DOMINATION NUMBER OF EDGE CYCLE GRAPHS

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Abstract

Let $G = (V, E)$ be a simple connected graph.A set $S \subset V$ is a dominating set of G if every vertex in $V \setminus S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. An edge cycle graph of a graph G is the graph $G(C_k)$ formed from one copy of G and $|E(G)|$ copies of P_k, where the ends of the i^{th} edge are identified with the ends of i^{th} copy of P_k. In this paper, we investigate the domination number of $G(C_k)$, $k \geq 3$.

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1 Introduction

Let $G = (V, E)$ be a simple connected, finite, undirectd graph with no loops and multiple edges. The degree of a vertex of a graph is the number of edges incident to the vertex. The degree of a vertex v is denoted by $deg(v)$. The maximum and minimum degree of a graph is denoted by $\Delta(G)$ and $\delta(G)$ respectively. We denote $N(v)$ and $N[v]$ as the open and closed neighbors of a vertex v respectively. A vertex $v \in G$ is called pendent vertex or end vertex of G if $deg(v) = 1$. A covering of a graph G is a subset K of V

such that every line of G is incident with a vertex in K. A vertex cover in a graph G is a subset K of vertices such tht every edge of G is incident with at least one vertex of K. The minimum cardinality taken over all minimal vertex covers of G is the vertex covering number of G and is denoted by $\alpha(G)$.

A set S of vertices in a graph G is a dominating set if every vertex in $V \setminus S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G.

J.P and N.S introduced edge cycle graph in [4]. An edge cycle graph of a graph G is the graph $G(C_k)$ formed from one copy of G and $|E(G)|$ copies of P_k, where the ends of the i^{th} edge are identified with the ends of i^{th} copy of P_k. A graph G and its edge cycle graph $G(C_k)$ are shown in Fig 1.1.

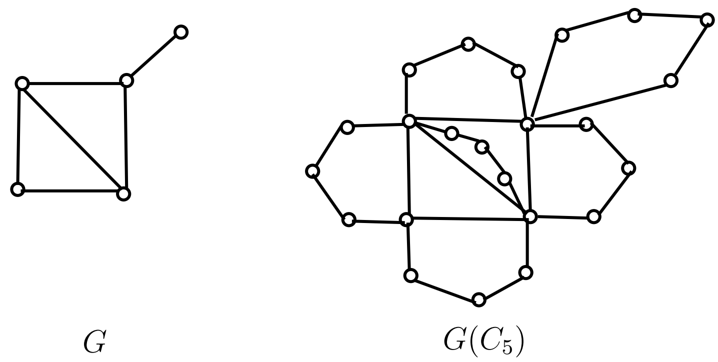

Fig 1.1 *A graph* G *and its edge cycle graph*

In this paper, we investigate the domination number of $G(C_k)$, $k \geq 3$.

2 Domination in Edge Cycle Graphs

Theorem 2.1. *Let G be a graph of order $n \geq 2$. Then $\gamma(G(C_3)) = \alpha(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be the edges of G. Then $C_1, C_2, \ldots, C_m$ be the edge cycles of $e_1, e_2, \ldots, e_m$ respectively.

We have to prove that $\gamma(G(C_3)) \leq \alpha(G)$.

Let $e_i = v_i v_j$ be in G. Then, let v_{ij} be the new vertex in $G(C_3)$ corresponding to the edge $v_i v_j$.

Let S be any covering set of G.

Since each covering set of G is a dominating set of G and G is the induced subgraph of $G(C_3)$, $v_1, v_2, \ldots v_n$ are dominate by S in $G(C_3)$.

Also, since each new vertex in $G(C_3)$ is adjacent to a S, $\{v_{ij}/1 \leq i \leq m\}$ are dominated by S.

Thus $\gamma(G(C_3)) \leq \alpha(G)$.

Next, we have to prove that $\gamma(G(C_3)) \geq \alpha(G)$.

Suppose that $\gamma(G(C_3)) \leq \alpha(G) - 1$.

Let S be a dominating set of $G(C_3)$. Since $\gamma(G(C_3)) \leq \alpha(G) - 1$, there exists at least one edge in G which is incident with no vertex of S. Let e_m be a such edge. Let $e_m = v_i v_j$. Then v_{ij} is dominated by no vertes of S, which is a contradiction.

Thus $\gamma(G(C_3)) \geq \alpha(G)$.

Hence $\gamma(G(C_3)) = \alpha(G)$. □

Theorem 2.2. *Let G be a graph of order $n \geq 2$. Then $\gamma(G(C_4)) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_2, e_2, \ldots, e_m\}$.

Initially, we show that $\gamma(G(C_4)) \leq n$.

Let $C_1, C_2, \ldots, C_m$ be the edge cycles of $e_1, e_2, \ldots, e_m$ respectively.

Let $S = \{v_1, v_2, \ldots, v_n\}$. Then clearly, $N[v_1, v_2, \ldots, v_n] = V(G(C_4))$. Therefore each vertex of $G(C_4)$ is adjacent to at least one vertex of S. It follows that S is a dominating set of G. Thus $\gamma(G(C_4)) \leq n$.

Next, we have to prove that $\gamma(G(C_4)) \geq n$.

Let S be a dominating set of $G(C_4)$.

Since G is connected, $d(v_i) \geq 1$ for all $1 \leq i \leq n$.

Now, let $d(v_i) = d_i$.

Let $v_{i1}, v_{i2}, \ldots, v_{id_i}$ be the new neighbors of v_i in $G(C_4)$.

Then $\langle\{v_i, v_{i1}, v_{i2}, \ldots, v_{id_i}\}\rangle = K_{1,d_i}$ for all $1 \leq i \leq n$ and $V(G(C_4)) = \{v_i, v_{i1}, v_{i2}, \ldots, v_{id_i}/1 \leq i \leq n\}$. Thus $|S \cap \{v_i, v_{i1}, v_{i2}, \ldots, v_{id_i}\}| \geq 1$ for all $1 \leq i \leq n$. It follows that $\gamma(G(C_4)) \geq n$.

Hence $\gamma(G(C_4)) = n$.

□

Theorem 2.3. *Let G be a graph of order $n \geq 2$ and m be the number of edges of G. Let $k \geq 6$ and $k \equiv 0 (mod\, 3)$. Then $\gamma(G(C_k)) = \alpha(G) + m(\frac{k-3}{3})$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$.

Let $C_1, C_2, \ldots, C_m$ be the corresponding edge cycles of $e_1, e_2, \ldots e_m$.

Let $V(C_i) = \{v_{i1}, v_{i2}, \ldots, v_{ik}\}$ and let $e_i = v_{i1}v_{ik}$ and $v_{i2}, v_{i3}, \ldots, v_{i(k-1)}$ are the new consecutive two degree vertices in $G(C_k)$. Here v_{i1} is adjacent to v_{i2} and v_{ik} is adjacent to $v_{i(k-1)}$.

Then we have $\langle\{v_{i1}, v_{ik}/1 \leq i \leq m\}\rangle \cong G$. Let $\{v_1, v_2, \ldots, v_n\} = \{v_{i1}, v_{ik}/1 \leq i \leq m\}$.

First, we have to prove that $\gamma(G(C_k)) \leq \alpha(G) + m(\frac{k-3}{3})$.

Let $X = \{v_1, v_2, \ldots, v_{\alpha(G)}\}$ be the minimum covering set of G.

Since X is a covering set of G, all the edges of G covered by a vertex of X. Therefore each edge in G is incident with a vertex of X.

Since every covering set is a dominating set, X is a dominating set of $\{v_{i1}, v_{ik}/1 \leq i \leq m\}$.

We observe that $\langle V(G(C_k)) \setminus K\rangle \cong mP_{k-3}$ and we know that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.

Let $G_1, G_2, \ldots, G_m$ be the union of m paths of $\langle V(G(C_k)) \setminus K\rangle$. Then $\gamma(G_i) = (\frac{k}{3} - 1)$ for all $1 \leq i \leq m$. Let S_i be the minimum dominating set of G_i for all $1 \leq i \leq m$.

Consequently, we have $S \cup S_1 \cup S_2 \cup \ldots \cup S_m$ is a dominating set of

$G(C_k)$.

Therefore $\gamma(G(C_k)) \leq |S| + |S_1| + |S_2| + \ldots + S_m$.
Thus $\gamma(G(C_k)) \leq \alpha(G) + m(\frac{k-3}{3})$.

Next, we have to prove that $\gamma(G(C_k)) \geq \alpha(G) + (\frac{k-3}{3})$.

Let S be a dominating set of $G(C_k)$.

We observe that all the new vertiecs are of degree two. Therefore $S \cap \{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}/1 \leq i \leq m\} \geq \frac{k}{3} - 1$ for all $1 \leq i \leq m$.

Next, we claim that $\left|G(C_k) \setminus S \cap \{v_{i2}, \ldots, v_{i(k-1)}/1 \leq i \leq m\}\right| \geq \alpha(G)$.

Suppose $\left\langle G(C_k)\ S \cap \{v_{i2}, \ldots, v_{i(k-1)/1 \leq i \leq m\}}\right\rangle \leq \alpha(G) - 1$. Let X be a such set. Then at least one edge of $\langle\{v_{i1}, v_{i2}/1 \leq i \leq m\}\rangle$ is not covered by X. Let $e_1 = u_1u_2$ be such an edge. Then new two degree vertex which is adjacent to u_1 or u_2 is not dominated by X when S is a minimum dominating set of $G(C_k)$.

Thus $\gamma(G(C_k)) \geq \alpha(G) + m(\frac{k-3}{3})$.
$\gamma(G(C_k)) = \alpha(G) + m(\frac{k-3}{3})$.

□

Theorem 2.4. *Let G be a graph of order $n \geq 2$ and m be the number of edges of G. Let $k \geq 7$ and $k \equiv 1(mod\ 3)$. Then $\gamma(G(C_k)) = n + m(\frac{k-1}{3})$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$.

Let $C_1, C_2, \ldots, C_m$ be the corresponding edge cycles of $e_1, e_2, \ldots e_m$.

Let $V(C_i) = \{v_{i1}, v_{i2}, \ldots, v_{ik}\}$ and let $e_i = v_{i1}v_{ik}$ and $v_{i2}, v_{i3}, \ldots, v_{i(k-1)}$ are the new consecutive two degree vertices in $G(C_k)$. Here v_{i1} is adjacent to v_{i2} and v_{ik} is adjacent to $v_{i(k-1)}$.

Then we have $< \{v_{i1}, v_{ik}/1 \leq i \leq m\} >\cong G$. Let $\{v_1, v_2, \ldots, v_n\} = \{v_{i1}, v_{ik}/1 \leq i \leq m\}$.

First, we have to prove that $\gamma(G(C_k)) \leq n + m(\frac{k-1}{3})$.

Let $S = V(G) \cup X_1 \cup X_2 \cup \ldots X_m$, where $X_i = \{v_{i4}, v_{i7}, \ldots, v_{i(k-3)}\}$ for all $1 \leq i \leq m$.

Then the vertices of $N[V(G)]$ are dominated by $V(G)$ and the vertices of $V(G(C_k)) \setminus (V(G)$ are dominate $\cup_{i=}^{m} X_i$.

$\gamma(G(C_k)) \leq |V(G)| + |X_1| + \ldots + |X_m|$.

But we have $\left|X_i = \{v_{i4}, v_{i7}, \ldots, v_{i(k-3)}\}\right| = \frac{k-1}{3}$ for all $1 \leq i \leq m$.

It follows that $\gamma(G(C_k)) \leq n + m(\frac{k-1}{3})$.

Next, we have to prove that $\gamma(G(C_k)) \geq n + m(\frac{k-1}{3})$.

Let $V(C_i) = \{v_{i1}, v_{i2}, \ldots, v_{ik}\}$, where $v_{i1}, v_{ik} \in V(G)$.

It follows that $|S| \geq n + m(\frac{k-1}{3})$

Thus $\gamma(G(C_k)) \geq n + m(\frac{k-1}{3})$.

Hence $\gamma(G(C_k)) = n + m(\frac{k-1}{3})$.

□

Theorem 2.5. *Let G be a graph of order $n \geq 2$ and m be the number of edges of G. Let $k \geq 5$ and $k \equiv 2(mod\, 3)$. Then $\gamma(G(C_k)) = \gamma(G) + m(\frac{k-2}{3})$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$.

Let $C_1, C_2, \ldots, C_m$ be the corresponding edge cycles of $e_1, e_2, \ldots e_m$.

Let $V(C_i) = \{v_{i1}, v_{i2}, \ldots, v_{ik}\}$ and let $e_i = v_{i1}v_{ik}$ and $v_{i2}, v_{i3}, \ldots, v_{i(k-1)}$ are the new consecutive two degree vertices in $G(C_k)$. Here v_{i1} is adjacent to v_{i2} and v_{ik} is adjacent to $v_{i(k-1)}$.

Let $V(G) = \{v_i, v_{ik}/1 \leq i \leq m\}$.

Then $V(G(C_k)) = V(G) \cup \{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}/1 \leq i \leq m\}$.

Let X be a minimum dominating set of G and S_i be the minimum dominating set of $\langle\{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}\}\rangle$ for all $1 \leq i \leq m$.

Then $\gamma(G(C_k)) \leq |X| + |S_1| + |S_2| + \ldots |S_m| = \gamma(G) + m(\frac{k-2}{3})$.

Hence $\gamma(G(C_k)) \leq \gamma(G) + m(\frac{k-2}{3})$.

Next, we have to prove that $\gamma(G(C_k)) \geq \gamma(G) + m(\frac{k-2}{3})$.

We observe that all the new vertices in $G(C_k)$ are of degree two and $\langle\{v_{i3}, v_{i4}, \ldots, v_{i(k-2)}\}\rangle \cong P_{k-4}$ for all $1 \leq i \leq m$.

We know that $\gamma(P_{k-4}) = \lceil \frac{k-4}{3} \rceil$.

Also $\left\langle\{v_{i2}, v_{i4}, \ldots, v_{i(k-1)}\}\right\rangle \cong P_{k-2}$ and $\gamma(P_{k-2}) = \left\lceil\frac{k-2}{3}\right\rceil = \frac{k-2}{3}$. Since $\left\lceil\frac{k-4}{3}\right\rceil = \frac{k-2}{3}$, $\left|S \cap \{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}\right\} \geq \frac{k-2}{3}$ for all $1 \leq i \leq m$. We observe that $\left\langle G(C_k) \setminus \{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}/1 \leq i \leq m\}\right\rangle \cong G$. Therefore, $\left|S \cap G(C_k) \setminus \{v_{i2}, v_{i3}, \ldots, v_{i(k-1)}/1 \leq i \leq m\}\right| \geq \gamma(G)$. Consequently, $|S \cap V(G(C_k))| \geq \gamma(G) + m(\frac{k-2}{3})$. Thus $\gamma(G(C_k)) \geq \gamma(G) + m(\frac{k-2}{3})$. Hence $\gamma(G(C_k)) = \gamma(G) + m(\frac{k-2}{3})$. □

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