On Prime Numbers Between kn And $(k+1)n$

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Abstracts

In this paper along with three previous studies on analyzing the binomial coefficients, we will complete the proof of a theorem. The theorem states that for two positive integers $n \geq 1$ and $k \geq 1$, if $n \geq k - 1$, then there always exists at least a prime number p such that $kn > p \geq (k+1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$.

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1. Introduction

The Bertrand-Chebyshev's theorem States that for any positive integer n, there is always a prime number p such that $n < p \le 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \le 3n$. In 2011, Andy Loo [3] expanded the theorem to prove that there is a prime number in the interval $(3n, 4n)$ when $n \ge 2$. It comes up with a question: Does any positive integer k make $kn < p \le (k+1)n$ stand? If it does, in what conditions? Previously, the author partially answered these questions by analyzing the binomial coefficients $\binom{3n}{n}$, $\binom{4n}{n}$, and $\binom{\lambda n}{n}$ where λ is a positive integer [4] [5] [6]. In this paper, we will complete the work with the above methodology. In this section, we will cite some important concepts from the previous papers. Then in section 2 and section 3, we will fill up the gaps of λ from 5 to 25. And in section 4, we will convert λ to k to complete this paper.

From [4]:

Definition: $\Gamma_{a \ge p > b}\{n\}$ denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer n or a positive integer expression. In this operator, p is a prime number, a and b are real numbers, and $n \ge a \ge p > b \ge 1$.

It has some properties:

It is always true that $\Gamma_{a \ge p \ge b}\{n\} \ge 1$. **— (1.1)**

If no prime number in $\Gamma_{a \ge p > b}\{n\}$, then $\Gamma_{a \ge p > b}\{n\} = 1$, or vice versa, if $\Gamma_{a \ge p > b}\{n\} = 1$, then no prime number in $\Gamma_{a \ge p > b}\{n\}$ as in $\Gamma_{12 \ge p > 4}\{12\} = 11^0 \cdot 7^0 \cdot 5^0 = 1$. **— (1.2)**

If there is at least one prime number in $\Gamma_{a \ge p > b}\{n\}$, then $\Gamma_{a \ge p > b}\{n\} > 1$, or vice versa, if $\Gamma_{a \ge p > b}\{n\} > 1$, then there is at least one prime number in $\Gamma_{a \ge p > b}\{n\}$ as in $\Gamma_{4 \ge p > 2}\{12\} = 3 > 1$. **— (1.3)**

For every positive integer n, there exists at least a prime number p such that $2n < p \le 3n$. **— (1.4)**

From [5]:

For every integer $n > 1$, there exists at least a prime number p such that $3n < p \le 4n$. **— (1.5)**

From [6 pp4-5]:

For $n \ge 2$ and $\lambda \ge 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ **— (1.6)**

If p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le \log_p(\lambda n)$, or $p^{v_p\left(\binom{\lambda n}{n}\right)} \le p^{R(p)} \le \lambda n$ **— (1.7)**

If $\lambda n \ge p > \lfloor\sqrt{\lambda n}\rfloor$, then $0 \le v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le 1$ **— (1.8)**

For $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (1.9)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For counting any successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6).

Thus, $\pi(n) \leq \left\lfloor\frac{n}{3}\right\rfloor + 2 \leq \frac{n}{3} + 2$ **— (1.10)**

when $n > \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

when $n \leq \lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$

Thus, $\binom{\lambda n}{n} \leq \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \cdot \Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\}$ **— (1.11)**

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to **(1.8)**, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} < \prod_{n \geq p} p$. It has been proved [7] that for $n \geq 3$, $\prod_{n \geq p} p \leq 2^{2n-3}$. Thus, for $n \geq 3$ and $\lambda \geq 3$, $\Gamma_{n \geq p > \lfloor\sqrt{\lambda n}\rfloor}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} < \prod_{n \geq p} p < 2^{2n-3}$.

Referred to **(1.7)** and **(1.10)**, $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \geq p}\{\frac{(\lambda n)!}{n! \cdot ((\lambda-1)n)!}\} \leq (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$

Thus for $n \geq 3$ and $\lambda \geq 3$, $\binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$ **— (1.12)**

Applying **(1.6)** to **(1.12)**, when $n \geq 3$ and $\lambda \geq 3$, we have

$\frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda-1)^{(\lambda-1)n - \lambda + 1}} < \binom{\lambda n}{n} < \Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2}$.

Since $2^{2n-3} > 0$ and $(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} > 0$, when $n \geq 3$, and $\lambda \geq 3$,

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{\lambda^{\lambda n - \lambda + 1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 2} \cdot 2^{2n-3} \cdot n(\lambda-1)^{(\lambda-1)n - \lambda + 1}} = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3} + 3}} \quad \text{— (1.13)}$$

2. A Prime Number Between (λ-1)n and λn when 5 ≤ λ ≤ 7 and n ≥ λ-2

Proposition 1: For $n \geq 36$ and $5 \leq \lambda \leq 7$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.1)**

Referring to **(1.13)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, we have

$$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \textbf{— (2.2)}$$

Let $f_1(x) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where x is a real number, the variable, and λ is a constant at one of the 3 integers from 5 to 7.

$f_1{}'(x) = f_1(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_1(x) \cdot f_2(x)$ where

$f_2(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}$

$f_2{}'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}\,ln(x)}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 0$ and $\lambda > 0$. Thus, $f_2(x)$ is a strictly increasing function.

When $x = 36$ and $\lambda = 5$, $f_2(x) = ln\left(\frac{5}{4}\right) + ln\left(\frac{5}{5-1}\right)^{5-1} - \frac{\sqrt{5}\,(ln(36)+ln(5)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.5859 > 0.$

When $x = 36$ and $\lambda = 6$, $f_2(x) = ln\left(\frac{6}{4}\right) + ln\left(\frac{6}{6-1}\right)^{6-1} - \frac{\sqrt{6}\,(ln(36)+ln(6)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 0.7155 > 0.$

When $x = 36$ and $\lambda = 7$, $f_2(x) = ln\left(\frac{7}{4}\right) + ln\left(\frac{7}{7-1}\right)^{7-1} - \frac{\sqrt{7}\,(ln(36)+ln(7)+2)}{6\sqrt{36}} - \frac{3}{36} \approx 08522 > 0.$

Since $f_2(x) > 0$ when $x = 36$ and $5 \leq \lambda \leq 7$, and since $f_2(x)$ is a strictly increasing function, then when $x \geq 36$ and $5 \leq \lambda \leq 7$, we have $f_2(x) > 0$. **— (2.3)**

Since when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 0$ and $f_2(x) > 0$, then $f_1{}'(x) = f_1(x) \cdot f_2(x) > 0$. Thus, when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x)$ is a strictly increasing function. $f_1(x+1) > f_1(x)$. **— (2.4)**

When $\lambda = 5$ and $x = 36$, $f_1(x) = \frac{50 \cdot \left(\left(\frac{5}{4}\right) \cdot \left(\frac{5}{5-1}\right)^{5-1}\right)^{(36-1)}}{(180)^{\frac{\sqrt{180}}{3}+3}} = \frac{4.5522\text{E}+18}{7.1073\text{E}+16} > 1.$

When $\lambda = 6$ and $x = 36$, $f_1(x) = \frac{72 \cdot \left(\left(\frac{6}{4}\right) \cdot \left(\frac{6}{6-1}\right)^{6-1}\right)^{(36-1)}}{(216)^{\frac{\sqrt{216}}{3}+3}} = \frac{7.5378\text{E}+21}{2.7530\text{E}+18} > 1.$

When $\lambda = 7$ and $x = 36$, $f_1(x) = \frac{98 \cdot \left(\left(\frac{7}{4}\right) \cdot \left(\frac{7}{7-1}\right)^{7-1}\right)^{(36-1)}}{(252)^{\frac{\sqrt{252}}{3}+3}} = \frac{3.6007\text{E}+24}{8.1511\text{E}+19} > 1.$

Referring to **(2.4)**, when $x \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(x) > 1$.

Let $x = n$, then when $n \geq 36$ and $5 \leq \lambda \leq 7$, $f_1(n) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(2.2)**, when $n \geq 36$ and $5 \leq \lambda \leq 7$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.5)**

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n + 1 = \sqrt{n^2 + 2n + 1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, $v_p = \sum_{i=1}^{\lambda-2}(i - i) = 0$ when any p in $\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$

Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$.

Therefore, when $n \geq 36$ and $5 \leq \lambda \leq 7$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$.

— (2.6)

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(2.6)** at last one of these two parts is greater than 1.

When $n \geq 36$ and $5 \leq \lambda \leq 7$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (2.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (2.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor in $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When a factor in $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{36}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number p such that $(\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{36}{i+1}$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$ > 1, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ > 1. **— (2.9)**

Referring to **(2.8)** and **(2.9)**, when n ≥ 36 and 5 ≤ λ ≤ 7, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$ ≥ 1, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ > 1. Thus, referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n$ < p ≤ λn. **— (2.10)**

In conclusion from **(2.5)**, **(2.7)**, **(2.10)**, when n ≥ 36 and 5 ≤ λ ≤ 7, then $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ > 1. When $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ > 1, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$ > 1, and there exists at least a prime number p such that $(\lambda-1)n$ < p ≤ λn. Thus, **Proposition 1** is proven.

Proposition 2: For 35 ≥ n ≥ λ −2 and 5 ≤ λ ≤ 7, there exists at least a prime number p such that (λ −1)n < p ≤ λn. **— (2.11)**

We use tables to prove **(2.11)**. **Table 1**, **Table 2**, and **Table 3** show that when λ = 5, 6, and 7, **Proposition 2** is correct. Thus, **(2.11)** is valid.

Table 1. When λ = 5 and 3 ≤ n ≤ 35, a prime number exists in the range of 4n < p ≤ 5n

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p	13	17	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
p	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	

Table 2. When λ = 6 and 4 ≤ n ≤ 35, a prime number exists in the range of 5n < p ≤ 6n

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
p	23	29	31	37	41	47	53	59	61	67	71	79	83	89	97	101
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
p	103	107	113	127	131	137	139	149	151	157	163	167	173	179	181	191

Table 3. When λ = 7 and 5 ≤ n ≤ 35, a prime number exists in the range of 6n < p ≤ 7n

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p	31	37	43	53	59	61	67	73	79	89	97	101	103	109	127	131
n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
p	137	149	151	157	163	167	173	179	181	191	193	197	211	223	227	

Combining **(2.1)** and **(2.11)**, we have proven that 5 ≤ λ ≤ 7 and n ≥ λ −2, there exists at least a prime number p such that (λ −1)n < p ≤ λn. **— (2.12)**

3. A Prime Number Between $(\lambda$-1)n and λn when 8 ≤ λ ≤ 25 and $n \geq \lambda$-2

Proposition 3: For $n \geq 24$ and $8 \leq \lambda \leq 25$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.1)**

Referring to **(1.13)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, we have

$$\Gamma_{\lambda n \geq p > n}\{\tfrac{(\lambda n)!}{((\lambda-1)n)!}\} > \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \quad \textbf{— (3.2)}$$

Let $f_3(x) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(x-1)}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}}$ where x is a real number, the variable, and λ is a constant at one of the 18 integers from 8 to 25.

$f_3'(x) = f_3(x) \cdot \left(ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x}\right) = f_3(x) \cdot f_4(x)$ where

$$f_4(x) = ln\left(\frac{\lambda}{4}\right) + ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\,(ln(x)+ln(\lambda)+2)}{6\sqrt{x}} - \frac{3}{x} \quad \textbf{— (3.2)}$$

$f_4'(x) = \frac{\sqrt{\lambda}\,ln(\lambda)}{12x\sqrt{x}} + \frac{\sqrt{\lambda}}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 0$ and $\lambda > 0$. Thus, $f_4(x)$ is a strictly increasing function.

We now calculate the $f_4(x)$ values and list them in **Table 4** for x = 24 and λ = 8, 9, 10, … 25.

Table 4. When x = 24 and λ from 8 to 25, $f_4(x) > 0$

λ	8	9	10	11	12	13	14	15	16
$f_4(x)$	0.805	0.876	0.935	0.985	1.028	1.064	1.096	1.124	1.148
λ	17	18	19	20	21	22	23	24	25
$f_4(x)$	1.168	1.186	1.202	1.215	1.227	1.237	1.246	1.253	1.259

Table 4 shows that when x = 24 and λ from 8 to 25, $f_4(x) > 0$. Since $f_3(x) > 0$ and $f_4(x) > 0$, thus when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_3'(x) = f_3(x) \cdot f_4(x) > 0$. Thus, under these conditions, $f_3(x)$ is a strictly increasing function, and $f_3(x+1) > f_3(x)$. **— (3.3)**

We now calculate the $f_3(x)$ values and list them in **Table 5** for x = 24 and λ = 8, 9, 10, … 25.

Table 5. When x = 24 and λ from 8 to 25, $f_5(x) > 1$

λ	8	9	10	11	12	13	14	15	16
$f_5(x)$	9.366	19.132	31.150	42.517	50.475	53.571	51.866	46.527	39.386
λ	17	18	19	20	21	22	23	24	25
$f_5(x)$	31.212	23.760	17.383	12.287	8.421	5.633	3.679	2.536	1.481

Table 5 shows when x = 24 and λ from 8 to 25, $f_5(x) > 1$. Since $f_3(x+1) > f_3(x)$, when $x \geq 24$ and $8 \leq \lambda \leq 25$, $f_5(x) > 1$. **— (3.4)**

Let $x = n$, then when $n \geq 24$ and $8 \leq \lambda \leq 25$, $f_3(n) = \dfrac{2\lambda^2 \cdot \left(\left(\frac{\lambda}{4}\right) \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1$.

Thus, referring to **(3.2)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.5)**

Thus, referring to **(1.3)**, there exists at least a prime number p such that $n < p \leq \lambda n$.

Since $n > \lambda - 2$, in $\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$, $p \geq n + 1 = \sqrt{n^2 + 2n + 1} > \sqrt{(n+2)n} > \lfloor\sqrt{\lambda n}\rfloor$.

Referring to **(1.8)**, we have $0 \leq v_p\left(\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \leq R(p) \leq 1$.

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} =$

$= \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right)$, $v_p = \sum_{i=1}^{\lambda-2}(i - i) = 0$ when any p in $\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}$.

Thus, referring to **(1.2)**, $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)n}{i} \geq p > \frac{\lambda n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$.

Therefore, when $n \geq 24$ and $8 \leq \lambda \leq 25$,

$\Gamma_{\lambda n \geq p > n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$.

— (3.6)

From **(1.1)**, $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} \geq 1$ and $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, and in **(3.6)** at last one of these two parts is greater than 1.

When $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then referring to **(1.3)**, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.7)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) = 1$, then $\Gamma_{\lambda n \geq p > (\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.8)**

If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then at least one factor in $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When a factor in $\Gamma_{\frac{\lambda n}{i+1} \geq p > \frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, let $y_{i+1} = \frac{n}{i+1}$, then $y_{i+1} \geq \frac{24}{i+1}$. We have

$\Gamma_{\lambda y_{i+1} \geq p > (\lambda-1)y_{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, when $y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1}$

Since $n > y_{i+1} \geq \frac{24}{i+1}$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, If $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. **— (3.9)**

Referring to **(3.8)** and **(3.9)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, if $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda n}{i+1}\geq p>\frac{(\lambda-1)n}{i+1}}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\}\right) \geq 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$. Thus, referring to **(1.3)**, there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. **— (3.10)**

In conclusion from **(3.5)**, **(3.7)**, **(3.10)**, when $n \geq 24$ and $8 \leq \lambda \leq 25$, then $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$.

When $\Gamma_{\lambda n\geq p>n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, then $\Gamma_{\lambda n\geq p>(\lambda-1)n}\{\frac{(\lambda n)!}{((\lambda-1)n)!}\} > 1$, and there exists at least a prime number p such that $(\lambda-1)n < p \leq \lambda n$. Thus, **Proposition 3** is proven.

Proposition 4: For $23 \geq n \geq \lambda - 2$ and $8 \leq \lambda \leq 25$, there exists at least a prime number p such that $(\lambda - 1)n < p \leq \lambda n$. **— (3.11)**

We use tables to prove **(3.11)**. **Table 6**, **Table 7**, and **Table 8** show that when $8 \leq \lambda \leq 25$, **Proposition 4** is correct. Thus, **(3.11)** is valid.

Table 6. When $8 \leq \lambda \leq 11$ and $\lambda - 2 \leq n \leq 23$, a prime number between $(\lambda - 1)n$ and λn

	n	6	7	8	9	10	11	12	13	14	
	$7n$	42	49	56	63	70	77	84	91	98	
	p	**47**	**53**	**59**	**67**	**73**	**83**	**89**	**97**	**101**	$\lambda = 8$
	$8n$	48	56	64	72	80	88	96	104	112	
$\lambda = 9$	p		**61**	**71**	**79**	**83**	**97**	**101**	**107**	**113**	
	$9n$		63	72	81	90	99	108	117	126	
	p			**73**	**83**	**97**	**101**	**109**	**127**	**131**	$\lambda = 10$
	$10n$			80	90	100	110	120	130	140	
$\lambda = 11$	p				**97**	**103**	**113**	**127**	**139**	**151**	
	$11n$				99	110	121	132	143	154	
	n	15	16	17	18	19	20	21	22	23	
	$7n$	105	112	119	126	133	140	147	154	161	
	p	**107**	**113**	**127**	**131**	**137**	**149**	**151**	**157**	**163**	$\lambda = 8$
	$8n$	120	128	136	144	152	160	168	176	184	
$\lambda = 9$	p	**127**	**131**	**139**	**149**	**157**	**167**	**173**	**179**	**191**	
	$9n$	135	144	153	162	171	180	189	198	207	
	p	**137**	**151**	**163**	**167**	**181**	**191**	**193**	**199**	**211**	$\lambda = 10$
	$10n$	150	160	170	180	190	200	210	220	230	
$\lambda = 11$	p	**157**	**167**	**179**	**191**	**197**	**211**	**223**	**227**	**233**	
	$11n$	165	176	187	198	209	220	231	242	253	

Table 7. When 12 ≤ λ ≤ 15 and λ −2 ≤ n ≤ 23, a prime number between (λ −1)n and λn

	$\boldsymbol{n}$	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	$\mathbf{11}\boldsymbol{n}$	110	121	132	143	154	165	176	187	198	209	220	231	242	253
$\boldsymbol{\lambda} = 12$	$\boldsymbol{p}$	**113**	**127**	**137**	**149**	**157**	**167**	**181**	**193**	**199**	**223**	**229**	**239**	**257**	**269**
	$\mathbf{12}\boldsymbol{n}$	120	132	144	156	168	180	192	204	216	228	240	252	264	276
$\boldsymbol{\lambda} = 13$	$\boldsymbol{p}$		**139**	**151**	**163**	**173**	**183**	**197**	**211**	**227**	**233**	**241**	**263**	**271**	**281**
	$\mathbf{13}\boldsymbol{n}$		143	156	169	182	195	208	221	234	247	260	273	286	299
$\boldsymbol{\lambda} = 14$	$\boldsymbol{p}$			**167**	**179**	**191**	**199**	**223**	**223**	**239**	**257**	**269**	**277**	**293**	**307**
	$\mathbf{14}\boldsymbol{n}$			168	182	196	210	224	238	252	266	280	294	308	322
$\boldsymbol{\lambda} = 15$	$\boldsymbol{p}$				**191**	**199**	**211**	**229**	**239**	**263**	**271**	**283**	**307**	**311**	**331**
	$\mathbf{15}\boldsymbol{n}$				195	210	225	240	255	270	285	300	315	330	345

Table 8. When 16 ≤ λ ≤ 25 and λ −2 ≤ n ≤ 23, a prime number between (λ −1)n and λn

	$\boldsymbol{n}$	14	15	16	17	18	19	20	21	22	23	
	$\mathbf{15}\boldsymbol{n}$	210	225	240	255	270	285	300	315	330	345	
$\boldsymbol{\lambda} = 16$	$\boldsymbol{p}$	**223**	**227**	**241**	**257**	**277**	**293**	**313**	**317**	**331**	**347**	
	$\mathbf{16}\boldsymbol{n}$	224	240	256	272	288	304	320	336	352	368	
	$\boldsymbol{p}$		**251**	**263**	**281**	**293**	**307**	**337**	**349**	**353**	**373**	$\boldsymbol{\lambda} = 17$
	$\mathbf{17}\boldsymbol{n}$		255	272	289	306	323	340	357	374	391	
$\boldsymbol{\lambda} = 18$	$\boldsymbol{p}$			**277**	**293**	**311**	**331**	**347**	**359**	**379**	**397**	
	$\mathbf{18}\boldsymbol{n}$			288	306	324	342	360	378	396	414	
	$\boldsymbol{p}$				**307**	**337**	**349**	**373**	**383**	**397**	**419**	$\boldsymbol{\lambda} = 19$
	$\mathbf{19}\boldsymbol{n}$				323	342	361	380	399	418	437	
$\boldsymbol{\lambda} = 20$	$\boldsymbol{p}$					**347**	**367**	**389**	**401**	**421**	**439**	
	$\mathbf{20}\boldsymbol{n}$					360	380	400	420	440	460	
	$\boldsymbol{p}$						**283**	**409**	**431**	**443**	**461**	$\boldsymbol{\lambda} = 21$
	$\mathbf{21}\boldsymbol{n}$						399	420	441	462	483	
$\boldsymbol{\lambda} = 22$	$\boldsymbol{p}$							**433**	**449**	**463**	**487**	
	$\mathbf{22}\boldsymbol{n}$							440	462	484	506	
	$\boldsymbol{p}$								**467**	**491**	**509**	$\boldsymbol{\lambda} = 23$
	$\mathbf{23}\boldsymbol{n}$								483	506	529	
$\boldsymbol{\lambda} = 24$	$\boldsymbol{p}$									**521**	**541**	
	$\mathbf{24}\boldsymbol{n}$									528	552	
	$\boldsymbol{p}$										**563**	$\boldsymbol{\lambda} = 25$
	$\mathbf{25}\boldsymbol{n}$										575	

Combining **(3.1)** and **(3.11)**, we have proven that 8 ≤ λ ≤ 25 and n ≥ λ −2, there exists at least a prime number p such that (λ −1)n < p ≤ λn. **— (3.12)**

4. A Prime Number Between kn and $(k+1)n$

From **(1.4),** for every positive integer n, there exists at least a prime number p such that $2n < p \le 3n$.

From **(1.5),** for every integer $n > 1$, there exists at least a prime number p such that $3n < p \le 4n$.

From **(2.12)**, when $5 \le \lambda \le 7$ and $n \ge \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

From **(3.12)**, when $8 \le \lambda \le 25$ and $n \ge \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

From **(1.9),** for $n \ge (\lambda - 2) \ge 24$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

Combining **(1.4)**, **(1.5)**, **(2.12)**, **(3.12)**, and **(1.9)**, we have that for $n \ge \lambda - 2 \ge 1$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$. **— (4.1)**

Let $k = \lambda - 1$, **(4.1)** becomes that for $n \ge k - 1 \ge 1$, there exists at least a prime number p such that $kn < p \le (k+1)n$. **— (4.2)**

Since the Bertrand-Chebyshev's theorem states that for any positive integer n, there is always a prime number p such that $n < p \le 2n$, we can state that for two positive integers $n \ge 1$ and $k \ge 1$, if $n \ge k - 1$, then there always exists at least a prime number p such that $kn < p \le (k+1)n$. The Bertrand's postulate / Chebyshev's theorem is the special case when $k = 1$.

5. References

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