A DECOMPOSITION FORMULA FOR THIRD-ORDER REAL ANTISYMMETRIC MATRICES

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ABSTRACT

A decomposition formula for an antisymmetric matrix $\mathbf{A}_{\boldsymbol{\omega}} \in \mathcal{A}_3(\mathbb{R})$ is provided, where its axial vector is expressed as $\boldsymbol{\omega} = \mathbf{M}\boldsymbol{\nu}$, with $\mathbf{M}$ symmetric and $\boldsymbol{\nu} \in \mathbb{R}^3$. The proof is based mainly on vector projection through Frobenius inner product. In the end, a vectorial identity involving cross product is proved as a corollary of the decomposition formula.

Keywords Antisymmetric Matrices · Cross Product · Frobenius Inner Product

1 Introduction

Let $\mathcal{A}_3(\mathbb{R}) = \{\mathbf{A} \in \mathcal{M}_3(\mathbb{R}) : \mathbf{A} = -\mathbf{A}^{\mathrm{T}}\}$ be the set of third-order real antisymmetric matrices, where $\mathcal{M}_3(\mathbb{R})$ is the vector space of square real matrices of order 3. Then $\mathcal{A}_3$ is a vector subspace of $\mathcal{M}_3$. In fact, given two antisymmetric matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{A}_3$, it is easy to show the closure with respect to sum:

$$\mathbf{A}_1 + \mathbf{A}_2 = -\mathbf{A}_1^{\mathrm{T}} - \mathbf{A}_2^{\mathrm{T}} = -(\mathbf{A}_1 + \mathbf{A}_2)^{\mathrm{T}}$$

Similarly, for any given $\lambda \in \mathbb{R}$, we can show the closure with respect to multiplication by a scalar:

$$\lambda\mathbf{A}_1 = -\lambda\mathbf{A}_1^{\mathrm{T}} = -(\lambda\mathbf{A}_1)^{\mathrm{T}}$$

Proposition 1.1 *$\mathcal{A}_3$ has canonical base $\mathcal{B} = \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, where:*

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Proof: Any $\mathbf{A} \in \mathcal{A}_3$ can be expressed as a linear combination of $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$. In fact:

$$\mathbf{A} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = a_{12}\mathbf{E}_1 + a_{13}\mathbf{E}_2 + a_{23}\mathbf{E}_3$$

therefore $\mathcal{A}_3 = \mathrm{Span}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$. Now consider the two following linear combinations:

$$\mathbf{A} = \gamma_1'\mathbf{E}_1 + \gamma_2'\mathbf{E}_2 + \gamma_3'\mathbf{E}_3$$
$$\mathbf{A} = \gamma_1''\mathbf{E}_1 + \gamma_2''\mathbf{E}_2 + \gamma_3''\mathbf{E}_3$$

By definition, we know that any antisymmetric matrix $\mathbf{A} \in \mathcal{A}_3$ is such that $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$, therefore $\mathbf{A} + \mathbf{A}^{\mathrm{T}} = 0$. In light of this, we can write:

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} = (\gamma_1'\mathbf{E}_1 + \gamma_2'\mathbf{E}_2 + \gamma_3'\mathbf{E}_3) + (\gamma_1''\mathbf{E}_1 + \gamma_2''\mathbf{E}_2 + \gamma_3''\mathbf{E}_3)^{\mathrm{T}} =$$
$$= (\gamma_1' - \gamma_1'')\mathbf{E}_1 + (\gamma_2' - \gamma_2'')\mathbf{E}_2' + (\gamma_3' - \gamma_3'')\mathbf{E}_3 = 0$$

The latter is satisfied if and only if $\gamma_i' = \gamma_i''$ for $i = 1, 2, 3$, which means that there is a unique linear combination to express $\mathbf{A}$, hence $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a set of linearly independent vectors. Therefore, $\mathcal{B} = \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a base of $\mathcal{A}_3$.
$\square$

An immediate consequence of this is that $\dim(\mathcal{A}_3) = 3$. Antisymmetric matrices are useful to express cross products in terms of matrix-vector products. In fact, given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, their cross product $\mathbf{a} \times \mathbf{b}$ can be expressed as:

$$\mathbf{a} \times \mathbf{b} = \mathbf{A_a}\mathbf{b} \tag{1}$$

where $\mathbf{A_a}$ is antisymmetric. Given (a_1, a_2, a_3) the coordinates of $\mathbf{a}$, the matrix $\mathbf{A_a}$ reads as follows:

$$\mathbf{A_a} = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \tag{2}$$

Given any antisymmetric matrix, it is always possible to associate it with a vector $\mathbf{a} \in \mathbb{R}^3$, which is called axial vector. Let us now consider the following set of antisymmetric matrices:

$$\mathbf{X}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{X}_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly to the last result, it can be easily shown that $\mathcal{B}' = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is a basis of $\mathcal{A}_3$, and that given an axial vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, it is always possible to write its associated antisymmetric matrix $\mathbf{A}_{\boldsymbol{\omega}}$ simply as:

$$\mathbf{A}_{\boldsymbol{\omega}} = \omega_1 \mathbf{X}_1 + \omega_2 \mathbf{X}_2 + \omega_3 \mathbf{X}_3 \tag{3}$$

Definition 1.1 *Given two real square matrices of order n $\mathbf{A}$, $\mathbf{B}$, the Frobenius inner product is a bilinear form $\langle \cdot, \cdot \rangle_F : \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ defined as:*

$$\langle \mathbf{A}, \mathbf{B} \rangle_F = \mathrm{Tr}(\mathbf{A}^{\mathrm{T}} \mathbf{B})$$

The norm induced by this product is given by:

$$\|\mathbf{A}\|_F = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle_F}$$

Proposition 1.2 *$\mathcal{B}' = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is an orthogonal basis with respect to the Frobenius inner product.*

Proof: We have to prove that $\langle \mathbf{X}_i, \mathbf{X}_j \rangle_F = 0$, for $i, j = 1, 2, 3$, $i \neq j$. It is straightforward to see that multiplying each row of $\mathbf{X}_i^{\mathrm{T}}$ with the correspondent column of $\mathbf{X}_j$ (i.e. first row with first column, second row with second column, and so on), one gets a null-diagonal matrix, hence the product is identically zero for any $i \neq j$, proving the statement. □

To conclude with, we can report the following theorem on vector projection [1] applied to antisymmetric matrices expressed with respect to $\mathcal{B}' = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$.

Theorem 1.1 *Given $\mathbf{C} \in \mathcal{A}_3(\mathbb{R})$ and the orthogonal basis $\mathcal{B}' = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ with respect to the Frobenius inner product, it holds that:*

$$\mathbf{C} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 + c_3 \mathbf{X}_3$$

where

$$c_i = \frac{\langle \mathbf{C}, \mathbf{X}_i \rangle_F}{\langle \mathbf{X}_i, \mathbf{X}_i \rangle_F} \tag{4}$$

are called Fourier's coefficients.

2 Decomposition Formula

Theorem 2.1 *Given two axial vector $\boldsymbol{\nu}, \boldsymbol{\omega} \in \mathbb{R}^3$, where $\boldsymbol{\omega}$ is expressible as $\boldsymbol{\omega} = \mathbf{M}\boldsymbol{\nu}$ with $\mathbf{M}$ symmetric, the following equality holds:*

$$\mathbf{A}_{\boldsymbol{\omega}} = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\boldsymbol{\nu}} - 2\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}) \tag{5}$$

where $\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{A}_{\boldsymbol{\omega}}$ are the antisymmetric matrices associated to the axial vectors $\boldsymbol{\nu}, \boldsymbol{\omega}$ respectively, and $\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$ is the antisymmetric part of $\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}$.

Proof: Consider the following antisymmetric matrix $\mathbf{A}_{\boldsymbol{\omega}}$, where $\boldsymbol{\omega} = \mathbf{M}\boldsymbol{\nu}$ and $\mathbf{M}$ is symmetric. We know that we can express $\mathbf{A}_{\boldsymbol{\omega}}$ through (3). Being ν_i and m_{ij} for $i, j = 1, 2, 3$ the components of respectively $\boldsymbol{\nu}$ and $\mathbf{M}$, we have:

$$
\begin{aligned}
\mathbf{A}_{\boldsymbol{\omega}} =& \omega_1\mathbf{X}_1 + \omega_2\mathbf{X}_2 + \omega_3\mathbf{X}_3 = \\
& (m_{11}\nu_1 + m_{12}\nu_2 + m_{13}\nu_3)\mathbf{X}_1 + \\
& (m_{12}\nu_1 + m_{22}\nu_2 + m_{23}\nu_3)\mathbf{X}_2 + \\
& (m_{13}\nu_1 + m_{23}\nu_2 + m_{33}\nu_3)\mathbf{X}_3
\end{aligned}
$$

Introducing $\sigma_{ij} = 1 - \delta_{ij}$, i.e. a tensor whose components are 0 on the diagonal ($i = j$) and 1 elsewhere, and recalling that $m_{ji} = m_{ij}$ for the symmetry of $\mathbf{M}$, we can express the last equality in Einstein's notation as:

$$
\mathbf{A}_{\boldsymbol{\omega}} = m_{jj}\nu_j\mathbf{X}_j + \sigma_{ij}m_{ij}\nu_i\mathbf{X}_j \tag{6}
$$

Now consider the following quantity:

$$
\mathbf{B} = \sigma_{ij}m_{ii}\nu_j\mathbf{X}_j
$$

Adding and subtracting it to (6), one has:

$$
\mathbf{A}_{\boldsymbol{\omega}} = \underbrace{(m_{jj}\nu_j + \sigma_{ij}m_{ii}\nu_j)\mathbf{X}_j}_{\text{Ⓘ}} + \underbrace{(\sigma_{ij}m_{ij}\nu_i - \sigma_{ij}m_{ii}\nu_j)\mathbf{X}_j}_{\text{Ⓘⓘ}} \tag{7}
$$

Let us show that Ⓘ corresponds to $\mathrm{Tr}(\mathbf{M})\mathbf{A}_{\boldsymbol{\nu}}$. In fact:

$$
(m_{jj}\nu_j + \sigma_{ij}m_{ii}\nu_j)\mathbf{X}_j = \sum_{j=1}^{3}\left(m_{jj}\nu_j + \sum_{i=1}^{3}\sigma_{ij}m_{ii}\nu_j\right)\mathbf{X}_j =
$$

$$
= \left(m_{11}\nu_1 + \nu_1\sum_{i=1}^{3}\sigma_{i1}m_{ii}\right)\mathbf{X}_1 + \left(m_{22}\nu_2 + \nu_2\sum_{i=1}^{3}\sigma_{i2}m_{ii}\right)\mathbf{X}_2 + \left(m_{33}\nu_1 + \nu_3\sum_{i=1}^{3}\sigma_{i3}m_{ii}\right)\mathbf{X}_3 =
$$

$$
= (m_{11} + m_{22} + m_{33})\nu_1\mathbf{X}_1 + (m_{11} + m_{22} + m_{33})\nu_2\mathbf{X}_2 + (m_{11} + m_{22} + m_{33})\nu_3\mathbf{X}_3 =
$$

$$
= \mathrm{Tr}(\mathbf{M})\left(\nu_1\mathbf{X}_1 + \nu_2\mathbf{X}_2 + \nu_3\mathbf{X}_3\right) = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\boldsymbol{\nu}}
$$

where the last step is obtained using (3). We now need to characterize Ⓘⓘ, call it $\mathbf{C} = (\sigma_{ij}m_{ij}\nu_i - \sigma_{ij}m_{ii}\nu_j)\mathbf{X}_j$. First of all, let us observe that we can remove σ_{ij}. In fact, for $i = j$, the term $m_{ij}\nu_i - m_{ii}\nu_j = 0$, hence we can simply put:

$$
\mathbf{C} = (m_{ij}\nu_i - m_{ii}\nu_j)\mathbf{X}_j. \tag{8}
$$

We want to find out who $\mathbf{C}$ is. Since $\mathrm{Tr}(\mathbf{M})\mathbf{A}_{\boldsymbol{\nu}}$ is antisymmetric, $\mathbf{C}$ must be forcedly antisymmetric in order to enforce (6) and have $\mathbf{A}_{\boldsymbol{\omega}} \in \mathcal{A}_3(\mathbb{R})$. Let us observe from (8) that the components of $\mathbf{C}$ are obtained from some linear operation between $\mathbf{M}$ and $\boldsymbol{\nu}$. We cannot choose $\mathbf{C} = \mathbf{A}_{\mathbf{M}\boldsymbol{\nu}}$ because it already appears at the left-hand member of (7), so a hint for $\mathbf{C}$ would be:

$$\mathbf{C} = \lambda\,\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$$

with λ opportunely chosen. Observe that this intuition makes sense since the components of $\mathbf{C}$ would consist of a sum of addenda where each of them is a product of some m_{ij} multiplying some ν_i (eventually with a shifted sign), as predicated by (8). In addition, taking the antisymmetric part will ensure the requirement of antisymmetry of $\mathbf{C}$. Also this choice is well-defined because:

$$(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})^{\mathrm{T}} = \mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}} = -\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}$$

which means $\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}$ is neither symmetric nor antisimmetryc. Moreover:

$$\begin{aligned}\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}) &= \frac{1}{2}\Big[\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}} - (\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})^{\mathrm{T}}\Big] = \frac{1}{2}\Big[\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}\Big] = \\ &= \frac{1}{2}\Big[\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}\Big] = \frac{1}{2}\Big[\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} - \mathbf{M}^{\mathrm{T}}\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{T}}\Big] = \\ &= \frac{1}{2}\Big[\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} - (\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M})^{\mathrm{T}}\Big] = \mathrm{Asym}(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M})\end{aligned}$$

In order to show this intuition is actually true, we will take $\mathbf{C} = \lambda\,\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$, project it on $\mathcal{B}' = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$, and check if the projection coefficients are actually corresponding to the components of $\mathbf{C}$ as expressed in (8). Before continuing, we need to introduce the following lemma.

Lemma 2.1 *Given a symmetric matrix* $\mathbf{M}$ *and an axial vector* $\boldsymbol{\nu}$ *with associated antisymmetric matrix* $\mathbf{A}_{\boldsymbol{\nu}}$*, it holds that:*

$$\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i\rangle_F = 2\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F \qquad i = 1,2,3 \tag{9}$$

where $\mathbf{X}_i \in \mathcal{B}'$.

Proof: Calculate $\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F$ first:

$$\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F = \mathrm{Tr}\left((\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})^{\mathrm{T}}\mathbf{X}_i\right) = \mathrm{Tr}\left(\mathbf{A}_{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{X}_i\right) = -\mathrm{Tr}(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}\mathbf{X}_i)$$

By the commutation property of the trace operator applied to a matrix product, for real square matrices we have $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$, which allows us to express the Frobenius inner product of two matrices alternatively as:

$$\langle \mathbf{A}, \mathbf{B}\rangle_F = \langle \mathbf{B}, \mathbf{A}\rangle_F = \mathrm{Tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \mathrm{Tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$$

Therefore, considering $\langle \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i\rangle_F$:

$$\langle \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i\rangle_F = \mathrm{Tr}(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}\mathbf{X}_i^{\mathrm{T}}) = -\mathrm{Tr}(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}\mathbf{X}_i) = \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F$$

Therefore:

$$\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i\rangle_F = \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F + \langle \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i\rangle_F = \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F + \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F = 2\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i\rangle_F$$

which proves the lemma. □

Now we can use this lemma to compute the Fourier's coefficients of $\mathbf{C} = \lambda \mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$ along $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$. We have that:

$$\mathbf{C} = \lambda \, \mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}) = \frac{\lambda}{2}\Big[\mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}\Big]$$

and

$$c_i = \frac{\langle \mathbf{C}, \mathbf{X}_i \rangle_F}{\langle \mathbf{X}_i, \mathbf{X}_i \rangle_F} = \frac{\lambda}{2} \frac{\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\boldsymbol{\nu}}, \mathbf{X}_i \rangle_F}{\langle \mathbf{X}_i, \mathbf{X}_i \rangle_F} = \lambda \frac{\langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i \rangle_F}{\langle \mathbf{X}_i, \mathbf{X}_i \rangle_F} \tag{10}$$

It is easy to calculate that $\langle \mathbf{X}_i, \mathbf{X}_i \rangle_F = \|\mathbf{X}_i\|_F^2 = 2$ for $i = 1, 2, 3$. In fact, take $i = 1$:

$$\langle \mathbf{X}_1, \mathbf{X}_1 \rangle_F = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right) = \mathrm{Tr} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2$$

It is easy to show that also for $\mathbf{X}_2$ and $\mathbf{X}_3$, allowing us to rewrite (10) as:

$$c_i = \frac{\lambda}{2} \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_i \rangle_F$$

which we need to explicit for $i = 1, 2, 3$. Consider $i = 1$:

$$\begin{aligned} c_1 = \frac{\lambda}{2} \langle \mathbf{A}_{\boldsymbol{\nu}}\mathbf{M}, \mathbf{X}_1 \rangle_F = \\ = \frac{\lambda}{2} \mathrm{Tr}\left(\begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}} \right) = \\ = \frac{\lambda}{2} \mathrm{Tr}\left(\begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & m_{13} & -m_{12} \\ 0 & m_{23} & -m_{22} \\ 0 & m_{33} & -m_{23} \end{bmatrix} \right) = \\ = \frac{\lambda}{2} \left(-m_{13}\nu_3 + m_{33}\nu_1 - m_{12}\nu_2 + m_{22}\nu_1 \right) = \\ = \frac{\lambda}{2} \Big[(m_{22} + m_{33})\nu_1 - m_{12}\nu_2 - m_{12}\nu_3 \Big] \end{aligned} \tag{11}$$

In a similar way, we can find out that:

$$c_2 = \frac{\lambda}{2} \Big[(m_{11} + m_{33})\nu_2 - m_{12}\nu_1 - m_{23}\nu_3 \Big] \tag{12}$$

$$c_3 = \frac{\lambda}{2} \Big[(m_{11} + m_{22})\nu_3 - m_{13}\nu_1 - m_{23}\nu_2 \Big] \tag{13}$$

Now, let us write explicitly the coordinates $\mathbf{C}$ as expressed in (8). Still using Einstein's notation, it reads:

$$\mathbf{C} = \underbrace{(m_{i1}\nu_i - m_{ii}\nu_1)}_{=c_1}\mathbf{X}_1 + \underbrace{(m_{i2}\nu_2 - m_{ii}\nu_2)}_{=c_2}\mathbf{X}_2 + \underbrace{(m_{i3}\nu_2 - m_{ii}\nu_3)}_{=c_3}\mathbf{X}_3$$

Marking summation explicitly and using $m_{ij} = m_{ji}$, we have:

$$c_1 = \sum_{i=1}^{3} m_{i1}\nu_i - m_{ii}\nu_1 = -(m_{22} + m_{33})\nu_1 + (m_{12}\nu_2 + m_{13}\nu_3) \tag{14}$$

$$c_2 = \sum_{i=1}^{3} m_{i2}\nu_i - m_{ii}\nu_2 = -(m_{11} + m_{33})\nu_2 + (m_{12}\nu_1 + m_{23}\nu_3) \tag{15}$$

$$c_3 = \sum_{i=1}^{3} m_{i3}\nu_i - m_{ii}\nu_3 = -(m_{11} + m_{22})\nu_3 + (m_{13}\nu_1 + m_{23}\nu_2) \tag{16}$$

Thus, (14), (15) and (16) coincide with (11), (12) and (13) respectively for $\lambda = -2$. This allows us to finally express $\mathbf{C}$ as:

$$\mathbf{C} = -2\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$$

Therefore, putting all together in (7), it yields:

$$\mathbf{A}_{\boldsymbol{\omega}} = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\boldsymbol{\nu}} - 2\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\boldsymbol{\nu}})$$

□

Since it is always possible to associate an antisymmetric matrix to the axial vector $\boldsymbol{\omega}$ and viceversa, this formula holds as long as the axial vector is expressible as a matrix-vector product through $\mathbf{M}$ and $\boldsymbol{\nu}$ ($\mathbf{M}$ symmetric). From this decomposition formula, we can immediately deduce the following result.

Corollary 2.1 *Given* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ *and a symmetric matrix* $\mathbf{M}$*, the following relationship is true:*

$$\mathbf{M}(\mathbf{a} \times \mathbf{b}) = \mathrm{Tr}(\mathbf{M})\,\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{M}\mathbf{b} + \mathbf{b} \times \mathbf{M}\mathbf{a} \tag{17}$$

Proof: Consider $\boldsymbol{\nu} \equiv \mathbf{a}$ and $\boldsymbol{\omega} = \mathbf{M}\mathbf{a}$. Then, using (5), we have:

$$\begin{aligned}\mathbf{A}_{\mathbf{Ma}} &= \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\mathbf{a}} - 2\mathrm{Asym}(\mathbf{M}\mathbf{A}_{\mathbf{a}}) = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\mathbf{a}} - \left[\mathbf{M}\mathbf{A}_{\mathbf{a}} - (\mathbf{M}\mathbf{A}_{\mathbf{a}})^{\mathrm{T}}\right] = \\ &= \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\mathbf{a}} - \mathbf{M}\mathbf{A}_{\mathbf{a}} + \mathbf{A}_{\mathbf{a}}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}} = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\mathbf{a}} - \mathbf{M}\mathbf{A}_{\mathbf{a}} - \mathbf{A}_{\mathbf{a}}\mathbf{M}\end{aligned} \tag{18}$$

Applying $\mathbf{b}$ to both members of (18), one gets:

$$\mathbf{A}_{\mathbf{Ma}}\mathbf{b} = \mathrm{Tr}(\mathbf{M})\mathbf{A}_{\mathbf{a}}\mathbf{b} - \mathbf{M}\mathbf{A}_{\mathbf{a}}\mathbf{b} - \mathbf{A}_{\mathbf{a}}\mathbf{M}\mathbf{b}$$

Using (2), we can write further:

$$(\mathbf{Ma}) \times \mathbf{b} = \mathrm{Tr}(\mathbf{M})\, \mathbf{a} \times \mathbf{b} - \mathbf{M}(\mathbf{a} \times \mathbf{b}) - \mathbf{a} \times (\mathbf{Mb})$$

If we reorganize the members and rewrite $(\mathbf{Ma}) \times \mathbf{b} = -\mathbf{b} \times (\mathbf{Ma})$, we obtain exactly:

$$\mathbf{M}(\mathbf{a} \times \mathbf{b}) = \mathrm{Tr}(\mathbf{M})\, \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{Mb} + \mathbf{b} \times \mathbf{Ma}$$

□

3 Conclusion

In the previous section, we have shown how a generic antisymmetric matrix of axial vector $\boldsymbol{\omega}$ can be decomposed. While it is always trivial to associate any $\mathbf{A} \in \mathcal{A}_3(\mathbb{R})$ with a vector of $\boldsymbol{\omega} \in \mathbb{R}^3$, it is not obvious how to find $\mathbf{M}$ and $\boldsymbol{\nu}$ such that $\boldsymbol{\omega} = \mathbf{M}\boldsymbol{\nu}$, under the symmetry constraint of $\mathbf{M}$. Future work may consist of showing the existence of the couple $(\mathbf{M}, \boldsymbol{\nu})$ for any given $\boldsymbol{\omega} \in \mathbb{R}^3$. Moreover, on the basis of that, one could seek for an optimal procedure of determining a three-dimensional vector $\boldsymbol{\omega}$ from 9 degrees of freedom (6 accounting for $\mathbf{M}$, and 3 for $\boldsymbol{\nu}$). Finally, given the vectorial form of equation (17), one could investigate its prospective applications in fields like Vector Calculus, Differential Geometry and Mechanics.

References

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