Gravidynamics of an Affine Connection on a Minkowski Background

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Abstract

Modern gravitation theory is couched in Riemannian geometry. In this paper, a post-Riemannian formalism is constructed based on a minimalistic set of modifications and principles (such as that of manifest covariance) and suggested as the framework for a classical alternative to General Relativity which, notably, can be formulated in Minkowski spacetime. Following the purely geometrical exposition, a Lagrangean quadratic in the gravitational field strengths is considered, and some of the properties of the resulting field equations, including their limiting Newtonian behavior, are analyzed in brief, as well as continuum phenomenology. Some tentative arguments are presented towards a description of the coupling of matter with gravitation within the proposed formalism, and a few issues with the paradigm are discussed.

1 Introduction

The ongoing search for a fully-developed quantum theory of gravity, probably the most renowned open problem in theoretical physics, has by now become a staple to a broad audience of experts and nonexperts alike, as it resisted over a century of attempts at quantizing it [1]; however, in spite of its prominence, it also seems to have overshadowed a number of other issues of a purely classical nature, as illustrated by a fairly recent (meta)list published by Coley [2], which includes over seventy (!) open problems rooted in plain General Relativity (GR). The author comments on the situation thus: "GR problems have typically been under-represented in lists of problems in mathematical physics [...], perhaps due to their advanced technical nature"; yet at the same time, some of the problems listed include examples such as "Show that a solution of the linearised (about Minkowski space) Einstein equations is close to a (non-flat) exact solution" and "Prove rigorously the existence of a limit in which solutions of the Einstein equations reduce to Newtonian spacetimes" (RB19 and RB21 respec.), which are surprisingly basic considering the level of maturity of the field.

Aside from unsolved technical problems, however, we can still point to known (and often well-established) features of GR such as the convoluted treatment of singularities, and causal and spinor structures [3], the possibility of closed timelike curves [4], and the lack of a true local conservation law [5] - features that strike one more like bugs. It is not my point to deny that these (perceived) bugs can be handled within the current state of the art - indeed, there is extensive literature concerned with each of these issues; rather, it is to suggest that the conceptual and technical problems that we're faced with even at the classical level are due to the idiosyncractic mathematical formulation of GR that has dominated the field since its inception in 1915 - an overly formal one that tends to alienate a larger audience of physicists, but which, in light of modern developments in our understanding of differential geometry, might be exchanged with a simpler, more intuitive apparatus without incurring any loss in any of the fundamental structures or ideas of physical import.

It may be briefly pointed, from a historical perspective, that the conventional view of the physical nature of the metric seems to have taken shape ca. 1907-1912 - i.e., between the publication of the famous special-relativistic review that introduced the equivalence principle, and the fateful reencounter of Einstein and Grossman in Zürich; in particular, Born's work on the rigid body seems to have had a major theoretical influence in the development of GR, both directly and indirectly [6]. After General Relativity was established, it enjoyed empirical prestige via its prediction of several effects such as the bending of light by massive bodies - in fact, so much so that by the mid-1970s, alternatives to the theory weren't taken very seriously except as possible PPN foils to GR [8], or something of the sort. At the same time, however, a countercurrent of ideas inspired by the then-nascent gauge formalism gave rise to a particular family of theories under the umbrella metric-affine (gauge) gravity (MAG), which is not so much an *alternative* to GR as an *augmentation* thereof - in which not only the metric, but also the linear connection and the coframe (or alternatively the soldering) are taken as dynamical variables [9–11].

Although we recognize that the MAG programme introduced important insights as to the nature of the gravitational field, it is still plagued by difficulties perhaps best expressed by Mielke [11]: "With reference to the proper foundation of a gauge theory of gravity, however, there is no absolute agreement among the members of the scientific community. It is the incorporation of a *dynamical geometry* as realized by Einstein via the pseudo-Riemannian metric that seems to prevent a direct transfer of the Yang–Mills gauge program." Here we see a sharp conflict: our best theory for all the non-gravitational interactions is not only successfully quantized, but it's also *gauged* - whereas our best theory for gravitational interactions is neither; could this be a clue to explain the continued clash between GR and QFT - and possibly guide us to a better approach?

It was thinking on those lines that lead to the proposal here that this geometrodynamical (i.e. "gravity-as-metric") view is fundamentally misguided, in that the metric need not be taken as a dynamical degree of freedom, and may be satisfactorily separated from the main machinery of the linear connection; such an arrangement not only brings about mathematical simplifications, but is also rich with physical implications, such as the restoration of the old "gravity-as-force" outlook - a viewpoint we will refer to as *gravidynamics*, to distinguish it from the previous one. To the best of my knowledge, however, no attempt has ever been definitely forwarded in the literature to pursue such a theory (no doubt due to the said prestige accumulated by GR over the years, which made investigators wary of tinkering too much with it); the purpose of the present work, thus, is towards filling that gap.

In section II, we review some basic concepts of tensor calculus and develop an argument leading to the introduction and interpretation of the covariant derivative; in section III, we augment this covariant machinery by the introduction of some tensorial objects and discuss several different aspects of their structure. Section IV introduces physics by means of a definite Lagrangean, as well as a phenomenological exploration of relativistic hydrodynamics in the present formalism. Section V, then, mentions several loose ends preventing the present treatment from being a complete theory of gravity, highlighting some challenges - particularly related to (minimal) coupling.

2 Basic Tensor Calculus

Since the reader is assumed to already have some familiarity with tensors and the relevant multilinear algebra, as well as exterior calculus, we will for the most part skip several technical definitions[1]; for our purposes, it will suffice to recall just a few. Given a n-dimensional manifold M, with corresponding tangent space $T(M)$ and cotangent space $T^*(M)$; the meaning of these objects is readily intuited from observing that, for any point $p \in M$, $T_p(M)$ forms a n-dimensional vector space, and $T_p^*(M)$ is its dual. Vectors belonging to $T_p(M)$ can be expanded in terms of a basis $\{\mathbf{e}_i\}$ as $\mathbf{v} := v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + ... + v^n\mathbf{e}_n \equiv v^i\mathbf{e}_i$, with the v^i being the components of the vector, and the familiar Einstein summation convention is used in the second equality; likewise for covectors, which are expanded as $\mathbf{c} := c_i\mathbf{e}^i$, in terms of a (co)basis $\{\mathbf{e}^i\}$; furthermore, the outer product $\otimes$ allows us to write down the general expression of a tensor (that itself may be defined over any $p \in M$ or the whole M) as

$$\begin{aligned}\mathbf{T} &= T^{i_1...i_p}{}_{j_1...j_q}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q} \\ &=: T^{i_1...i_p}{}_{j_1...j_q} \bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m}, p, q \in \mathbb{N}\end{aligned}$$

We say that $\mathbf{T}$ is the tensor itself, the $T^{i_1...i_p}{}_{j_1...j_q}$ are its components, and the $\{\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q}\}$ are the generators of its basis[2]. It displays what is sometimes coloquially referred to as the "tensorial property" of transforming under chartwise well-defined coordinate transformations $(x')^i :=$

[1] For a brush-up on these mathematical preliminaries, see, e.g., [12–14].

[2] In physics, we write differential equations involving tensors simply because of their nice, convenient, space-saving properties; as such, the ordering of each *individual* outer product $\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_p} \otimes \mathbf{e}^{j_1} \otimes ... \otimes \mathbf{e}^{j_q}$ may, *a priori*, always be taken to be "normal-ordered" in the manner shown here.

$x^{i'} = x^{i'}\left(x^i\right)$ over M as follows:

$$T^{i'_1 \dots i'_p}{}_{j'_1 \dots j'_q} = J^{i'_1}{}_{i_1} \dots J^{i'_p}{}_{i_p} J^{j_1}{}_{j'_1} \dots J^{j_q}{}_{j'_q} T^{i_1 \dots i_p}{}_{j_1 \dots j_q} \tag{1}$$

with a similar expression holding for its generators; here we denote the components of the Jacobian matrix of the transformation $x^{i'} = x^{i'}\left(x^i\right)$ as $J^{i'_n}{}_{i_n} := \frac{\partial x^{i'_n}}{\partial x^{i_n}}$, whereas $J^{j_m}{}_{j'_m} := \frac{\partial x^{j_m}}{\partial x^{j'_m}}$ are the components of the inverse matrix, as easily checked using the chain rule of ordinary calculus. Strictly speaking, this is valid only for coordinate bases $\left\{\mathbf{e}_i = \frac{\partial}{\partial \mathbf{x}^i}, \mathbf{e}^i = d\mathbf{x}^i\right\}$, but the above is readily extended to noncoordinate bases as well, which we denote as $\left\{\mathbf{e}_{\tilde{i}} := e^i_{\tilde{i}} \frac{\partial}{\partial \mathbf{x}^i}, \mathbf{e}^{\tilde{i}} := e^{\tilde{i}}_i d\mathbf{x}^i\right\}$, with the $\left\{e^i_{\tilde{i}}, e^{\tilde{i}}_i\right\}$ assumed invertible ($e^k_{\tilde{i}} e^{\tilde{j}}_k = \delta^{\tilde{j}}_{\tilde{i}}, e^{\tilde{k}}_i e^j_{\tilde{k}} = \delta^j_i$). In this paper, unless explicitly stated, coordinate bases are always assumed when performing explicit computations - otherwise, we shall use the tilde notation, to emphasize that the bases in question are specifically noncoordinate (i.e., $e^i_{\tilde{i}} \neq \delta^i_{\tilde{i}}$).

This tensorial property, which may more properly be called coordinate-invariance, or *covariance*, makes tensors natural objects for the mathematical description of physical quantities; however, as easily checked from 1, partial derivatives $\partial_{k'} T^{i_1 \dots i_p}{}_{j_1 \dots j_q} = \frac{\partial}{\partial x^{k'}} T^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ are, in general, nontensorial - which poses a problem for the use of tensors in differential equations. The problem is simply disposed of in the case of a Riemann space $(M, \mathbf{g})$; this new tensor $\mathbf{g} = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j$ we call the *metric*, and it has basically three uses: first, is it can be used to define a notion of distance in the manifold; this is easily illustrated with the special case of semi-Euclidian metrics (i.e., metrics that can be put as $g_{ij} = \eta_{ij} = diag\left[\pm 1, \pm 1, ..., \pm 1\right]$ in some global chart): given two points $\mathbf{x}, \mathbf{y}$, the distance between them may be written as $d\left(\mathbf{x}, \mathbf{y}\right) = \sqrt{\eta_{ij}\left(x^i - y^i\right)\left(x^j - y^j\right)}$ - which indeed corresponds to our ordinary notion of length for strictly Euclidian metrics (i.e., equal to $diag\left[+1, +1, ..., +1\right]$ in some global chart). A second one is that, along with its inverse, $\mathbf{g}^{-1} = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, it allows for raising and lowering indexes; for example:

$$T_i{}^{kl\dots}{}_{j\dots} = g_{in} g^{mk} T^n{}_{m}{}^{l\dots}{}_{j\dots}$$

A third use will be the construction of the correcting factor we need, by the introduction of the operation defined by

$$\begin{aligned}\left(T^{i_1 \dots i_p}{}_{j \dots j_q}\right)_{,k} \quad : \quad &= \nabla_k T^{i_1 \dots i_p}{}_{j_1 \dots j_q} := \partial_k T^{i_1 \dots i_p}{}_{j_1 \dots j_q} + \begin{Bmatrix} i_1 \\ ik \end{Bmatrix} T^{i \dots i_p}{}_{j_1 \dots j_q} + \dots \\ &\dots + \begin{Bmatrix} i_p \\ ik \end{Bmatrix} T^{i_1 \dots i}{}_{j_1 \dots j_q} - \left(\begin{Bmatrix} j \\ j_1 k \end{Bmatrix} T^{i_1 \dots i_p}{}_{j \dots j_q} + \dots + \begin{Bmatrix} j \\ j_q k \end{Bmatrix} T^{i_1 \dots i_p}{}_{j_1 \dots j}\right)\end{aligned} \tag{2}$$

where the Christoffel symbol of the second kind $\left\{ {l \atop ij} \right\}$ is related to the symbol of the first kind $\{ij|k\}$ by

$$\begin{Bmatrix} l \\ ij \end{Bmatrix} := g^{lk}\left\{k|ij\right\} := \frac{1}{2} g^{lk}\left(\frac{\partial}{\partial x^j} g_{ik} + \frac{\partial}{\partial x^i} g_{kj} - \frac{\partial}{\partial x^k} g_{ij}\right) \tag{3}$$

and, contrary to common use, we employ a comma rather than a semicolon to denote the ∇-operation, for reasons that will be clear later on. After effecting a change of coordinates $g_{i'j'} = g_{ml} J^m{}_{i'} J^l{}_{j'}$ from some generic curvilinear system to another one, we can show the following by straightforward manipulation (mod standard analytical conditions):

$$\begin{aligned}
\left\{ {l' \atop i'j'} \right\} &= \frac{1}{2} g^{l'k'} \left(\frac{\partial}{\partial x^{j'}} g_{i'k'} + \frac{\partial}{\partial x^{i'}} g_{k'j'} - \frac{\partial}{\partial x^{k'}} g_{i'j'} \right) \\
&= J^{l'}{}_l J^i{}_{i'} J^j{}_{j'} \left\{ {l \atop ij} \right\} + \frac{1}{2} J^{l'}{}_l \left(J^j{}_{j'} \frac{\partial}{\partial x^j} J^l{}_{i'} + J^i{}_{i'} \frac{\partial}{\partial x^i} J^l{}_{j'} \right) \\
&\quad + \frac{1}{2} g^{l'k'} g_{ij} \left[J^i{}_{i'} \left(J^k{}_{j'} \frac{\partial}{\partial x^k} J^j{}_{k'} - J^k{}_{k'} \frac{\partial}{\partial x^k} J^j{}_{j'} \right) + J^j{}_{j'} \left(J^k{}_{i'} \frac{\partial}{\partial x^k} J^i{}_{k'} - J^k{}_{k'} \frac{\partial}{\partial x^k} J^i{}_{i'} \right) \right] \\
&= J^{l'}{}_l J^i{}_{i'} J^j{}_{j'} \left\{ {l \atop ij} \right\} + J^{l'}{}_l J^j{}_{j'} \frac{\partial}{\partial x^j} J^l{}_{i'}
\end{aligned}$$

where in the last equality use was made of the analytical property $\frac{\partial}{\partial x^{i'}} J^l{}_{j'} = \frac{\partial}{\partial x^{j'}} J^l{}_{i'}$ to effect the simplification. This well-known transformation rule not only shows (explicitly) that the Christoffel symbols does not form a tensor, but also allow us to see (and prove by induction) why the ∇-operation works: it's because the extra piece $J^{l'}{}_l J^j{}_{j'} \frac{\partial}{\partial x^j} J^l{}_{i'}$ in the last equality exactly cancels out the one that appears due to the derivation of the tensor components.

At this juncture, one must have a very clear picture of what has been established, and for why: that is, in order to maintain the covariance of our physical theories, we introduced a "generalized partial derivative", or *covariant derivative*, for the *exclusive* purpose of bookkeeping coordinate changes in tensor components; this requirement, per se, has nothing to do with physics of any kind - it's just part of our *a priori* mathematical framework - in the same vein of number sets, algebraic structures, and so on.

As for the properties of the Christoffels, mere inspection of 3 shows that $\left\{ {l \atop ij} \right\} = \left\{ {l \atop ji} \right\}$; however, arguably more important is its metric compatibility:

$$\begin{aligned}
g_{ab,c} &= \frac{\partial}{\partial x^c} g_{ab} - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{ak} + \frac{\partial}{\partial x^a} g_{kc} - \frac{\partial}{\partial x^k} g_{ac} \right) g_{lb} - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{bk} + \frac{\partial}{\partial x^b} g_{kc} - \frac{\partial}{\partial x^k} g_{bc} \right) g_{al} = 0, \\
\delta^b_{a,c} &= \frac{\partial}{\partial x^c} \delta^b_a + \frac{1}{2} g^{bk} \left(\frac{\partial}{\partial x^c} g_{lk} + \frac{\partial}{\partial x^l} g_{kc} - \frac{\partial}{\partial x^k} g_{lc} \right) \delta^l_a - \frac{1}{2} g^{lk} \left(\frac{\partial}{\partial x^c} g_{ak} + \frac{\partial}{\partial x^a} g_{kc} - \frac{\partial}{\partial x^k} g_{ac} \right) \delta^b_l = 0
\end{aligned}$$

These relations suffice to show that $g^{ab}{}_{,c} = 0$ as well.

After the introduction of covariant differentiation, it is customary in GR and/or tensor calculus textbooks to define the Riemann-Christoffel (RC) tensor - typically in terms of transport of a vector along a closed circuit. Such an undertaking, however, is made considerably more expedient (not to mention geometrically clear) in the formalism of tensor-valued (multi)forms pioneered by Cartan [15]. With it, computation of the curvature, as well as other objects of geometrical interest, becomes quite efficient; indeed, ordinarily, this method "surpasses in efficiency every other known method for calculating the curvature 2-forms." [8].

A general tensor-valued r-form is defined as

$$\mathbf{T} := \mathbf{T}^{i_1\ldots i_p}{}_{j_1\ldots j_q} \bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} = T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r} \left(\bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \otimes \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right), p, q, r \in \mathbb{N}$$

Since this formula reduces to our previous definition of tensor for $r = 0$, we see it is a straightforward generalization of the concept. Also, it is important to note that, albeit the newly introduced $\boldsymbol{\theta}^k$ are "soldered" to the $\mathbf{e}^k$ in the sense that they transform identically under coordinate transformations (e.g., $\boldsymbol{\theta}^{\bar{k}} = \theta^{\bar{k}}_k \boldsymbol{\theta}^k \equiv e^{\bar{k}}_k \boldsymbol{\theta}^k$), nonetheless the spaces spanned by the $\boldsymbol{\theta}'s$ and $\mathbf{e}'s$ are to be treated differently - as we shall see below.

With these tensor-valued forms we define the covariant exterior derivative $\mathbf{d}$ by the operation

$$\begin{aligned}
\mathbf{dT} &= \left(\nabla_l T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r} \right) \left(\boldsymbol{\theta}^l \wedge \bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \otimes \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right) \qquad (4) \\
&\equiv \left(\mathbf{d}T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r} \right) \wedge \left(\bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \otimes \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right) \\
&\quad + \left(T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r} \right) \left[\mathbf{d} \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right) \right] \wedge \left(\bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \\
&\quad + \left(T^{i_1\ldots i_p}{}_{j_1\ldots j_q k_1\ldots k_r} \right) \left(\bigotimes_{n=1}^{p} \mathbf{e}_{i_n} \bigotimes_{m=1}^{q} \mathbf{e}^{j_m} \right) \otimes \left[\mathbf{d} \left(\bigwedge_{s=1}^{r} \boldsymbol{\theta}^{k_s} \right) \right]
\end{aligned}$$

In order to satisfy the second equality, one defines a 1-form

$$\boldsymbol{\gamma}^b_a := \gamma^b_{ak} \boldsymbol{\theta}^k \qquad (5)$$

that in a coordinate basis is given by $\gamma^b_{ak} = \left\{ {b \atop ak} \right\}$, so that

$$\begin{aligned}
\mathbf{de}_a \quad &: \quad = \boldsymbol{\gamma}^b_a \otimes \mathbf{e}_b, &(6) \\
\mathbf{de}^b \quad &: \quad = -\boldsymbol{\gamma}^b_a \otimes \mathbf{e}^a, &(7) \\
\mathbf{d}\boldsymbol{\theta}^c \quad &\equiv \quad -\boldsymbol{\gamma}^c_i \wedge \boldsymbol{\theta}^i &(8)
\end{aligned}$$

The last relation deserves some comment; in a coordinate basis, it is easily seen to be true: from Poincaré's lemma, we have $\mathbf{d}^2\boldsymbol{\theta}^c \equiv \mathbf{0}$ - but it is also true that $\boldsymbol{\gamma}^c_i \wedge \boldsymbol{\theta}^i = \left\{ {c \atop ik} \right\} \boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^i \equiv \mathbf{0}$, because of $\left\{ {c \atop ik} \right\} = \left\{ {c \atop ki} \right\}$. To see that it holds even in noncoordinate bases, we will introduce components $\gamma^{\tilde{l}}_{\tilde{i}\tilde{j}}$ so that

$$\mathbf{dv} = \left[\mathbf{e}_j \left(v^l \right) + \gamma^l_{ij} v^i \right] \boldsymbol{\theta}^j \otimes \mathbf{e}_l \equiv \left[\mathbf{e}_{\tilde{j}} \left(v^{\tilde{l}} \right) + \gamma^{\tilde{l}}_{\tilde{i}\tilde{j}} v^{\tilde{i}} \right] \boldsymbol{\theta}^{\tilde{j}} \otimes \mathbf{e}_{\tilde{l}}$$

is covariant (and where the notation $\mathbf{e}_{\tilde{\imath}}\left(f\right) := e^i_{\tilde{\imath}}\frac{\partial}{\partial x^i}f$ was introduced). This simplifies to

$$\gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}} = e^{\tilde{l}}_l e^i_{\tilde{\imath}} e^j_{\tilde{\jmath}} \gamma^l_{ij} + e^{\tilde{l}}_l e^j_{\tilde{\jmath}} \left(\partial_j e^l_{\tilde{\imath}}\right)$$

from which we get the commutator

$$\gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}} - \gamma^{\tilde{l}}_{\tilde{\jmath}\tilde{\imath}} = e^{\tilde{l}}_l \left[e^j_{\tilde{\jmath}}\left(\partial_j e^l_{\tilde{\imath}}\right) - e^i_{\tilde{\imath}}\left(\partial_i e^l_{\tilde{\jmath}}\right)\right] \equiv -c^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}} \tag{9}$$

where the $c^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}}$ are identified with the structure coefficients associated with the Lie bracket

$$\left[\mathbf{e}_{\tilde{\imath}}, \mathbf{e}_{\tilde{\jmath}}\right] = e^i_{\tilde{\imath}}\frac{\partial}{\partial \mathbf{x}^i}\left(e^j_{\tilde{\jmath}}\frac{\partial}{\partial \mathbf{x}^j}\right) - e^j_{\tilde{\jmath}}\frac{\partial}{\partial \mathbf{x}^j}\left(e^i_{\tilde{\imath}}\frac{\partial}{\partial \mathbf{x}^i}\right) = \left[e^i_{\tilde{\imath}}\left(\partial_i e^l_{\tilde{\jmath}}\right) - e^j_{\tilde{\jmath}}\left(\partial_j e^l_{\tilde{\imath}}\right)\right] e^{\tilde{l}}_l \mathbf{e}_{\tilde{l}} =: c^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}}\mathbf{e}_{\tilde{l}}$$

But this just happens to match the derivative of $\boldsymbol{\theta}^{\tilde{l}}$, as well:

$$\mathbf{d}\boldsymbol{\theta}^{\tilde{l}} = \left(\partial_i e^{\tilde{l}}_l\right)\boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^l = \left(\partial_i e^{\tilde{l}}_l\right)\left(e^i_{\tilde{\imath}}\boldsymbol{\theta}^{\tilde{\imath}} \wedge e^l_{\tilde{\jmath}}\boldsymbol{\theta}^{\tilde{\jmath}}\right) \equiv -e^i_{\tilde{\imath}} e^{\tilde{l}}_l \left(\partial_i e^l_{\tilde{\jmath}}\right)\boldsymbol{\theta}^{\tilde{\imath}} \wedge \boldsymbol{\theta}^{\tilde{\jmath}} \tag{10}$$

where in the third equality we used integration by parts and the fact that $\partial_i\left(e^{\tilde{l}}_l e^l_{\tilde{\jmath}}\right) = \partial_i \delta^{\tilde{l}}_{\tilde{\jmath}} = 0$. So, comparing eqs. 9 and 10, the result 8 follows.

As a sanity check, let us compute the metric compatibility in the new formalism:

$$\begin{aligned}
\mathbf{dg} &= \left(\mathbf{d}g_{ab}\right)\otimes\mathbf{e}^a\otimes\mathbf{e}^b + g_{cb}\left(\mathbf{de}^c\right)\otimes\mathbf{e}^b + g_{ac}\mathbf{e}^a\otimes\left(\mathbf{de}^c\right) \\
&\equiv \left(\partial_c g_{ab}\right)\boldsymbol{\theta}^c\otimes\mathbf{e}^a\otimes\mathbf{e}^b - \left(g_{cb}\boldsymbol{\gamma}^c_a + g_{ac}\boldsymbol{\gamma}^c_b\right)\otimes\mathbf{e}^a\otimes\mathbf{e}^b \\
&\equiv \left(\nabla_c g_{ab}\right)\boldsymbol{\theta}^c\otimes\mathbf{e}^a\otimes\mathbf{e}^b
\end{aligned} \tag{11}$$

We thus see the consistency with our previous computation; however, since this is a tensorial operation, we're able to rewrite the exact same thing in a noncoordinate basis

$$\mathbf{0} \equiv \mathbf{dg} = \mathbf{e}_{\tilde{c}}\left(g_{\tilde{a}\tilde{b}}\right)\boldsymbol{\theta}^{\tilde{c}}\otimes\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}} - \left(g_{\tilde{c}\tilde{b}}\boldsymbol{\gamma}^{\tilde{c}}_{\tilde{a}} + g_{\tilde{a}\tilde{c}}\boldsymbol{\gamma}^{\tilde{c}}_{\tilde{b}}\right)\otimes\mathbf{e}^{\tilde{a}}\otimes\mathbf{e}^{\tilde{b}}$$

from which it follows we can perform the linear combination

$$\left\{\tilde{k}|\tilde{\imath}\tilde{\jmath}\right\} := \frac{1}{2}\left[\mathbf{e}_{\tilde{\jmath}}\left(g_{\tilde{\imath}\tilde{k}}\right) + \mathbf{e}_{\tilde{\imath}}\left(g_{\tilde{k}\tilde{\jmath}}\right) - \mathbf{e}_{\tilde{k}}\left(g_{\tilde{\imath}\tilde{\jmath}}\right)\right] = \frac{1}{2}\left[\left(\gamma_{\tilde{k}\tilde{\imath}\tilde{\jmath}} + \gamma_{\tilde{\imath}\tilde{k}\tilde{\jmath}}\right) + \left(\gamma_{\tilde{\jmath}\tilde{k}\tilde{\imath}} + \gamma_{\tilde{k}\tilde{\jmath}\tilde{\imath}}\right) - \left(\gamma_{\tilde{\jmath}\tilde{\imath}\tilde{k}} + \gamma_{\tilde{\imath}\tilde{\jmath}\tilde{k}}\right)\right] \tag{12}$$

from which, after some algebra, we recuperate the expression of the Levi-Civita (LC) connection $\gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}}$ in *any* basis:

$$g_{\tilde{k}\tilde{l}}\gamma^{\tilde{l}}_{\tilde{\imath}\tilde{\jmath}} = \gamma_{\tilde{k}\tilde{\imath}\tilde{\jmath}} = \left\{\tilde{k}|\tilde{\imath}\tilde{\jmath}\right\} + \frac{1}{2}\left(c_{\tilde{\jmath}\tilde{k}\tilde{\imath}} + c_{\tilde{\imath}\tilde{k}\tilde{\jmath}} - c_{\tilde{k}\tilde{\imath}\tilde{\jmath}}\right) \tag{13}$$

The combinations of structure coefficients in parenthesis are often called the Ricci (rotation) coefficients; by inspection, they are seen to be antisymmetric in $\tilde{k}, \tilde{\imath}$.

As shown by these examples, the properties of $\mathbf{d}$ allow us to breeze through otherwise laborious calculations - for instance:

$$\begin{aligned}
\mathbf{d}^2\left(v^a\mathbf{e}_a\right) &= \mathbf{d}\left[(\mathbf{d}v^a)\otimes\mathbf{e}_a + v^a\mathbf{d}\mathbf{e}_a\right] = \left[\left(\mathbf{d}^2v^a\right)\otimes\mathbf{e}_a - (\mathbf{d}v^a)\wedge\mathbf{d}\mathbf{e}_a\right] + \left[(\mathbf{d}v^a)\wedge\mathbf{d}\mathbf{e}_a + v^a\mathbf{d}^2\mathbf{e}_a\right] \\
&\equiv \left[\mathbf{d}\left(\partial_j v^a\boldsymbol{\theta}^j\right)\right]\otimes\mathbf{e}_a + v^b\mathbf{d}\left(\boldsymbol{\gamma}^a_b\otimes\mathbf{e}_a\right) = \left[\left(\partial_i\partial_j v^a\right)\left(\boldsymbol{\theta}^i\wedge\boldsymbol{\theta}^j\right) + v^b\left(\mathbf{d}\boldsymbol{\gamma}^a_b - \boldsymbol{\gamma}^c_b\wedge\boldsymbol{\gamma}^a_c\right)\right]\otimes\mathbf{e}_a \\
&=: v^b\mathbf{R}^a_b\otimes\mathbf{e}_a \qquad (14)
\end{aligned}$$

where in last line the RC tensor is defined - from a tensor-valued 2-form. We can check that this is indeed the same quantity from the textbooks by simply writing it explicitly:

$$\begin{aligned}
\mathbf{R}^a_b &= \left[\mathbf{d}\left(\begin{Bmatrix} a \\ bj \end{Bmatrix}\boldsymbol{\theta}^j\right)\right] - \begin{Bmatrix} c \\ bi \end{Bmatrix}\begin{Bmatrix} a \\ cj \end{Bmatrix}\left(\boldsymbol{\theta}^i\wedge\boldsymbol{\theta}^j\right) \qquad (15) \\
&= \left[\partial_i\begin{Bmatrix} a \\ bj \end{Bmatrix} - \begin{Bmatrix} c \\ bi \end{Bmatrix}\begin{Bmatrix} a \\ cj \end{Bmatrix}\right]\left(\boldsymbol{\theta}^i\wedge\boldsymbol{\theta}^j\right)
\end{aligned}$$

From the last line, we see the RC tensor is antisymmetric in i, j, meaning that, in n dimensions, it has $\frac{n^3(n-1)}{2}$ independent components; to diminish this, we can derive 11 again, and, using $\mathbf{d}^2\mathbf{e}^c = -\mathbf{d}\left(\boldsymbol{\gamma}^c_b\otimes\mathbf{e}^b\right) = -\mathbf{R}^c_b\otimes\mathbf{e}^b$, obtain the "Ricci identity"

$$\mathbf{0}\equiv\mathbf{d}^2\mathbf{g} = g_{cb}\left(\mathbf{d}^2\mathbf{e}^c\right)\otimes\mathbf{e}^b + g_{ac}\mathbf{e}^a\otimes\left(\mathbf{d}^2\mathbf{e}^c\right) = -\left(g_{cb}\mathbf{R}^c_a + g_{ac}\mathbf{R}^c_b\right)\otimes\mathbf{e}^a\otimes\mathbf{e}^b \qquad (16)$$

which lowers the components down to $\left[\frac{n(n-1)}{2}\right]^2$; we may, however, further down their number with the help of the algebraic identity, as computed from

$$\mathbf{d}^2\boldsymbol{\theta}^a = \mathbf{d}\left(\boldsymbol{\gamma}^a_b\wedge\boldsymbol{\theta}^b\right)\equiv\left(\mathbf{R}^a_b + \boldsymbol{\gamma}^c_b\wedge\boldsymbol{\gamma}^a_c\right)\wedge\boldsymbol{\theta}^b = \mathbf{R}^a_b\wedge\boldsymbol{\theta}^b - \boldsymbol{\gamma}^a_c\wedge\left(\boldsymbol{\gamma}^c_b\wedge\boldsymbol{\theta}^b\right) \qquad (17)$$

otherwise known as $R^a{}_{[bij]} = 0$. This property further reduces the remaining independent components of $R^a{}_{bij}$ down to $\left[\frac{n(n-1)}{2}\right]^2 - n\left[\frac{n(n-1)(n-2)}{3!}\right] = \frac{n^2\left(n^2-1\right)}{12}$; so, for $n = 4$, this means we've made quite the economy, going from 256 components to just 20 - not too shabby. Finally, we get the so-called Bianchi identity by a similar procedure:

$$\mathbf{d}\mathbf{R}^a_b = \mathbf{d}^2\boldsymbol{\gamma}^a_b - (\mathbf{d}\boldsymbol{\gamma}^c_b)\wedge\boldsymbol{\gamma}^a_c + \boldsymbol{\gamma}^c_b\wedge(\mathbf{d}\boldsymbol{\gamma}^a_c) = -\left(\mathbf{R}^c_b + \boldsymbol{\gamma}^d_b\wedge\boldsymbol{\gamma}^c_d\right)\wedge\boldsymbol{\gamma}^a_c + \boldsymbol{\gamma}^c_b\wedge\left(\mathbf{R}^a_c + \boldsymbol{\gamma}^d_c\wedge\boldsymbol{\gamma}^a_d\right) \qquad (18)$$

otherwise known (in this paper's notation) as $R^a{}_{b[ij,k]} = 0$; on its turn, this relation is famous as the starting point in the derivation of the Einstein tensor.

Before closing this section, a final word on notation: we can use the metric to freely lower and raise indexes and rewrite tensor-valued forms however we prefer, but we have to be careful when nontensorial objects are involved; for instance, in the case of the RC tensor, its defining expression is given by the structural eq. 14 - but, *a posteriori*, we may introduce $\mathbf{R}_{ab} = g_{ac}\mathbf{R}^c_b$, etc. As another example, consider the covariant derivative $\mathbf{U} = \mathbf{du}$ of a vector-valued 1-form $\mathbf{u}$; we can read off its components $\mathbf{U}^a$ from

$$\mathbf{U}^a \otimes \mathbf{e}_a = \mathbf{d}\left(\mathbf{u}^a \otimes \mathbf{e}_a\right) = \left(\mathbf{du}^a - \mathbf{u}^c \wedge \boldsymbol{\gamma}^a_c\right) \otimes \mathbf{e}_a \tag{19}$$

However, if we wish to treat $\mathbf{U}$ as a *co*vector-valued 2-form instead, its components will change to

$$\mathbf{U}_b \otimes \mathbf{e}^b = \mathbf{d}\left(\mathbf{u}_b \otimes \mathbf{e}^b\right) = \left(\mathbf{du}_b + \mathbf{u}_c \wedge \boldsymbol{\gamma}^c_b\right) \otimes \mathbf{e}^b \tag{20}$$

So, if we keep these distinctions in mind, there'll be no problem with the (admittedly language-abusing) notation $\mathbf{U} = \mathbf{U}^a \otimes \mathbf{e}_a = \mathbf{U}_b \otimes \mathbf{e}^b$ that'll be employed later on, because the ambiguity can be eliminated based on the context.

3 Differential Affine Geometry

The concepts thus introduced suffice to formulate a pragmatic, multipurpose tensor calculus framework fully integrated with exterior algebra, which is particularly important for problems involving integration and provides a modern, more elegant reformulation of the old vector calculus that can be readily generalized to any dimensionality. Nonetheless, up to now, no explicit mention has been made of any gravitational phenomena; in particular, the metric was introduced as an *ad hoc*, nondynamical mathematical device for the purposes of providing 1) a formalization of our intuition of "length", 2) a means to "raise and lower indices", and 3) an explicit expression of the covariant derivative, via its introduction in the Christoffels. None of these, we see, has any obvious gravitational connotation; in fact, since any physical system (whether under gravitational influences or not) may be described in terms of this formalism, we can appreciate their significance as being purely operational - as part of the general toolbox of mathematical concepts that we introduce in order to frame and quantify generic physical phenomena. How, then, can we characterize gravitational phenomena as separate from such a toolbox? To this problem we turn next.

Fortunately, a simple fix is available to us, thanks to the tensor-valued formalism: we propose introducing a new operator $\mathbf{D}$ that represents a slight generalization of our previous $\mathbf{d}$ by its effect on tensor-valued forms: for $\mathbf{d} \to \mathbf{D}$, substitute in eq. 4

$$\nabla_l T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} \rightarrow \overset{\omega}{\nabla}_l T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} \tag{21}$$

$$T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r , l} \rightarrow T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r ; l} \tag{22}$$

with $\mathbf{D}T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r} = \mathbf{d}T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q k_1 \ldots k_r}$, and where now

$$\overset{\omega}{\boldsymbol{\nabla}}\mathbf{e}_a \equiv \mathbf{D}\mathbf{e}_a := \mathbf{d}\mathbf{e}_a + \boldsymbol{\omega}^b_a \otimes \mathbf{e}_b, \tag{23}$$

$$\overset{\omega}{\boldsymbol{\nabla}}\mathbf{e}^b \equiv \mathbf{D}\mathbf{e}^b := \mathbf{d}\mathbf{e}^b - \boldsymbol{\omega}^b_a \otimes \mathbf{e}^a, \tag{24}$$

$$\mathbf{D}\boldsymbol{\theta}^c \; : \; = \mathbf{d}\boldsymbol{\theta}^c - \boldsymbol{\Theta}^c \tag{25}$$

with the alternative notation for the components $T^{i_1 \dots i_p}{}_{j_1 \dots j_q k_1 \dots k_r ; l} := \overset{\omega}{\nabla}_l T^{i_1 \dots i_p}{}_{j_1 \dots j_q k_1 \dots k_r}$ (thus justifying our previous choice of notation). As seen from these definitions, the tensor-valued 1-form[3] $\boldsymbol{\omega}_{\alpha}^{\beta}$ and the vector-valued 2-form $\boldsymbol{\Theta}^{\sigma}$ account for all the deviation between $\mathbf{D}$ and $\mathbf{d}$ in a manifestly covariant way; furthermore, it will also prove useful to define from the general expression of $\mathbf{D}$ another operator $\mathbf{D_0}$, obtained from the former by putting $\boldsymbol{\Theta}^{\sigma} = \mathbf{0}$.

Now that $\mathbf{D}$ (respec. $\mathbf{D_0}$) has been defined, we proceed to once again calculate the second derivative of the vector $\mathbf{v}$:

$$\begin{aligned}
\mathbf{D}^2 \left(v^a \mathbf{e}_a \right) &= \mathbf{D} \left[\left(\mathbf{d} v^a \right) \otimes \mathbf{e}_a + v^a \mathbf{D} \mathbf{e}_a \right] \equiv \left(\mathbf{D} \mathbf{d} v^a \right) \otimes \mathbf{e}_a + v^b \mathbf{D} \left(\mathbf{d} \mathbf{e}_b + \boldsymbol{\omega}_b^a \otimes \mathbf{e}_a \right) \\
&\equiv \left\{ \left[\mathbf{d}^2 v^a - \left(\partial_c v^a \right) \boldsymbol{\Theta}^c \right] + v^b \left[\mathbf{d} \left(\boldsymbol{\gamma}_b^a + \boldsymbol{\omega}_b^a \right) - \left(\gamma_{bc}^a + \omega_{bc}^a \right) \boldsymbol{\Theta}^c - \left(\boldsymbol{\gamma}_b^c + \boldsymbol{\omega}_b^c \right) \wedge \left(\boldsymbol{\gamma}_c^a + \boldsymbol{\omega}_c^a \right) \right] \right\} \otimes \mathbf{e}_a \\
&\equiv \left\{ v^b \left[\mathbf{R}_b^a + \left(\mathbf{d} \boldsymbol{\omega}_b^a - \boldsymbol{\gamma}_b^c \wedge \boldsymbol{\omega}_c^a - \boldsymbol{\omega}_b^c \wedge \boldsymbol{\gamma}_c^a \right) - \boldsymbol{\omega}_b^c \wedge \boldsymbol{\omega}_c^a \right] - v^a{}_{;c} \boldsymbol{\Theta}^c \right\} \otimes \mathbf{e}_a \\
&=: \left[v^b \left(\mathbf{R}_b^a + \boldsymbol{\Omega}_b^a \right) - v^a{}_{;c} \boldsymbol{\Theta}^c \right] \otimes \mathbf{e}_a
\end{aligned} \tag{26}$$

where in the last step we defined the tensor-valued 2-form, $\boldsymbol{\Omega}_b^a$. In keeping with our previous steps, it is straightforward to obtain a new Bianchi-like identity associated with it:

$$\begin{aligned}
\mathbf{d} \boldsymbol{\Omega}_b^a &= \left(\mathbf{d}^2 \boldsymbol{\omega}_b^a + \mathbf{d} \boldsymbol{\gamma}_c^a \wedge \boldsymbol{\omega}_b^c - \boldsymbol{\gamma}_c^a \wedge \mathbf{d} \boldsymbol{\omega}_b^c + \mathbf{d} \boldsymbol{\omega}_c^a \wedge \boldsymbol{\gamma}_b^c - \boldsymbol{\omega}_c^a \wedge \mathbf{d} \boldsymbol{\gamma}_b^c \right) + \mathbf{d} \boldsymbol{\omega}_c^a \wedge \boldsymbol{\omega}_b^c - \boldsymbol{\omega}_c^a \wedge \mathbf{d} \boldsymbol{\omega}_b^c \\
&\equiv \boldsymbol{\Omega}_c^a \wedge \left(\boldsymbol{\gamma}_b^c + \boldsymbol{\omega}_b^c \right) - \left(\boldsymbol{\gamma}_c^a + \boldsymbol{\omega}_c^a \right) \wedge \boldsymbol{\Omega}_b^c + \left(\mathbf{R}_c^a \wedge \boldsymbol{\omega}_b^c - \boldsymbol{\omega}_c^a \wedge \mathbf{R}_b^c \right)
\end{aligned} \tag{27}$$

which, upon rearranging, can be written

$$\mathbf{D_0} \boldsymbol{\Omega} \equiv \left(\boldsymbol{\omega}_b^c \wedge \mathbf{R}_c^a - \boldsymbol{\omega}_c^a \wedge \mathbf{R}_b^c \right) \otimes \mathbf{e}_a \otimes \mathbf{e}^b \tag{28}$$

where $\boldsymbol{\Omega} := \boldsymbol{\Omega}_b^a \otimes \mathbf{e}_a \otimes \mathbf{e}^b$.

Up to here, our analysis has led to three sets of objects: $\mathbf{e}_a, \mathbf{e}^b, \boldsymbol{\theta}^c$; $\mathbf{g}, \boldsymbol{\gamma}_b^a, \mathbf{R}_b^a$; and $\boldsymbol{\omega}_b^a, \boldsymbol{\Theta}^c, \boldsymbol{\Omega}_b^a$ - all of which, with the exception of $\boldsymbol{\gamma}_b^a$, tensorial in nature. Consider the first three of these: even in semi-Euclidian spaces, they can't be made to vanish nontrivially, because they're necessary to define the tensor-valued forms that'll be interpreted as physical objects (even in the absence of gravity); the situation is similar with the *Riemannian* objects $\mathbf{g}, \boldsymbol{\gamma}_b^a, \mathbf{R}_b^a$, because they're required to maintain the general covariance of all tensor(-valued) expressions. This suggests that these objects are not suitable candidates for being gravitational variables, which we'd like to be able to freely take to vanish in a manifestly covariant way; so, we're left with the *post-Riemannian* $\boldsymbol{\omega}_b^a, \boldsymbol{\Theta}^c, \boldsymbol{\Omega}_b^a$ as prime candidates, and to their detailed geometrical properties we now turn, after introducing some nomenclature: without going to bundles [18], we shall simply refer to them as the *linear connection, torsion and (linear) curvature*, respec.

[3] Incidentally, this object has already appeared under the name "distortion 1-form" in the MAG literature at least as early as 1997 [17] - as well as the decomposition of the "total curvature" into the RC tensor and the "post-Riemannian pieces".

Let us first study the metric compatibility of $\mathbf{D}$ (which is identical to that of $\mathbf{D_0}$).

$$\begin{aligned} \mathbf{Dg} &= (\mathbf{d}g_{ab}) \otimes \mathbf{e}^a \otimes \mathbf{e}^b + g_{cb} (\mathbf{De}^c) \otimes \mathbf{e}^b + g_{ac}\mathbf{e}^a \otimes (\mathbf{De}^c) \\ &\equiv (\nabla_c g_{ab}) \boldsymbol{\theta}^c \otimes \mathbf{e}^a \otimes \mathbf{e}^b - (g_{cb}\boldsymbol{\omega}_a^c + g_{ac}\boldsymbol{\omega}_b^c) \otimes \mathbf{e}^a \otimes \mathbf{e}^b \end{aligned} \tag{29}$$

Since it was already established that $g_{ab,c} \equiv 0$, this shows that the *non-metricity* $\mathbf{n}$ has a simple dependence on $\boldsymbol{\omega}_{(ab)} = \frac{1}{2!} (\boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{ba})$:

$$\mathbf{n} := \boldsymbol{\omega}_{(ab)} \otimes \mathbf{e}^a \otimes \mathbf{e}^b \equiv -\frac{1}{2}\mathbf{Dg} \tag{30}$$

This observation induces the handy decomposition $\boldsymbol{\omega}_{ab} = \mathring{\boldsymbol{\omega}}_{ab} + \mathbf{n}_{ab}$ of the linear connection in terms of the antisymmetric $\mathring{\boldsymbol{\omega}}_{ab} := \boldsymbol{\omega}_{[ab]} = \frac{1}{2!} (\boldsymbol{\omega}_{ab} - \boldsymbol{\omega}_{ba})$ - which in turn leads to a similar decomposition of the curvature. After eq. 26, define

$$\mathring{\boldsymbol{\Omega}}_b^a := \mathbf{d}\mathring{\boldsymbol{\omega}}_b^a - \boldsymbol{\gamma}_b^c \wedge \mathring{\boldsymbol{\omega}}_c^a - \mathring{\boldsymbol{\omega}}_b^c \wedge \boldsymbol{\gamma}_c^a - \mathring{\boldsymbol{\omega}}_b^c \wedge \mathring{\boldsymbol{\omega}}_c^a \tag{31}$$

with $\mathring{\boldsymbol{\Omega}} := \mathring{\boldsymbol{\Omega}}_b^a \otimes \mathbf{e}_a \otimes \mathbf{e}^b$ - and substitute it back in that equation:

$$\begin{aligned} \boldsymbol{\Omega}_b^a &\equiv \mathring{\boldsymbol{\Omega}}_b^a + [\mathbf{dn}_b^a - (\boldsymbol{\gamma}_b^c + \mathring{\boldsymbol{\omega}}_b^c) \wedge \mathbf{n}_c^a - \mathbf{n}_b^c \wedge (\boldsymbol{\gamma}_c^a + \mathring{\boldsymbol{\omega}}_c^a) - \mathbf{n}_b^c \wedge \mathbf{n}_c^a] \\ &=: \mathring{\boldsymbol{\Omega}}_b^a + \mathbf{N}_b^a \end{aligned} \tag{32}$$

where in the last equality $\mathbf{N}_b^a$ was defined. The above development makes it natural to define a new operator $\mathring{\mathbf{D}}$ (respec. $\mathring{\mathbf{D}}_\mathbf{0}$) as being the same as $\mathbf{D}$ (respec. $\mathbf{D_0}$) but with the $\boldsymbol{\omega}_b^a$ restricted to $\mathring{\boldsymbol{\omega}}_b^a$ only; this will prove useful later on - though we can already adapt the Ricci identity argument of the previous section to $\mathring{\mathbf{D}}_\mathbf{0}$ and show that $\mathring{\boldsymbol{\Omega}}_{ab} = -\mathring{\boldsymbol{\Omega}}_{ba}$ too - just like the RC tensor.

Continuing this process, the nonmetricity can be further decomposed as

$$\boldsymbol{\phi} := \boldsymbol{\omega}_c{}^c \equiv \mathbf{n}_c{}^c, \tag{33}$$

$$\mathbf{n}_{ab} =: \frac{1}{n} g_{ab}\boldsymbol{\phi} + \boldsymbol{\sigma}_{ab} \tag{34}$$

from which the curvature can be further decomposed, as well:

$$\begin{aligned} \mathbf{N}_b^a &\equiv \frac{1}{n}\delta_b^a \mathbf{d}\boldsymbol{\phi} + [\mathbf{d}\boldsymbol{\sigma}_b^a - (\boldsymbol{\gamma}_b^c + \mathring{\boldsymbol{\omega}}_b^c) \wedge \boldsymbol{\sigma}_c^a - \boldsymbol{\sigma}_b^c \wedge (\boldsymbol{\gamma}_c^a + \mathring{\boldsymbol{\omega}}_c^a) - \boldsymbol{\sigma}_b^c \wedge \boldsymbol{\sigma}_c^a] \\ &=: \frac{1}{n}\delta_b^a \mathbf{d}\boldsymbol{\phi} + \boldsymbol{\Sigma}_b^a \end{aligned} \tag{35}$$

on account of the appearance of terms such as $\boldsymbol{\phi} \wedge \boldsymbol{\phi}$, $(\boldsymbol{\phi} \wedge \boldsymbol{\sigma}_b^a + \boldsymbol{\sigma}_b^a \wedge \boldsymbol{\phi})$, etc., which vanish identically.

Next, we must make sense of the vector-valued 2-form $\boldsymbol{\Theta}^c$; for while it is quite natural to regard $\boldsymbol{\Omega}_b^a$ as the "field strength" associated to the "potential" $\boldsymbol{\omega}_b^a$, no such association is apparent concerning the torsion. At this point, we must go beyond linear structures and investigate *affine* ones; thankfully, this is easily done in terms of the former by the use of a Möbius map [9] that imbeds tensors over our target n-dimensional base space M onto ones over a $(n+1)$-dimensional base instead. This is accomplished via a foliation

$$\mathbf{g}^{(n+1)} = g_{-1-1}\mathbf{e}^{-1} \otimes \mathbf{e}^{-1} + g_{ab}^{(n)}\left(x^c\right)\mathbf{e}^a \otimes \mathbf{e}^b \tag{36}$$

of a fictitious $(n+1)$-dimensional metric space whose slices are isomorphic to our target n-dimensional metric space (and where, just for convenience, we allow the indexes the value -1, associated to an extra coordinate x^{-1} in the $(n+1)$-manifold). If, for simplicity, we keep to a coordinate basis, and put $g_{-1-1} = \pm 1$, the only nonvanishing Christoffels are seen to be the same $\left\{ {l \atop ij} \right\}$ of the n-slices; furthermore, if we restrict attention to coordinate transformations $x^{-1} \to x^{-1}, x^c \to x^{c'}\left(x^c\right)$, this result maintains - and it's trivial to extend it for noncoordinate bases, as well. The point to be taken is this: since the general covariance of the n-slices is all that matters to us (as opposed to that of the entire foliation), we may define a modified LC connection $\tilde{\boldsymbol{\gamma}}$ in this space characterized by $\begin{pmatrix} \tilde{\gamma}^{-1}_{-1} \equiv 0 & \tilde{\boldsymbol{\gamma}}^a_{-1} \equiv \mathbf{0} \\ \tilde{\boldsymbol{\gamma}}^{-1}_b \equiv \mathbf{0} & \tilde{\boldsymbol{\gamma}}^a_b \equiv \boldsymbol{\gamma}^a_b \end{pmatrix}$. Based on this representation, we can make sense of covariant differentiation in the $(n+1)$-space via a $\tilde{\mathbf{D}}$ that is essentially the same as the $\mathbf{D_0}$ defined previously in $(n+1)$ dimensions, but with the modified $\tilde{\boldsymbol{\gamma}}$ in place of $\boldsymbol{\gamma}$, and an *affine connection* $\tilde{\boldsymbol{\omega}}$ given by $\begin{pmatrix} \tilde{\boldsymbol{\omega}}^{-1}_{-1} & \tilde{\boldsymbol{\omega}}^a_{-1} \\ \tilde{\boldsymbol{\omega}}^{-1}_b & \tilde{\boldsymbol{\omega}}^a_b \equiv \boldsymbol{\omega}^a_b \end{pmatrix}$; with these, we write explicitly the components $\begin{pmatrix} \tilde{\boldsymbol{\Omega}}^{-1}_{-1} & \tilde{\boldsymbol{\Omega}}^a_{-1} \\ \tilde{\boldsymbol{\Omega}}^{-1}_b & \tilde{\boldsymbol{\Omega}}^a_b \end{pmatrix}$ of the associated *affine curvature*, and notice *two* particularly interesting choices in fully determining $\tilde{\boldsymbol{\omega}}$: for the first one, by putting $\tilde{\boldsymbol{\omega}}^{-1}_{-1} \equiv 0, \tilde{\boldsymbol{\omega}}^{-1}_b \equiv \mathbf{0}$, the only nonvanishing components will be $\tilde{\boldsymbol{\Omega}}^a_b \equiv \boldsymbol{\Omega}^a_b$ and $\tilde{\boldsymbol{\Omega}}^a_{-1} = \left(\mathbf{d}\tilde{\boldsymbol{\omega}}^a_{-1} - \tilde{\boldsymbol{\omega}}^c_{-1} \wedge \boldsymbol{\gamma}^a_c\right) - \tilde{\boldsymbol{\omega}}^c_{-1} \wedge \tilde{\boldsymbol{\omega}}^a_c$. There's a strong suggestion here that the latter may be identified with the torsion; indeed, this idea gains weight if $\tilde{\boldsymbol{\omega}}^a_{-1}$ is identified with some covector-valued 1-form $\boldsymbol{\omega}^a$ in the original n-space, whose derivative is given by

$$\mathbf{D_0}\boldsymbol{\theta} = \left[\left(\mathbf{d}\boldsymbol{\omega}^a + \boldsymbol{\gamma}^a_c \wedge \boldsymbol{\omega}^c\right) + \boldsymbol{\omega}^a_c \wedge \boldsymbol{\omega}^c\right] \otimes \mathbf{e}_a =: \boldsymbol{\Omega}^a \otimes \mathbf{e}_a \tag{37}$$

where we defined the shorthand $\boldsymbol{\theta} := \boldsymbol{\omega}^a \otimes \mathbf{e}_a$; from here, the by now familiar procedure will yield

$$\begin{aligned} \mathbf{d}\boldsymbol{\Omega}^a &= \mathbf{d}^2\boldsymbol{\omega}^a + \left(\mathbf{d}\boldsymbol{\gamma}^a_c + \mathbf{d}\boldsymbol{\omega}^a_c\right) \wedge \boldsymbol{\omega}^c - \left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right) \wedge \mathbf{d}\boldsymbol{\omega}^c \\ &\equiv \left(\mathbf{R}^a_c + \boldsymbol{\Omega}^a_c\right) \wedge \boldsymbol{\omega}^c - \left(\boldsymbol{\gamma}^a_c + \boldsymbol{\omega}^a_c\right) \wedge \boldsymbol{\Omega}^c \end{aligned} \tag{38}$$

which, using $\boldsymbol{\Theta} := \boldsymbol{\Omega}^a \otimes \mathbf{e}_a$, rearranges to

$$\mathbf{D_0}\boldsymbol{\Theta} \equiv \left(\mathbf{R}^a_c + \boldsymbol{\Omega}^a_c\right) \wedge \boldsymbol{\omega}^c \otimes \mathbf{e}_a \tag{39}$$

and may be interpreted as another Bianchi identity. This looks very promising, but we must not forget the second choice alluded previously: putting $\tilde{\boldsymbol{\omega}}^{-1}_{-1} \equiv 0, \tilde{\boldsymbol{\omega}}^a_{-1} \equiv \mathbf{0}$ - in this case, we're left with $\tilde{\boldsymbol{\Omega}}^a_b \equiv \boldsymbol{\Omega}^a_b$ and $\tilde{\boldsymbol{\Omega}}^{-1}_b =$

$\left(\mathbf{d}\tilde{\boldsymbol{\omega}}_b^{-1} - \gamma_b^c \wedge \tilde{\boldsymbol{\omega}}_c^{-1}\right) - \tilde{\boldsymbol{\omega}}_b^c \wedge \tilde{\boldsymbol{\omega}}_c^{-1}$. With analogous reasoning, we can show that this $\tilde{\boldsymbol{\omega}}_b^{-1}$ can be identified with a vector-valued 1-form $\left(\boldsymbol{\omega}^\sharp\right)_b$ in the n-space, whose derivative is identical to $\tilde{\boldsymbol{\Omega}}$; in fact, it introduces a new geometrical object in our space, which we'll call the *dual torsion* $\boldsymbol{\Theta}^\sharp := \left(\boldsymbol{\Theta}^\sharp\right)_b \otimes \mathbf{e}^b$. This quantity, we stress, is not merely a rewriting of $g_{ba}\boldsymbol{\Theta}^a$ (hence the special label); however, under this understanding, we shall often drop the "$\sharp$" notation for the sake of brevity, while at the same time employing omegas $\boldsymbol{\omega}, \boldsymbol{\Omega}$ in a manner similar to Cartan's.

This completes the description of our differential affine formalism. So far in this and the previous section, we've exclusively talked about pure mathematics; now is the time to transfer these theoretical results into the arena of physics - but before we do, a last digression will prove useful for bookkeeping purposes: if we reflect back on the literature [9–11], we see references to different "geometries" or "spaces" based on criteria such as the (non)vanishing of curvature, torsion, nonmetricity, etc. - i.e., starting from the most generic (metric-)affine description, one then picks some of the objects describing the geometry to be dynamical, and such a choice yields a physical theory of gravitation. In our present formalism, a similar approach can be pursued in terms of the four objects $\boldsymbol{\omega}_a, \mathring{\boldsymbol{\omega}}_{ab}, \boldsymbol{\phi}, \boldsymbol{\sigma}_{ab}$ comprising the affine connection; chosing whether or not any of these vanish yields a total of 16 geometries - some of which I've named in Table 1.

Table 1. Selected geometries for theories of gravitation.

geometry	non-dynamical object
Weyl-Cartan	$\boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weyl-Weitzenböck	$\mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weyl	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
pre-Weyl	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Cartan	$\boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Weitzenböck	$\mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Ricci	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$
Riemann	$\boldsymbol{\omega}_a \equiv \mathbf{0}, \mathring{\boldsymbol{\omega}}_{ab} \equiv \mathbf{0}, \boldsymbol{\phi} \equiv \mathbf{0}, \boldsymbol{\sigma}_{ab} \equiv \mathbf{0}$

The attempt was made to introduce a nomenclature that mirrors the historical contributions of several eminent mathematicians; it is necessarily imperfect, due to the fact these authors did not employ the present schema - its value being mostly as a mnemonic device, that should be read with care (specially when comparing with the literature).

In the above list, we deliberately excluded, w.l.o.g., all eight geometries with $\boldsymbol{\sigma}_{ab} \neq \mathbf{0}$ guided by physical intuition; however, as the theoretical need for such geometries may rise, one can easily extend our naming conventions to include such fields - though in the present paper, we will not concern ourselves with them. (Also, the reader will notice the similarity of our taxonomy with that of the "MAGic cube" of [9]; in fact, by including the $\boldsymbol{\sigma}_{ab}$ field, we'd have a "mAGic tesseract" - an amusing touch.)

At this point, finally, I will abandon mathematical generality and make physical commitments in order to obtain a theory of gravity in four-dimensional spacetime, which will be signaled by the switch to Greek indices. In order to obtain agreement with Special Relativity, the (nondynamical) metric is taken to be Minkowski's (with signature $\eta_{\mu\nu} = diag\,[-1,+1,+1,+1]$) - and as a consequence, we have $\mathbf{R}^{\alpha}_{\beta} = \mathbf{0}$, which simplifies some of the previous identities obtained with a generic $\mathbf{g}$; as such, the role of *gravitational potential* is transferred wholly to the many pieces of the affine connection that have been introduced above. Before discussing their dynamics, it is of notice that, from a purely geometrical perspective, we can "translate" the conventional GR picture into this framework - which may be surprising to some; but it can be simply achieved via the assignments

$$\boldsymbol{\omega}^{\beta} = \mathbf{0}, \boldsymbol{\omega}^{\beta}_{\alpha} = \mathbf{\Gamma}^{\beta}_{\alpha} - \boldsymbol{\gamma}^{\beta}_{\alpha} \tag{40}$$

where the $\mathbf{\Gamma}^{\beta}_{\alpha}$ refer to the LC connection computed from the Lorentzian metric $\mathbf{g}$ of Einstein's theory; it follows from these that the RC tensor in GR is equivalent to the curvature in this geometry. This alone might prove the usefulness of this formalism, for example, in the semiclassical regime - albeit it doesn't shed much light on the dynamics, as they're constrained by the Einstein field equations; henceforth we shall drop this equivalence formalism and instead deliberately experiment with an approach different from classic geometrodynamics, yet closer in philosophy to the gauge field theories used in modern physics.

4 Gravidynamics

In order to make specific claims about gravitational dynamics, one needs a Lagrangean. After all the trouble separating ourselves from the Riemannian formalism, it is now quite natural to turn to a Lagrangean quadratic in the field strengths, rather than just linear, after all the desirable features that made them a mainstay in the Standard Model of Particle Physics - not the least of which being agreeable to quantization. (Another important aspect of such Lagrangeans pertains to *gauge-invariance*, but we will not get into this topic - except to comment it is pretty clear that the general affine group $GA(4,\mathbb{R}) = GL(4,\mathbb{R}) \ltimes T(4)$ and its subgroups are intimately related to the symmetries here.)

Reverting to the Möbius foliation of the previous section, as well as to Latin indexes (now taken to extend to the "-1" coordinate as well), we write for the gravitational Lagrangean scalar-valued 5-form

$$\mathbf{Lag}_{GD} = \frac{1}{4\kappa_0}\left[\left(\tilde{\mathbf{\Omega}}^{\sharp}\right)^{a}_{b} \wedge \star\tilde{\mathbf{\Omega}}^{b}_{a} + \tilde{\mathbf{\Omega}}^{a}_{b} \wedge \star\left(\tilde{\mathbf{\Omega}}^{\sharp}\right)^{b}_{a}\right] \tag{41}$$

where κ_0 is the gravitational constant, $\star$ is the Hodge star[4], and the "$\sharp$" indicates the presence of the dual torsion in the formula - without which the

[4] The Hodge operator requires a metric. Two observations here: first, this metric is not to be confused with $\mathbf{g} = g_{ab} \otimes \mathbf{e}^{a} \otimes \mathbf{e}^{b}$ - but thanks to the soldering, we may map this $\mathbf{g}$ into $\mathbf{g}_{\boldsymbol{\theta}} := g_{ab} \otimes \boldsymbol{\theta}^{a} \otimes \boldsymbol{\theta}^{b}$; second, the star defined over the 5-foliation has a simple relation with that over 4-spacetime, so we use the same symbol for both.

torsion-related pieces would be seen to drop off the indicial sum; performing said sum, and simplifying, this can be written over spacetime as

$$\mathbf{Lag}_{GD} = \frac{1}{2\kappa_0}\left(\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \boldsymbol{\Omega}^\alpha_\beta \wedge \star\boldsymbol{\Omega}^\beta_\alpha\right) \tag{42}$$

Researchers interested in the rigorous variational treatment are referred to Bleecker [19]. Here, a quick heuristic will suffice: Taylor-expanding the Lagrangean up to first order in ϵ around a perturbation $\delta\tilde{\boldsymbol{\Omega}}$ of the affine curvature

$$\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}} + \epsilon\delta\tilde{\boldsymbol{\Omega}}\right) =: \mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}\right) + \epsilon\delta\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}, \delta\tilde{\boldsymbol{\Omega}}\right) + \mathbf{O}\left(\epsilon^2\right) \tag{43}$$

we can immediately read off the first variation $\delta\mathbf{Lag}_{GD}$ and simplify it further:

$$\begin{aligned}
\delta\mathbf{Lag}_{GD} &= \frac{1}{\kappa_0}\left[\frac{1}{2}\left(\boldsymbol{\Omega}_\beta \wedge \star\delta\boldsymbol{\Omega}^\beta + \delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta\right) + \frac{1}{2}\left(\boldsymbol{\Omega}^\alpha_\beta \wedge \star\delta\boldsymbol{\Omega}^\beta_\alpha + \delta\boldsymbol{\Omega}^\alpha_\beta \wedge \star\boldsymbol{\Omega}^\beta_\alpha\right)\right] \\
&\equiv \frac{1}{\kappa_0}\left[\delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \delta\left(\mathring{\boldsymbol{\Omega}}^\alpha_\beta + \frac{1}{4}\delta^\alpha_\beta \mathbf{d}\boldsymbol{\phi}\right) \wedge \star\left(\mathring{\boldsymbol{\Omega}}^\beta_\alpha + \frac{1}{4}\delta^\beta_\alpha \mathbf{d}\boldsymbol{\phi}\right)\right] \\
&\equiv \frac{1}{\kappa_0}\left(\delta\boldsymbol{\Omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta + \delta\mathring{\boldsymbol{\Omega}}^\alpha_\beta \wedge \star\mathring{\boldsymbol{\Omega}}^\beta_\alpha + \frac{1}{4}\delta\mathbf{d}\boldsymbol{\phi} \wedge \star\mathbf{d}\boldsymbol{\phi}\right)
\end{aligned} \tag{44}$$

where in the second equality we explicitly constrained ourselves to Weyl-Cartan geometry. Now, in order to proceed, we have to rearrange this quantity to

$$\delta\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\Omega}}, \delta\tilde{\boldsymbol{\Omega}}\right) =: \delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD}\left(\tilde{\boldsymbol{\omega}}, \mathbf{d}\tilde{\boldsymbol{\omega}}\right) \wedge \delta\tilde{\boldsymbol{\omega}} \tag{45}$$

where the newly-introduced notation is a self-evident shorthand. Thus, to obtain $\delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD}$, there is a need for the explicit expression for the variations of the field strengths

$$\begin{aligned}
\delta\boldsymbol{\Omega}_\beta &= \left(\delta\mathbf{d}\boldsymbol{\omega}_\beta - \boldsymbol{\gamma}^\sigma_\beta \wedge \delta\boldsymbol{\omega}_\sigma\right) - \left(\mathring{\boldsymbol{\omega}}^\sigma_\beta + \frac{1}{4}\delta^\sigma_\beta\boldsymbol{\phi}\right) \wedge \delta\boldsymbol{\omega}_\sigma - \left(\delta\mathring{\boldsymbol{\omega}}^\sigma_\beta + \frac{1}{4}\delta^\sigma_\beta\delta\boldsymbol{\phi}\right) \wedge \boldsymbol{\omega}_\sigma \\
\delta\mathring{\boldsymbol{\Omega}}^\alpha_\beta &= \delta\mathbf{d}\mathring{\boldsymbol{\omega}}^\alpha_\beta - \boldsymbol{\gamma}^\sigma_\beta \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\sigma - \delta\mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \boldsymbol{\gamma}^\alpha_\sigma - \delta\mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \mathring{\boldsymbol{\omega}}^\alpha_\sigma - \mathring{\boldsymbol{\omega}}^\sigma_\beta \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\sigma
\end{aligned}$$

Notice how, consistent with our philosophy, no term like $\delta\boldsymbol{\gamma}^\alpha_\beta$ appears above, as such variations are meaningless. The road is now clear: putting as usual $\delta\mathbf{d}\mathring{\boldsymbol{\omega}}^\alpha_\beta = \mathbf{d}\delta\mathring{\boldsymbol{\omega}}^\alpha_\beta$, etc., then integrating by parts, putting the field variations in evidence and dropping the boundary terms, we're left, after algebraic manipulation, with

$$\begin{aligned}
\delta_{\tilde{\boldsymbol{\omega}}}\mathbf{Lag}_{GD} \wedge \delta\tilde{\boldsymbol{\omega}} \equiv\ & -\frac{1}{\kappa_0}\frac{1}{4}\left(\mathbf{d}\star\mathbf{d}\boldsymbol{\phi} - \boldsymbol{\omega}_\beta \wedge \star\boldsymbol{\Omega}^\beta\right) \wedge \delta\boldsymbol{\phi} - \frac{1}{\kappa_0}\left[\mathbf{d}\star\boldsymbol{\Omega}^\sigma + \left(\boldsymbol{\gamma}^\sigma_\beta + \mathring{\boldsymbol{\omega}}^\sigma_\beta + \frac{1}{4}\delta^\sigma_\beta\boldsymbol{\phi}\right) \wedge \star\boldsymbol{\Omega}^\beta\right] \wedge \delta\boldsymbol{\omega}_\sigma \\
& -\frac{1}{\kappa_0}\left[\mathbf{d}\star\mathring{\boldsymbol{\Omega}}^\beta_\alpha - \left(\boldsymbol{\gamma}^\sigma_\alpha + \mathring{\boldsymbol{\omega}}^\sigma_\alpha\right) \wedge \star\mathring{\boldsymbol{\Omega}}^\beta_\sigma + \left(\boldsymbol{\gamma}^\beta_\sigma + \mathring{\boldsymbol{\omega}}^\beta_\sigma\right) \wedge \star\mathring{\boldsymbol{\Omega}}^\sigma_\alpha - \boldsymbol{\omega}_\alpha \wedge \star\boldsymbol{\Omega}^\beta\right] \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\beta
\end{aligned} \tag{47}$$

The Euler-Lagrange (EL) equations are finally obtained by equating the above term with

$$\mathbf{J}_{\tilde{\boldsymbol{\omega}}} \wedge \delta\tilde{\boldsymbol{\omega}} := -\frac{1}{4}\mathbf{C} \wedge \delta\boldsymbol{\phi} - \mathbf{T}^\sigma \wedge \delta\boldsymbol{\omega}_\sigma - \mathbf{L}^\beta_\alpha \wedge \delta\mathring{\boldsymbol{\omega}}^\alpha_\beta \tag{48}$$

where $\mathbf{C}, \mathbf{T} := \mathbf{T}^{\sigma} \otimes \mathbf{e}_{\sigma}, \mathbf{L} := \mathbf{L}_{\alpha}^{\beta} \otimes \mathbf{e}_{\beta} \otimes \mathbf{e}^{\alpha}$ are obviously the current 3-forms associated with the potentials. All of this can be put in a very compact (and elegant) form:

$$\mathbf{d} \star \mathbf{d}\boldsymbol{\phi} - \boldsymbol{\omega}_{\beta} \wedge \star \boldsymbol{\Omega}^{\beta} = \kappa_0 \mathbf{C} \tag{49a}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}} \star \boldsymbol{\Theta} + \frac{1}{4} \boldsymbol{\phi} \wedge \star \boldsymbol{\Theta} = \kappa_0 \mathbf{T} \tag{49b}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}} \star \mathring{\boldsymbol{\Omega}} - \boldsymbol{\theta} \wedge \star \boldsymbol{\Theta} = \kappa_0 \mathbf{L} \tag{49c}$$

The equations above showcase the degree of coupling between the different pieces of the connection - therefore presenting an opportunity to examine their asymptotic flatness (i.e., the regimes under which one or more potentials are taken to zero), which in turn effectively ("weakly") change the underlying geometry from Weyl-Cartan to one of the other geometries listed in Table 1. It is apparent from our dynamical laws that this procedure will consistently leave the currents associated with the vanishing fields to vanish also - unless $\boldsymbol{\omega}_a \neq \mathbf{0}$ (thus, for the Weyl-Weitzenböck, Cartan and Weitzenböck cases). The *residual currents* $\mathbf{C}_{res} = -\frac{1}{\kappa_0} \boldsymbol{\omega}_{\beta} \wedge \star \boldsymbol{\Omega}^{\beta}$ and/or $\mathbf{L}_{res} = -\frac{1}{\kappa_0} \boldsymbol{\theta} \wedge \star \boldsymbol{\Theta}$ we obtain this way could be a manifestation of spontaneous symmetry breaking, or simply of an illegal move; however, since their physical meaning is unclear to me, I refrain to comment further on their significance, beyond suggesting that they may possibly have cosmological interest.

Having said this, we can also work out the field equations by taking potentials to vanish at the Lagrangean level ("strongly"); if we do this to obtain a Weitzenböck geometry, we get

$$\mathbf{d} \star \boldsymbol{\Theta} = \kappa_0 \mathbf{T} \tag{50a}$$

$$\mathbf{d}\boldsymbol{\Theta} = \mathbf{0} \tag{50b}$$

Hopefully, the formal similarity with the Maxwell equations of electrodynamics will not be lost on the reader - specially as we show the Bianchi identity alongside the EL law. It is an unsurprising result, actually, due to the well-known fact that both Coulomb's and Newton's laws are derived from the same differential equation (namely, Poisson's), on the one hand, and the fact that both the translation subgroup $T(4)$ and the unitary $U(1)$ of quantum electrodynamics are Abelian, on the other. Not only that, but these equations allow an immediate interpretation in terms of known electrodynamical results, from which we readily establish the Newtonian limit, the existence of gravitational waves, and even an exact quadrupole formula [20]; they also hint at hitherto unexplored prospects, such as the possibility of a "macroscopic" formulation that accounts for a bound current contribution $\mathbf{T}_{bound}$ in terms of a "gravitational polarization", and that of "gravimagnetic monopoles" through the dynamization of the Bianchi identity. So, from these arguments, we see this (sub)theory offers a rich phenomenological testbed that can be explored with known theoretical tools, as well as a rather convenient starting point for a quantum theory of gravity; for these reasons, it'll be convenient to give to this special case its own name: we'll call it *teledynamics*, to honor also the old teleparallelism theory.

Analagous considerations for the case of only $\mathring{\boldsymbol{\omega}}^{\alpha}_{\beta}$ nonvanishing lead us to the formulae

$$\mathring{\mathbf{D}}_{\mathbf{0}} \star \mathring{\boldsymbol{\Omega}} = \kappa_0 \mathbf{L} \tag{51a}$$

$$\mathring{\mathbf{D}}_{\mathbf{0}} \mathring{\boldsymbol{\Omega}} = \mathbf{0} \tag{51b}$$

which are simply the Yang-Mills (YM) equations associated with the (non-Abelian) Lorentz group; for this reason, one may call this subtheory *orthodynamics*. Contrasted with the Maxwellian nature of teledynamics, the dynamics here doesn't have an immediate interpretation in Newtonian terms, but at the same time may be tied to the presence of a conserved current associated with, e.g., rotations. Unsurprising, again, if we interpret this in the light (*mutatis mutandis*) of the pioneering works of Uchiyama [21] and Kibble [22]; doubly so, if we also heed the no less prophetic words of Cartan [16]: "La translation révèle la *torsion*, la rotation révèle la *courbure* de la variété donnée."

Discussing the gravitational degrees of freedom is nice and all, but it'd be nicer if we could say something about the phenomenology of the currents, as well; thus, to furnish physical motivation, let us consider (in sketch) the set of three differential equations that comprise Euler's theory of inviscid hydrodynamics - that of conservation of energy, of mass, and finally the dynamical law of the 3-momentum ("Newton's 2^{nd} law for continua") [23]. Ignoring the first one, it is possible to glom the latter two into a single *relativistic* equation[5]

$$\mathbf{dt} = \mathbf{F} - \mathbf{dP} \tag{52}$$

where the vector-valued 3-form $\mathbf{t}$ is the intrinsic momentum *of* the fluid, the vector-valued 3-form $\mathbf{P}$ is the stress applied *to* the fluid, and finally the vector-valued 4-form $\mathbf{F}$ is the density of applied forces; one might refer to these as the *Euler-Einstein equations.* Consider now a system described by these equations but in absence of gravity; it is quite clear from this formula that, for a system for which $\mathbf{F}$ vanishes identically, the total stress-momentum $\mathbf{T} := \mathbf{t} + \mathbf{P}$ is a conserved quantity; now, if we "turn on" gravity, we will immediately obtain a generalized conservation law of the form

$$\begin{aligned} \mathbf{0} = \mathbf{DT} &\equiv \mathbf{d}\left(\mathbf{T}^{\beta}\mathbf{e}_{\beta}\right) - \left(\mathbf{T}^{\alpha} \wedge \boldsymbol{\omega}^{\beta}_{\alpha} + 3T^{\beta}_{\mu\nu\rho}\boldsymbol{\Omega}^{\mu} \wedge \boldsymbol{\theta}^{\nu} \wedge \boldsymbol{\theta}^{\rho}\right) \otimes \mathbf{e}_{\beta} \\ &=: \mathbf{dT} - \mathbf{F}_{grav} \end{aligned} \tag{53}$$

If we interpret the first equality as the condition for a free-falling fluid, it follows from the second one that this condition can be simulated by applying a "force" $\mathbf{F}_{grav}$ to the special-relativistic fluid; isn't this a little too reminiscent of the equivalence principle?

[5] For the sake of brevity, we skip this derivation; it is based on application of the Leibniz-Reynolds transport theorem to the Newtonian equations, which are then "relativized". As such, new terms are seen to appear, compared to the original equations - but those can be taken to vanish in the nonrelativistic limit. For an approach to continuum mechanics similar in spirit to my take here, cf. [24].

As a bonus, basing ourselves solely in complete analogy with eq. 52, we can immediately propose the formula

$$\mathbf{dl} = \boldsymbol{\tau} - \mathbf{dS} \tag{54}$$

where the *tensor*-valued 3-form $\mathbf{l}$ is the mechanical *angular* momentum of the fluid, the *tensor*-valued 3-form $\mathbf{S}$ is its *spin*, and finally the *tensor*-valued 4-form $\boldsymbol{\tau}$ is the density of applied *torques*; needless to say, the total angular momentum $\mathbf{L} := \mathbf{l} + \mathbf{S}$ is then also a conserved quantity, provided the system is torque-free - whereas the substitution $\mathbf{d} \to \mathbf{D}$ will lead to the appearance of an equivalent "torque" $\boldsymbol{\tau}^{\alpha\beta}_{grav}$. The liberality of language here is justified by the pure phenomenology of the description, but it already raises some interesting questions concerning the physical relation between $\mathbf{t}$ and $\mathbf{S}$ [25,26], as well as the classical symmetry condition[6] commonly imposed to $\mathbf{P}$. Indeed, one of the most peculiar features of the Einstein-Cartan theory is precisely a modification of the current term in the Einstein field equations dependent on the spin content of the system; this observation may prove not only illuminating towards a more precise discussion of the current situation, but also point to observational implications.

5 Miscellaneous questions

Despite the rather encouraging developments so far, one is still warily conscious of the Wheelerian maxim [8]: "Space tells matter how to move; matter tells space how to curve." In terms of the present work, we've already addressed the second part at length - but the first one, so far untouched, has proven stubborn. In standard geometrodynamics, it is well understood to refer to the geodetic equation, which covers a large class of motions[7] and therefore looks more favorably on the edge of Occam's razor; however, the difficulty here is not so much one of sheer simplicity, but rather of adapting electrodynamics, hydrodynamics and particle mechanics to the tensor-valued formalism, *as well as* describing how electromagnetism and matter (minimally) couple to gravity - all of which, in total, represents a non-negligible undertaking that nonetheless cannot be avoided if we want to apply the theory to problems of cosmological and astrophysical interest, as well as check how well it performs w.r.t. to the "classic tests" of GR.

I do not purport to have completed such a large-scale project; instead, the arguments that will be now presented are to be taken simply as possible pointers towards the full picture, and the equations that will be henceforth derived must be seen as strictly provisional, and illustrative.

About electrodynamics: writing the EM potential as $\mathbf{A} = A_\nu \boldsymbol{\theta}^\nu$ and the field strength (a.k.a. Faraday tensor) $\mathbf{F} := \mathbf{dA}$, the Maxwell equations may be

[6] See footnote in [23], pp. 14-15.

[7] I.e., such as the trajectories of both timelike particles (e.g. electrons) and null-like particles (e.g. photons), as well as associated effects like time dilation and red/blueshift; 'one equation to rule them all'.

written in the celebrated form

$$
\begin{aligned}
\mathbf{d}\star\mathbf{F} &= \mu_0 \mathbf{J}_{EM} && \text{(55a)}\\
\mathbf{dF} &= \mathbf{0} && \text{(55b)}
\end{aligned}
$$

Furthermore, if one defines $\mathbf{F_e} := F_{\mu\nu}\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}$, the Lorentz force acting upon a given particle can be obtained from the interior product of $\mathbf{F_e}$ with the particle's velocity [12]. This is all well and good, but it is not quite clear how this argument ought to be extended to continua, if we understand the Lorentz force density $\mathbf{F}_L$ to be a (co)vector-valued 4-form[8], and if, qualitatively (and empirically) speaking, we wish the formula to go like

$$\mathbf{F}_L \propto \text{current density} \times \text{field strength}; \tag{56}$$

for in the present case, $\mathbf{J}_{EM}$ is a scalar-valued 3-form, so that neither $\mathbf{F}$ nor $\mathbf{F_e}$ will fit. This complication also leads to trouble in writting down an explicit formula for the (four-dimensional) Maxwell stress $\mathbf{P}_M$, which is known [20] to behave as

$$\mathbf{P}_M \propto (\text{field strength})^2 \tag{57}$$

and must also be a source term in the gravidynamical/teledynamical 49b/50 - or in other words, a (co)vector-valued 3-form, which, on account of 52, we may expect to be related to the Lorentz force as $\mathbf{F}_L = \mathbf{dP}_M$. Thus, such considerations aren't just of electrodynamical concern, even as one still has to describe the coupling of gravitation to the EM field, as to be able to account for such phenomena of interest in GR as lensing; we may, for instance, try the same argument as the previous section and substitute $\mathbf{dP}_M \to \mathbf{DP}_M$ to get a modified Lorentz force in the presence of gravity. Without appeal to a quantum theory, another, perhaps obvious guess, is to simply modify the l.h.s. of 55a to $\mathbf{d}\star\mathbf{F} \to \mathbf{D}\star\mathbf{DA}$; this way, the presence of a gravitational field (in this case torsion) is immediately seen to introduce a deviation from the special-relativistic behavior which can already be discussed in light of established empirical knowledge.

This one does by introducing a weak-field linearization $\mathbf{A} \simeq \mathbf{A}_0 + \epsilon\mathbf{A}_1, \boldsymbol{\Theta} \simeq \epsilon\boldsymbol{\Theta}_1$, so that

$$
\begin{aligned}
\mathbf{0} &= \mathbf{D}\star\mathbf{DA} \simeq \mathbf{d}\star\mathbf{dA}_0 + \epsilon\left\{\mathbf{d}\star\mathbf{dA}_1 - \mathbf{d}\star\left[(A_0)_\mu\,\boldsymbol{\Omega}_1^\mu\right] - 2\,(\star\mathbf{d}A_0)_{\mu\nu}\,\boldsymbol{\Omega}_1^\mu\wedge\boldsymbol{\theta}^\nu\right\}\\
&=: \mathbf{d}\star\mathbf{dA}_0 + \epsilon\left(\mathbf{d}\star\mathbf{dA}_1 - \mu_0\mathbf{J}_{bend}\right) && \text{(58)}
\end{aligned}
$$

Since the "current" $\mathbf{J}_{bend}$ defined in the last equation does not depend on $\mathbf{A}_1$, this can be interpreted as a sort of refraction [20].

Next, we would also want a description of the motion of (massive) particles under an external gravitational field, which is the least obvious one so far; but, using the previous discussion of continua as a guidance, we recall that $\mathbf{F}_{grav}$ displays a coupling of the stress-momentum components with both the torsion

[8] Cf. [27–31].

$\boldsymbol{\Omega}^\mu$ and the linear potential $\boldsymbol{\omega}_\alpha^\beta$. Since $\mathbf{t}$ is proportional to the mass density, this implies that, in the present schema, the law of motion for point particles should include a torsion piece as well as a (linear) potential one; this is rather surprising, because it comes across as a confluence of intuitions from both the Lorentz force law (proportionality with a field strength), as well as the Einsteinian geodetic equation (proportionality with the connection). In light of this, it seems sensible to introduce the *ansatz*

$$m \left(v^\mu{}_{;\alpha} - v^\sigma \Omega_{\alpha\sigma}{}^\mu \right) \frac{dX^\alpha}{d\tau} = 0 \tag{59}$$

with m the mass, v^μ the components of the particle's velocity and X^α those of the particle's trajectory (which is parameterized by τ). As an educated guess, it fares a bit better than a mere dumbing down of hydrodynamics, because not only it incorporates the weak equivalence principle (i.e., the equality of inertial and gravitational masses) explicitly, but it also seems to have a promising Newtonian limit for teledynamics (i.e., with $v^\mu{}_{;\alpha} = v^\mu{}_{,\alpha}$).

The physical argument just given is borderline semiquantitative and has no pretentions otherwise; however, some tentative calculations within a Weitzenböck geometry suggest that the teledynamical equivalent of the Schwarzschild solution (i.e., one restricted by spherical symmetry, stationarity, etc.) might potentially reduce simply to the ordinary Newtonian theory; if this conjecture is shown to be correct, eq. 59 would then conflict with observation, because it wouldn't be able to account for the anomalous perihelion precession of Mercury. However, the problem also suggests the solution here, because such a conclusion requires the vanishing of the linear connection - but suppose we didn't impose this, and instead used a more general geometry - like Cartan's? By pure phenomenology, such a term *could* account for the effect, in principle; physically, this might be interpreted (curiously enough) as a kind of *frame-dragging* - which in turn, and contrary to current thinking, would likely be attributed to the Sun's *rotation*, as it contributes to the $\mathbf{L}$ current associated with the potential. My remarks are getting increasingly speculative, but the point here is not so much to stake controversial claims, as to illustrate the difficulties inherent to translating gravitational phenomena into the new formalism proposed in this paper - which on its turn translate into the difficulty of which models to pick for the classic tests, as well as any other empirical criteria we care to subject them to.

Finally, while still wading the waters of controversy, we may note in passing that the phenomenology just described might be appropriated to possibly explain the famous mystery of the anomalous galactic rotation curves: instead of introducing a halo of "matter" to fit the deviation from Keplerian behavior, one instead introduces "frame-dragging" of the sort discussed, and ties it to the angular momentum of the whole galaxy; stirred, not shaken. From where we stand now, though, this is but an aperitif.

6 Concluding remarks

Since the introduction of GR in the beginning of the 20^{th} century, the Riemannian paradigm has dominated the theoretical framework of classical gravitation - partly because it was then the state-of-art of differential geometry, and partly because of the tremendous empirical successes it undoubtedly enjoyed since. The geometrical alternative defined and discussed in this paper stands on its own mathematical merits regardless of any further physical considerations, and indeed may be studied on its own right - however, as we've shown, it also displays a list of potentially desirable features for a gauge theory of gravity - perhaps even as a competitor to the Einstein theory. Nonetheless, a complete understanding of how it can be used to describe real-world gravitational phenomena is still wanting and riddled with open problems which, in particular, currently preclude us from contrasting it empirically with GR. While this might at first sight seem like a fatal weakness dooming the whole enterprise, one should remind oneself that the perspective being advocated here is rather new and as such should not be readily dismissed without due consideration of its contents, incomplete they may be; indeed, I wanted to leave the matter of the gauge content as open-ended as possible for now to showcase the phenomenological power of the formalism - questions of symmetry pose a rich scenario to explore from here (not least those touching subjects such as reframing the "cosmological principle" in this language). Furthermore, the very nature of this incompleteness represents a definiteness in terms of a research programme exploring these questions systematically - not the least being that (apart from the caveats already addressed) these ideas do not seem to require a substantial revision of previous physics, as it seems to be the case with several modern attempts at quantizing gravity.

Given the current status of research, it is of interest to explore as many different venues as possible. The revision of our geometric intuition being here proposed comes with pros and cons, but it is hoped that the impression will be that the pros outweigh the cons - that there are enough nuggets, here and there, to afford sufficient motivation for researchers from various disciplines to pursue solutions to these problems, and give the theory its final classical touches - so that we can move on to better and quantum things.

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