(D_2) = Hom(A'_2,B'_2)$ in $\mathcal{C}_{\mathcal{H}}$. And for every morphism $f\colon D_1 \to D_2$ in $\mathcal{D}$ the following diagram is commutative where $\tau_{D_1} = \langle v_1, u_2 \rangle$, $\tau_{D_2} = \langle v_2, u_2 \rangle$, $G(f) = \langle \nu, \mu \rangle$ and $G'(f) = \langle \nu', \mu' \rangle$.

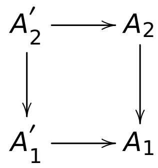
$$\begin{array}{ccc} Hom(A_1,B_1) & \xrightarrow{\tau_{D_1}} & Hom(A'_1,B'_1) \\ {\scriptstyle\langle\nu,\mu\rangle}\downarrow & & \downarrow{\scriptstyle\langle\nu',\mu'\rangle} \\ Hom(A_2,B_2) & \xrightarrow[\tau_{D_1}]{} & Hom(A'_2,B'_2) \end{array}$$

Hence the following diagram is commutative for all $f \in Hom(A_1,B_1)$.

$$\begin{array}{ccccccc} A'_1 & \longrightarrow & A_1 & \xrightarrow{f} & B_1 & \longrightarrow & B'_1 \\ \uparrow & & \uparrow & & \downarrow & & \downarrow \\ A'_2 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & B'_2 \end{array}$$

Then the following two diagrams are commutative.





By definition 3.1, we have that two natural transformations:

$$\omega : T \circ G \xrightarrow{\bullet} T \circ G' \text{ given by } \omega_D := T(\tau_D)$$

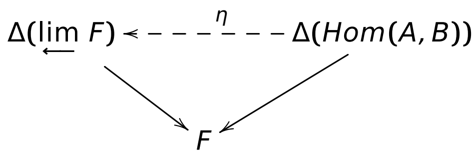
$$\eta : S \circ G \xrightarrow{\bullet} S \circ G' \text{ given by } \eta_D := S(\tau_D)$$

Therefore,

$$Nat(G, G') \cong Nat(T \circ G, T \circ G') \times Nat(S \circ G, S \circ G') \qquad \square$$

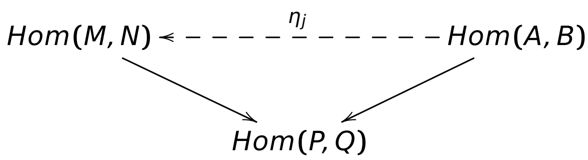
Let $\mathcal{J}$ be a categroy, F a functor from $\mathcal{J}$ to $\mathcal{C}_{\mathcal{H}}$. Suppose that there exists the limit of F in $\mathcal{C}_{\mathcal{H}}$. Then for all $Hom(A,B) \in \mathcal{C}$ with a natural transformation $\Delta(Hom(A,B)) \xrightarrow{\bullet} \varprojlim F$, there exists unique natural transformation $\eta : \Delta(Hom(A,B)) \xrightarrow{\bullet} \Delta(\varprojlim F)$ given by $(\eta_j := \langle \nu, \mu \rangle)_{j \in \mathcal{J}}$ such that the diagram(3.31) is commutative. That Δ is a diagnal functor[1].

(3.31)



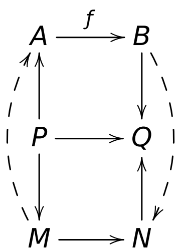
And there exists $Hom(M,N) \in \mathcal{C}_{\mathcal{H}}$ such that $Hom(M,N) \cong \varprojlim F$. Hence for every $j \in \mathcal{J}$ with $F(j) = Hom(P,Q)$, the diagram(3.32) is commutative.

(3.32)



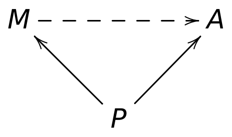
Hence for all $f \in Hom(A,B)$, the diagram(3.33) is commutative.

(3.33)

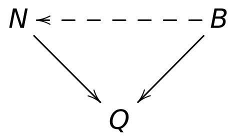


It follows that the diagrams (3.34) and (3.35) are commutative.

(3.34)



(3.35)



The two functors T and S are full by definition 3.1, hence we have that

(3.36) $$M \cong \varprojlim S \circ F$$

(3.37) $$N \cong \varprojlim T \circ F$$

Proposition 3.14. *There exists the limit of F if and only if there exist the limit of $T \circ F$ and the limit of $S \circ F$.*

Proof. Immediate from diagrams (3.31 to 3.35), definition 3.1 and proposition 3.13. □

Proposition 3.15. *There exists the colimit of F if and only if there exist the colimit of $T \circ F$ and the colimit of $S \circ F$.*

Proof. The proof of proposition 3.15 is similar to the proof of proposition 3.14. □

References

[1] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer, 1971.