# On Prime Numbers Between $kn$ And $(k+1)n$

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## Abstracts

In this paper along with three previous studies on analyzing the binomial coefficients, we will complete the proof of a theorem. The theorem states that for two positive integers $n \geq 1$ and $k \geq 1$, if $n \geq k - 1$, then there always exists at least a prime number $p$ such that $kn < p \leq (k + 1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k$ =1.

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# 1. Introduction

The Bertrand-Chebyshev's theorem States that for any positive integer $n$, there is always a prime number $p$ such that $n < p \le 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer $n$, there is a prime number $p$ such that $2n < p \le 3n$. In 2011, Andy Loo [3] expanded the theorem to prove that there is a prime number in the interval $(3n, 4n)$ when $n \ge 2$. It comes up with a question: Does any positive integer $k$ make $kn < p \le (k+1)n$ stand? If it does, in what conditions? Previously, the author partially answered these questions by analyzing the binomial coefficients $\binom{3n}{n}$, $\binom{4n}{n}$, and $\binom{\lambda n}{n}$ where $\lambda$ is a positive integer [4] [5] [6]. In this paper, we will complete the work with the above methodology. In this section, we will cite some important concepts from the previous papers. Then in section 2 and section 3, we will fill up the gaps of $\lambda$ from 5 to 25. And in section 4, we will convert $\lambda$ to $k$ to complete this paper.

From [4]:

**Definition**: $\Gamma_{a\ge p>b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer $n$ or a positive integer expression. In this operator, $p$ is a prime number, $a$ and $b$ are real numbers, and $n \ge a \ge p > b \ge 1$.

It has some properties:

It is always true that $\Gamma_{a\ge p\ge b}\{n\} \ge 1$. **— (1.1)**

If no prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} = 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} = 1$, then no prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{12\ge p>4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (1.2)**

If there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$, then $\Gamma_{a\ge p>b}\{n\} > 1$, or vice versa, if $\Gamma_{a\ge p>b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a\ge p>b}\{n\}$ as in $\Gamma_{4\ge p>2}\{12\} = 3 > 1$. **— (1.3)**

For every positive integer $n$, there exists at least a prime number $p$ such that $2n < p \le 3n$. **— (1.4)**

From [5]:

For every integer $n > 1$, there exists at least a prime number $p$ such that $3n < p \le 4n$. **— (1.5)**

From [6 pp4-5]:

For $n \ge 2$ and $\lambda \ge 3$, $\binom{\lambda n}{n} > \dfrac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (1.6)**

If $p$ divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \le p^{R(p)} \le \lambda n$ **— (1.7)**

If $\lambda n \ge p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \le v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le 1$ **— (1.8)**

For $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (1.9)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to $n$. For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For counting any successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6).

Thus, $\pi(n) \leq \left\lfloor \frac{n}{3} \right\rfloor + 2 \leq \frac{n}{3} + 2$ **— (1.10)**

when $n > \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$

when $n \leq \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$

Thus, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\}$ **— (1.11)**

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to **(1.8)**, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} < \prod_{n \geq p} p$. It has been proved [7] that for $n \geq 3$, $\prod_{n \geq p} p \leq 2^{2n-3}$. Thus, for $n \geq 3$ and $\lambda \geq 3$, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} < \prod_{n \geq p} p < 2^{2n-3}$.

Referred to **(1.7)** and **(1.10)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$

Thus for $n \geq 3$ and $\lambda \geq 3$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$ **— (1.12)**

Applying **(1.6)** to **(1.12)**, when $n \geq 3$ and $\lambda \geq 3$, we have

$$\frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

Since $2^{2n-3} > 0$ and $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 0$, when $n \geq 3$, and $\lambda \geq 3$,

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} \cdot 2^{2n-3} \cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \textbf{— (1.13)}$$

## 2. A Prime Number Between ($\lambda$-1)$n$ and $\lambda n$ when 5 ≤ $\lambda$ ≤ 7 and $n \geq \lambda$-2

**Proposition 1**: For $n \geq 36$ and $5 \leq \lambda \leq 7$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.1)**

Referring to **(1.13)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, we have

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \textbf{— (2.2)}$$

Let $f_1(x) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where $x$ is a real number, the variable, and $\lambda$ is a constant at one of the 3 integers from 5 to 7.

$f_1{}'(x) = f_1(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_1(x) \cdot f_2(x)$ where

$f_2(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}$

$f_2{}'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}\,ln(x)}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 0$ and $\lambda > 0$. Thus, $f_2(x)$ is a strictly increasing function.

When $x = 36$ and $\lambda = 5$, $f_2(x) = ln\left(\frac{5}{4}\right) + ln\left(\frac{5}{5-1}\right)^{5-1} - \frac{\sqrt{5}\,(ln(36)+ln(5)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.5859 > 0$.

When $x = 36$ and $\lambda = 6$, $f_2(x) = ln\left(\frac{6}{4}\right) + ln\left(\frac{6}{6-1}\right)^{6-1} - \frac{\sqrt{6}\,(ln(36)+ln(6)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.7155 > 0$.

When $x = 36$ and $\lambda = 7$, $f_2(x) = ln\left(\frac{7}{4}\right) + ln\left(\frac{7}{7-1}\right)^{7-1} - \frac{\sqrt{7}\,(ln(36)+ln(7)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 08522 > 0$.

Since $f_2(x) > 0$ when $x = 36$ and $5 \leq \lambda \leq 7$, and since $f_2(x)$ is a strictly increasing function, then when $x \geq 36$ and $5 \leq \lambda \leq 7$, we have $f_2(x) > 0$. **— (2.3)**

Since when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 0$ and $f_2(x) > 0$, then $f_1{}'(x) = f_1(x) \cdot f_2(x) > 0$. Thus, when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x)$ is a strictly increasing function. $f_1(x+1) > f_1(x)$. **— (2.4)**

When $\lambda = 5$ and $x = 36$, $f_1(x) = \frac{50 \cdot \left(\left(\frac{5}{4}\right) \cdot \left(\frac{5}{5-1}\right)^{5-1}\right)^{(36-1)}}{(180)^{\frac{\sqrt{180}}{3}+3}} = \frac{4.5522\text{E}+18}{7.1073\text{E}+16} > 1$.

When $\lambda = 6$ and $x = 36$, $f_1(x) = \frac{72 \cdot \left(\left(\frac{6}{4}\right) \cdot \left(\frac{6}{6-1}\right)^{6-1}\right)^{(36-1)}}{(216)^{\frac{\sqrt{216}}{3}+3}} = \frac{7.5378\text{E}+21}{2.7530\text{E}+18} > 1$.

When $\lambda = 7$ and $x = 36$, $f_1(x) = \frac{98 \cdot \left(\left(\frac{7}{4}\right) \cdot \left(\frac{7}{7-1}\right)^{7-1}\right)^{(36-1)}}{(252)^{\frac{\sqrt{252}}{3}+3}} = \frac{3.6007\text{E}+24}{8.1511\text{E}+19} > 1$.

Referring to **(2.4)**, when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 1$.

Let $x = n$, then when $n \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(n) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(2.2)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.5)**

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n + 1 = \sqrt{n^2 + 2n + 1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, $v_p = \sum_{i=1}^{\lambda-2}(i - i) = 0$ when any $p$ in $\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$

Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$.

Therefore, when $n \geq 36$ and $5 \leq \lambda \leq 7$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$.

**— (2.6)**

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(2.6)** at last one of these two parts is greater than 1.

When $n \geq 36$ and $5 \leq \lambda \leq 7$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When the factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{36}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number $p$ such that $(\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.9)**

Referring to **(2.8)** and **(2.9)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, referring to **(1.3)**, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.10)**

In conclusion from **(2.5)**, **(2.7)**, **(2.10)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, then $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. When $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, and there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. Thus, **Proposition 1** is proven.

**Proposition 2**: For $35 \geq n \geq \lambda - 2$ and $5 \leq \lambda \leq 7$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.11)**

We use tables to prove **(2.11)**. **Table 1**, **Table 2**, and **Table 3** show that when $\lambda$ = 5, 6, and 7, **Proposition 2** is correct. Thus, **(2.11)** is valid.

**Table 1.** When $\lambda = 5$ and $3 \leq n \leq 35$, a prime number exists in the range of $4n < p \leq 5n$

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | 13 | 17 | 23 | 29 | 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 | 73 | 79 | 83 |
| | | | | | | | | | | | | | | | | | |
| $n$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | |
| $p$ | 89 | 97 | 101 | 103 | 107 | 109 | 113 | 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | |

**Table 2.** When $\lambda = 6$ and $4 \leq n \leq 35$, a prime number exists in the range of $5n < p \leq 6n$

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | 23 | 29 | 31 | 37 | 41 | 47 | 53 | 59 | 61 | 67 | 71 | 79 | 83 | 89 | 97 | 101 |
| | | | | | | | | | | | | | | | | |
| $n$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| $p$ | 103 | 107 | 113 | 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 | 179 | 181 | 191 |

**Table 3.** When $\lambda = 7$ and $5 \leq n \leq 35$, a prime number exists in the range of $6n < p \leq 7n$

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | 31 | 37 | 43 | 53 | 59 | 61 | 67 | 73 | 79 | 89 | 97 | 101 | 103 | 109 | 127 | 131 |
| | | | | | | | | | | | | | | | | |
| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | |
| $p$ | 137 | 149 | 151 | 157 | 163 | 167 | 173 | 179 | 181 | 191 | 193 | 197 | 211 | 223 | 227 | |

Combining **(2.1)** and **(2.11)**, we have proven that when $5 \leq \lambda \leq 7$ and $n \geq \lambda - 2$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.12)**

## 3. A Prime Number Between ($\lambda$-1)$n$ and $\lambda n$ when 8 ≤ $\lambda$ ≤ 25 and $n$ ≥ $\lambda$-2

**Proposition 3**: For $n \geq 24$ and $8 \leq \lambda \leq 25$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.1)**

Referring to **(1.13)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, we have

$$\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \textbf{— (3.2)}$$

Let $f_3(x) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where $x$ is a real number, the variable, and $\lambda$ is a constant at one of the 18 integers from 8 to 25.

$f_3{}'(x) = f_3(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_3(x) \cdot f_4(x)$ where

$f_4(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}$

$f_4{}'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 0$ and $\lambda > 0$. Thus, $f_4(x)$ is a strictly increasing function.

We now calculate the $f_4(x)$ values and list them in **Table 4** for $x = 24$ and $\lambda$ = 8, 9, 10, … 25.

**Table 4.** When $x$ = 24 and $\lambda$ from 8 to 25, $f_4(x) > 0$

| $\lambda$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $f_4(x)$ | 0.805 | 0.876 | 0.935 | 0.985 | 1.028 | 1.064 | 1.096 | 1.124 | 1.148 |
| | | | | | | | | | |
| $\lambda$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| $f_4(x)$ | 1.168 | 1.186 | 1.202 | 1.215 | 1.227 | 1.237 | 1.246 | 1.253 | 1.259 |

**Table 4** shows that when $x$ = 24 and $\lambda$ from 8 to 25, $f_4(x) > 0$. Since $f_3(x) > 0$ and $f_4(x) > 0$, thus when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_3{}'(x) = f_3(x) \cdot f_4(x) > 0$. Thus, under these conditions, $f_3(x)$ is a strictly increasing function, and $f_3(x+1) > f_3(x)$. **— (3.3)**

We now calculate the $f_3(x)$ values and list them in **Table 5** for $x$ = 24 and $\lambda$ = 8, 9, 10, … 25.

**Table 5.** When $x$ = 24 and $\lambda$ from 8 to 25, $f_5(x) > 1$

| $\lambda$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $f_5(x)$ | 9.366 | 19.132 | 31.150 | 42.517 | 50.475 | 53.571 | 51.866 | 46.527 | 39.386 |
| | | | | | | | | | |
| $\lambda$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| $f_5(x)$ | 31.212 | 23.760 | 17.383 | 12.287 | 8.421 | 5.633 | 3.679 | 2.536 | 1.481 |

**Table 5** shows when $x$ = 24 and $\lambda$ from 8 to 25, $f_5(x) > 1$. Since $f_3(x+1) > f_3(x)$, when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_5(x) > 1$. **— (3.4)**

Let $x = n$, then when $n \geq 24$ and $8 \leq \lambda \leq 25$, $f_3(n) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(3.2)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.5)**

Thus, referring to **(1.3)**, there exists at least a prime number $p$ such that $n < p \leq \lambda n$.

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n + 1 = \sqrt{n^2 + 2n + 1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, $v_p = \sum_{i=1}^{\lambda-2}(i - i) = 0$ when any $p$ in $\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$.

Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$.

Therefore, when $n \geq 24$ and $8 \leq \lambda \leq 25$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$.

**— (3.6)**

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(3.6)** at last one of these two parts is greater than 1.

When $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When the factor $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{24}{i+1}$. We have $\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number $p$ such that $(\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.9)**

Referring to **(3.8)** and **(3.9)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, referring to **(1.3)**, there exists at least a prime number $p$ such that $(\lambda-1)n < p \leq \lambda n$. **— (3.10)**

In conclusion from **(3.5)**, **(3.7)**, **(3.10)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, then $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. When $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, and there exists at least a prime number $p$ such that $(\lambda-1)n < p \leq \lambda n$. Thus, **Proposition 3** is proven.

**Proposition 4**: For $23 \geq n \geq \lambda-2$ and $8 \leq \lambda \leq 25$, there exists at least a prime number $p$ such that $(\lambda-1)n < p \leq \lambda n$. **— (3.11)**

We use tables to prove **(3.11)**. **Table 6**, **Table 7**, and **Table 8** show that when $8 \leq \lambda \leq 25$, **Proposition 4** is correct. Thus, **(3.11)** is valid.

**Table 6.** When $8 \leq \lambda \leq 11$ and $\lambda-2 \leq n \leq 23$, a prime number between $(\lambda-1)n$ and $\lambda n$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\boldsymbol{n}$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | |
| | $\boldsymbol{7n}$ | 42 | 49 | 56 | 63 | 70 | 77 | 84 | 91 | 98 | |
| | $\boldsymbol{p}$ | **47** | **53** | **59** | **67** | **73** | **83** | **89** | **97** | **101** | $\boldsymbol{\lambda = 8}$ |
| | $\boldsymbol{8n}$ | 48 | 56 | 64 | 72 | 80 | 88 | 96 | 104 | 112 | |
| $\boldsymbol{\lambda = 9}$ | $\boldsymbol{p}$ | | **61** | **71** | **79** | **83** | **97** | **101** | **107** | **113** | |
| | $\boldsymbol{9n}$ | | 63 | 72 | 81 | 90 | 99 | 108 | 117 | 126 | |
| | $\boldsymbol{p}$ | | | **73** | **83** | **97** | **101** | **109** | **127** | **131** | $\boldsymbol{\lambda = 10}$ |
| | $\boldsymbol{10n}$ | | | 80 | 90 | 100 | 110 | 120 | 130 | 140 | |
| $\boldsymbol{\lambda = 11}$ | $\boldsymbol{p}$ | | | | **97** | **103** | **113** | **127** | **139** | **151** | |
| | $\boldsymbol{11n}$ | | | | 99 | 110 | 121 | 132 | 143 | 154 | |
| | | | | | | | | | | | |
| | $\boldsymbol{n}$ | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | |
| | $\boldsymbol{7n}$ | 105 | 112 | 119 | 126 | 133 | 140 | 147 | 154 | 161 | |
| | $\boldsymbol{p}$ | **107** | **113** | **127** | **131** | **137** | **149** | **151** | **157** | **163** | $\boldsymbol{\lambda = 8}$ |
| | $\boldsymbol{8n}$ | 120 | 128 | 136 | 144 | 152 | 160 | 168 | 176 | 184 | |
| $\boldsymbol{\lambda = 9}$ | $\boldsymbol{p}$ | **127** | **131** | **139** | **149** | **157** | **167** | **173** | **179** | **191** | |
| | $\boldsymbol{9n}$ | 135 | 144 | 153 | 162 | 171 | 180 | 189 | 198 | 207 | |
| | $\boldsymbol{p}$ | **137** | **151** | **163** | **167** | **181** | **191** | **193** | **199** | **211** | $\boldsymbol{\lambda = 10}$ |
| | $\boldsymbol{10n}$ | 150 | 160 | 170 | 180 | 190 | 200 | 210 | 220 | 230 | |
| $\boldsymbol{\lambda = 11}$ | $\boldsymbol{p}$ | **157** | **167** | **179** | **191** | **197** | **211** | **223** | **227** | **233** | |
| | $\boldsymbol{11n}$ | 165 | 176 | 187 | 198 | 209 | 220 | 231 | 242 | 253 | |

**Table 7.** When 12 ≤ $\lambda$ ≤ 15 and $\lambda$ −2 ≤ $n$ ≤ 23, a prime number between ($\lambda$ −1)$n$ and $\lambda n$

| | $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $11n$ | 110 | 121 | 132 | 143 | 154 | 165 | 176 | 187 | 198 | 209 | 220 | 231 | 242 | 253 |
| $\lambda = 12$ | $p$ | **113** | **127** | **137** | **149** | **157** | **167** | **181** | **193** | **199** | **223** | **229** | **239** | **257** | **269** |
| | $12n$ | 120 | 132 | 144 | 156 | 168 | 180 | 192 | 204 | 216 | 228 | 240 | 252 | 264 | 276 |
| $\lambda = 13$ | $p$ | | **139** | **151** | **163** | **173** | **183** | **197** | **211** | **227** | **233** | **241** | **263** | **271** | **281** |
| | $13n$ | | 143 | 156 | 169 | 182 | 195 | 208 | 221 | 234 | 247 | 260 | 273 | 286 | 299 |
| $\lambda = 14$ | $p$ | | | **167** | **179** | **191** | **199** | **223** | **223** | **239** | **257** | **269** | **277** | **293** | **307** |
| | $14n$ | | | 168 | 182 | 196 | 210 | 224 | 238 | 252 | 266 | 280 | 294 | 308 | 322 |
| $\lambda = 15$ | $p$ | | | | **191** | **199** | **211** | **229** | **239** | **263** | **271** | **283** | **307** | **311** | **331** |
| | $15n$ | | | | 195 | 210 | 225 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 |

**Table 8.** When 16 ≤ $\lambda$ ≤ 25 and $\lambda$ −2 ≤ $n$ ≤ 23, a prime number between ($\lambda$ −1)$n$ and $\lambda n$

| | $n$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $15n$ | 210 | 225 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | |
| $\lambda = 16$ | $p$ | **223** | **227** | **241** | **257** | **277** | **293** | **313** | **317** | **331** | **347** | |
| | $16n$ | 224 | 240 | 256 | 272 | 288 | 304 | 320 | 336 | 352 | 368 | |
| | $p$ | | **251** | **263** | **281** | **293** | **307** | **337** | **349** | **353** | **373** | $\lambda = 17$ |
| | $17n$ | | 255 | 272 | 289 | 306 | 323 | 340 | 357 | 374 | 391 | |
| $\lambda = 18$ | $p$ | | | **277** | **293** | **311** | **331** | **347** | **359** | **379** | **397** | |
| | $18n$ | | | 288 | 306 | 324 | 342 | 360 | 378 | 396 | 414 | |
| | $p$ | | | | **307** | **337** | **349** | **373** | **383** | **397** | **419** | $\lambda = 19$ |
| | $19n$ | | | | 323 | 342 | 361 | 380 | 399 | 418 | 437 | |
| $\lambda = 20$ | $p$ | | | | | **347** | **367** | **389** | **401** | **421** | **439** | |
| | $20n$ | | | | | 360 | 380 | 400 | 420 | 440 | 460 | |
| | $p$ | | | | | | **283** | **409** | **431** | **443** | **461** | $\lambda = 21$ |
| | $21n$ | | | | | | 399 | 420 | 441 | 462 | 483 | |
| $\lambda = 22$ | $p$ | | | | | | | **433** | **449** | **463** | **487** | |
| | $22n$ | | | | | | | 440 | 462 | 484 | 506 | |
| | $p$ | | | | | | | | **467** | **491** | **509** | $\lambda = 23$ |
| | $23n$ | | | | | | | | 483 | 506 | 529 | |
| $\lambda = 24$ | $p$ | | | | | | | | | **521** | **541** | |
| | $24n$ | | | | | | | | | 528 | 552 | |
| | $p$ | | | | | | | | | | **563** | $\lambda = 25$ |
| | $25n$ | | | | | | | | | | 575 | |

Combining **(3.1)** and **(3.11)**, we have proven that when 8 ≤ $\lambda$ ≤ 25 and $n$ ≥ $\lambda$ −2, there exists at least a prime number $p$ such that ($\lambda$ −1)$n$ < $p$ ≤ $\lambda n$. **— (3.12)**

## 4. A Prime Number Between $kn$ and $(k+1)n$

From **(1.4),** for every positive integer $n$, there exists at least a prime number $p$ such that $2n < p \le 3n$.

From **(1.5),** for every integer $n > 1$, there exists at least a prime number $p$ such that $3n < p \le 4n$.

From **(2.12)**, when $5 \le \lambda \le 7$ and $n \ge \lambda - 2$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \le \lambda n$.

From **(3.12)**, when $8 \le \lambda \le 25$ and $n \ge \lambda - 2$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \le \lambda n$.

From **(1.9),** for $n \ge (\lambda - 2) \ge 24$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \le \lambda n$.

Combining **(1.4)**, **(1.5)**, **(2.12)**, **(3.12)**, and **(1.9)**, we have that for $n \ge \lambda - 2 \ge 1$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \le \lambda n$. **— (4.1)**

Let $k = \lambda - 1$, **(4.1)** becomes that for $n \ge k - 1 \ge 1$, there exists at least a prime number $p$ such that $kn < p \le (k + 1)n$. **— (4.2)**

Since the Bertrand-Chebyshev's theorem states that for any positive integer $n$, there is always a prime number $p$ such that $n < p \le 2n$, we can state that for two positive integers $n \ge 1$ and $k \ge 1$, if $n \ge k - 1$, then there always exists at least a prime number $p$ such that $kn < p \le (k + 1)n$. The Bertrand's postulate / Chebyshev's theorem is a special case when $k = 1$.

## 5. References


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