On the $K-$continuity of a functor

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Abstract

We examine the concept of $K-$continuity of a functor from two perspectives: one considering $K-$continuity as given in some formulations of Shape theory and the other as a restriction of the usual definition of the continuity of a functor. We show that under a certain condition the concept of $K-$continuity from Shape theory includes the concept of $K-$continuity arising from the usual definition of continuity.

1 Introduction

Given functors $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ the $K-$continuity of the functor T is a concept developed in the categorical formulation of Shape theory given in the works of Bacon [B75] and Cordier and Porter [CP08]. Their construction is quite general in scope and has its place even when the limits of the functors K and TK don't exist.

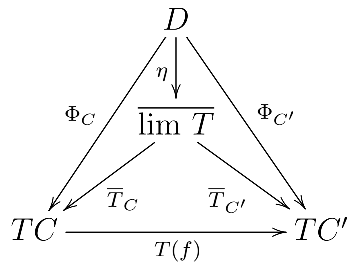
A similar concept of $K-$continuity arises as a particular case of the standard concept of the continuity of a functor, for example, as presented by Hofmann [Hf76] and, contrarily to the Shape theoretical construction, it demands the existence of the limits of K and TK.

It is the purpose of our study to examine how the concept of $K-$continuity arising from these two perspectives are related. We start by reviewing some aspects of both definitions.

The limit of a functor $T : \mathcal{C} \to \mathcal{D}$ may be seen as the cone $\{\overline{\lim\ T}, \overline{T}_C\}_{C \in \mathrm{Obj}_{\mathcal{C}}}$ with $\overline{\lim\ T} \in \mathrm{Obj}_{\mathcal{D}}$ and $\overline{T}_C \in \mathrm{Morf}_{\mathcal{D}}(\overline{\lim\ T}, TC)$ morphisms satisfying the universal property: $\forall \{D, \Phi_C\}_{C \in \mathrm{Obj}_{\mathcal{C}}}$ (cone of

2020 *Mathematics Subject Classification*: Primary 18A22; Secondary 18-02.
Key words and phrases: continuous functor, comma category, limit of a functor.

$D \in \mathrm{Obj}_{\mathcal{D}}$ over T), $\exists! \eta : D \overset{\mathcal{D}}{\to} \overline{\lim\, T}$ such that $\forall f : C \overset{\mathcal{C}}{\to} C'$ we have the diagram below commutative



(1)

with the relation $\Phi_C = \overline{T}_C\, \eta$ being a factorization for Φ_C.

Following Hofmann [Hf76] we may also conceive the limit of a functor $T : \mathcal{C} \to \mathcal{D}$ as a pair $\lim T = (\overline{\lim\, T}, \overline{T})$ with $\overline{T} : (\overline{\lim\, T})_{\mathcal{C}} \overset{\cdot}{\to} T$ being a natural transformation. This definition is equivalent to the one given previously in terms of the cone $\{\overline{\lim\, T}, \overline{T}_C\}_{C \in \mathrm{Obj}_{\mathcal{C}}}$ with the family of morphisms $\overline{T}_C$ being all grouped in $\overline{T}$. The fact that $\overline{T}$ is a natural transformation summarizes all relations between the morphisms $\overline{T}_C$ given in the commutative diagram of (1).

With this notion of limit let us consider a class of functors Ω. We say that T is $\Omega-$*continuous* if $\forall K \in \Omega$ ($K : \mathcal{B} \to \mathcal{C}$) it exists $\lim K = (\overline{\lim\, K}, \overline{K})$, $\lim TK = (\overline{\lim\, TK}, \overline{TK})$ and an invertible morphism $T_K : T\,\overline{\lim\, K} \overset{\mathcal{D}}{\to} \overline{\lim\, TK}$ satisfying (see theorem 2.7)

$$\overline{TK} = T\overline{K}\, T^{-1}_{K\mathcal{B}} \,. \tag{2}$$

We define T as *continuous* if we take Ω to be the class of all functors with small domain. We define T as *continuous for inverse systems* if we restrict the class Ω to include inverse systems. Then, since inverse systems are functors defined over domains that are directed sets they belong to the class Ω of all functors with small domain and, as a result, we see that the definition of continuity includes naturally the case of continuity for inverse systems.

Another particular case of $\Omega-$continuity, which we call $K-$continuity, appears if we take $\Omega = \{K\}$ for a fixed functor K. Then, due to this restriction, $K-$continuity of T doesn't imply necessarily the continuity of T for inverse systems.

In the Shape theory developed by Bacon [B75], and also by Cordier and Porter [CP08], the $K-$continuity of a functor $T : \mathcal{C} \to \mathcal{D}$ is developed without using the concept of the limit of a functor and relies on a class of functors belonging to $Func(C \downarrow K, D \downarrow TK)$, which is defined in terms of the comma categories $C \downarrow K$ and $D \downarrow TK$. The main motivation for this Shape theoretical formulation is to replace the limiting cone by the comma categories in cases where neither $\mathcal{C}$ nor $\mathcal{D}$ have small limits (see remark on pg. 29 in [CP08]). As we will see, the $K-$continuity of $T : \mathcal{C} \to \mathcal{D}$ is defined by the condition

$$V = g^* \delta_T \tag{3}$$

for certain functors $\delta_T : C \downarrow K \to TC \downarrow TK$ and $g^* : TC \downarrow TK \to D \downarrow TK$ where g^* is induced by a unique morphism $g : D \overset{\mathcal{D}}{\to} TC$.

Conditions (2) and (3) are the main elements on the two definitions given for the $K-$continuity of T and they exhibit a certain similarity provided we can associate $V \leftrightarrow \overline{TK}$, $\delta_T \leftrightarrow T\overline{K}$ and $g^* \leftrightarrow T^{-1}_{K\mathcal{B}}$.

This suggests us to investigate if there is a connection between these two definitions. In particular, we show that $K-$continuity in the sense of Bacon-Cordier-Porter garantees the $K-$continuity in the sense of Hofmann if we have $g : \overline{\lim\, TK} \overset{\mathcal{D}}{\to} T\,\overline{\lim\, K}$ invertible (see theorem 4.12). Another purpose of our study is to examine if in the perspective of the Shape theory of Bacon-Cordier-Porter we can redefine g^* and δ_T in such a way as to incorporate the limits of K and TK in cases those both limits exist.

Our work is organized as follows. In section 2 we review the standard definition of continuity as presented by Hofmann [Hf76]. In section 3 we review the concept of continuity for inverse systems where we consider inverse systems as functors. In section 4 we analyze the construction of $K-$continuity in the sense of Bacon, Cordier and Porter and show how it implies continuity for inverse systems. We also show how it includes the $K-$continuity in the sense of Hofmann provided we add one further restriction on the morphism g that induces g^* appearing in (3). In section 5 we establish a relation associating to each morphism used in the $K-$continuity definition of Hofmann a corresponding morphism used in the $K-$continuity definition of Bacon, Cordier, Porter. We show that under a certain condition we obtain a closer similarity of these two perspectives of $K-$continuity in the sense that the corresponding morphisms satisfy conditions having the same form. Section 6 is devoted to search if we can redefine the Bacon, Cordier, Porter construction in terms of the limits of K and TK, in case they both exist.

A word about notation. Given a functor $F : \mathcal{B} \to \mathcal{C}$ sometimes we write F_{ob} and F_{mo} to denote its action on the objects and morphisms of $\mathcal{B}$. A morphism $u \in \mathrm{Morf}_{\mathcal{B}}(B, B')$ is written as $u : B \overset{\mathcal{B}}{\to} B'$. Whenever we treat with inverses systems $\{X_\alpha, p_{\alpha\beta}\}_\Lambda$, Λ is a pre-ordered set where the indexes run. When we write relations like $p_\alpha = p_{\alpha\beta}\, p_\beta$, $u_\alpha = p_\alpha h$ etc. it is assumed they are valid $\forall\alpha \in \Lambda, \forall\beta \in \Lambda$, observing that $\alpha \leq \beta$ whenever it appears in $p_{\alpha\beta}$, therefore, for ease of notation we omit this information. We follow the convention of writing natural transformations putting a dot over the arrow, e.g. $\Psi : F \dot{\to} G$ denotes a natural transformation between functors F and G. We use BCP as a shorthand for Bacon-Cordier-Porter. Even though Hofmann doesn't explore the concept of $K-$continuity in his work [Hf76], we will say "$K-$continuity in the sense of Hofmann" to refer to this particular case of Ω-continuity when we take $\Omega = \{K\}$ and also to distinguish it from the concept of $K-$continuity in the sense of BCP.

2 The standard definition of the continuity of a functor

We analyze the concept of limit and continuity of a functor as presented by Hofmann and prove some results that were stated without proof in [Hf76]. The concept of $K-$continuity is established relative to a functor previously fixed and employs the standard definition of limit.

First we introduce the concept of a constant functor induced by an object.

Def. 2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $D \in Obj_{\mathcal{D}}$. We define a constant functor $D_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ as follows*

$D_{\mathcal{C}ob} : Obj_{\mathcal{C}} \to Obj_{\mathcal{D}}$ $\qquad\qquad$ $D_{\mathcal{C}mo} : Morf_{\mathcal{C}} \to Morf_{\mathcal{D}}$

$$C \to D_{\mathcal{C}ob}(C) := D \qquad\qquad h : C \xrightarrow{\mathcal{C}} C' \to D_{\mathcal{C}mo}(h) := 1_D$$

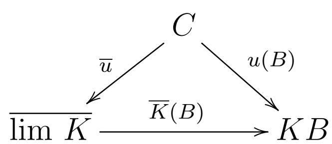
Given a morphism it induces a natural transformation as follows:

Def. 2.2. *Given categories $\mathcal{C}$ and $\mathcal{D}$ and a morphism $F : D \xrightarrow{\mathcal{D}} D'$ we define the natural transformation $F_{\mathcal{C}} : D_{\mathcal{C}} \dot{\to} D'_{\mathcal{C}}$ as*
$F_{\mathcal{C}} : Obj_{\mathcal{C}} \to Morf_{\mathcal{D}}$
$\quad C \to F_{\mathcal{C}}(C) := F.$

Now, we recall the standard definition of the limit of a functor, which is equivalent to the one given in terms of cones that we reviewed in the introduction.

Def. 2.3. *Let $K : \mathcal{B} \to \mathcal{C}$ be a functor. The* limit of K *consists of a pair $(\overline{\lim\, K}, \overline{K})$ with $\overline{\lim\, K} \in Obj_{\mathcal{C}}$ and $\overline{K} : (\overline{\lim\, K})_{\mathcal{B}} \dot{\to} K$ such that $\forall C \in Obj_{\mathcal{C}}$, $\forall u : C_{\mathcal{B}} \dot{\to} K$, $\exists! \overline{u} : C \xrightarrow{\mathcal{C}} \overline{\lim\, K}$ such that $\overline{K}\,\overline{u}_{\mathcal{B}} = u$.*

It is helpfull to see the condition $\overline{K}\,\overline{u}_{\mathcal{B}} = u$ expressed in terms of the commutative diagram below ($\forall B \in \mathrm{Obj}_{\mathcal{B}}$):



(4)

We say that $\overline{K}$ is the *limit morphism* and $\overline{\lim\, K}$ is the *limit object*. As a notation, we write $\lim K$ as a shorthand for the pair $(\overline{\lim\, K}, \overline{K})$.

Def. 2.4. *Let $\mathcal{C}, \mathcal{D}$ be categories. We define the category $\mathcal{D}^{\mathcal{C}}$ as follows. $Obj_{\mathcal{D}^{\mathcal{C}}}$ is the class having for elements functors $F : \mathcal{C} \to \mathcal{D}$. For functors $F, G : \mathcal{C} \to \mathcal{D}$ we define $Morf_{\mathcal{D}^{\mathcal{C}}}(F, G)$ as the set having for elements natural transformations $u : F \dot{\to} G$ with the composition being defined in terms of the composition of natural transformations. Given $F \in Obj_{\mathcal{D}^{\mathcal{C}}}$ the identity $1_F : F \dot{\to} F$ satisfies $1_F(C) = 1_{F(C)}, \forall C \in Obj_{\mathcal{C}}$.*

We will show that the association $u \leftrightarrow \overline{u}$ given in definition 2.3 is a bijection.

Theorem 2.5. *Let $K : \mathcal{B} \to \mathcal{C}$ be a functor with $\lim K = (\overline{\lim\, K}, \overline{K})$. The map $^{-} : Morf_{\mathcal{C}^{\mathcal{B}}}(C_{\mathcal{B}}, K) \to Morf_{\mathcal{C}}(C, \overline{\lim\, K})$ given by $u \to \overline{u}$ with $u(B) = \overline{K}(B)\overline{u}$ is a bijection.*

Proof. Given $u, v : C_{\mathcal{B}} \dot{\to} K$ we have associated $\overline{u}$ and $\overline{v}$ such that $u(B) = \overline{K}(B)\overline{u}$ and $v(B) = \overline{K}(B)\overline{v}$, $\forall B \in \mathcal{B}$, then $\overline{u} = \overline{v} \Rightarrow u = v$ and $^{-}$ is injective.

Let us now take $\varphi \in \mathrm{Morf}_{\mathcal{C}}(C, \overline{\lim\, K})$ and consider $\varphi_{\mathcal{B}} : C_{\mathcal{B}} \dot{\to} (\overline{\lim\, K})_{\mathcal{B}}$. Then $\overline{K} \circ \varphi_{\mathcal{B}} \in \mathrm{Morf}_{\mathcal{C}^{\mathcal{B}}}(C_{\mathcal{B}}, K)$ and we have a unique $\overline{\overline{K} \circ \varphi_{\mathcal{B}}} \in \mathrm{Morf}_{\mathcal{C}}(C, \overline{\lim\, K})$ satisfying $(\overline{K} \circ \varphi_{\mathcal{B}})(B) = \overline{K}(B)\overline{\overline{K} \circ \varphi_{\mathcal{B}}}$ $\therefore \overline{K}(B)\varphi = \overline{K}(B)\overline{\overline{K} \circ \varphi_{\mathcal{B}}}$ and the uniqueness of $\overline{\overline{K} \circ \varphi_{\mathcal{B}}}$ gives that $\overline{\overline{K} \circ \varphi_{\mathcal{B}}} = \varphi$, therefore $^{-}$ is surjective.

Then, $^{-}$ is bijective. □

We need a preliminary result.

Theorem 2.6. *Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors and let us assume that $\exists \lim K$. Then, $T \circ (\overline{\lim K})_{\mathcal{B}} = [T \overline{\lim K}]_{\mathcal{B}}$.*

Proof. It follows directly from definition 2.1. □

The next result garantees the existence of a unique funtor T_K provided it exists the limits of K and TK.

Theorem 2.7. *Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors. If $\exists \lim K$, $\exists \lim TK$ then $\exists! T_K : T \overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ such that $\overline{TK} T_{K\mathcal{B}} = T\overline{K}$, i.e. the diagram below is commutative*

$$\begin{array}{ccc} [T \overline{\lim K}]_{\mathcal{B}} & \xrightarrow{T_{K\mathcal{B}}} & (\overline{\lim TK})_{\mathcal{B}} \\ & {\scriptstyle T\overline{K}} \searrow \quad \swarrow {\scriptstyle \overline{TK}} & \\ & TK & \end{array} \tag{5}$$

Proof. Since it exists $\lim K$ it follows that there is a natural transformation $\overline{K} : (\overline{\lim K})_{\mathcal{B}} \dot{\to} K$. Using theorem 2.6 we consider the natural transformation $T\overline{K} : [T \overline{\lim K}]_{\mathcal{B}} \dot{\to} TK$. Since it also exists $\lim TK$ there is a natural transformation $\overline{TK} : (\overline{\lim TK})_{\mathcal{B}} \dot{\to} TK$ satisfying:

$$\forall v : D_{\mathcal{B}} \dot{\to} TK, \exists! \overline{v} : D \overset{\mathcal{D}}{\to} \overline{\lim TK} \ \text{ such that } \overline{TK}\, \overline{v}_{\mathcal{B}} = v \ . \tag{6}$$

Identifying in (6): $D \equiv T \overline{\lim K}$ and $v \equiv T\overline{K}$ we have that $\exists! \overline{v} : T \overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ such that $\overline{TK}\, \overline{v}_{\mathcal{B}} = T\overline{K}$. We identify the morphism T_K with $\overline{v}$ and this ends our proof. □

We are now equipped to define $K-$continuity in the sense of Hofmann.

Def. 2.8. *Let $K : \mathcal{B} \to \mathcal{C}$ be a functor. The functor $T : \mathcal{C} \to \mathcal{D}$ is K-*continuous *iff*
i. $\exists \lim K \Rightarrow \exists \lim TK$
ii. $T_K : T \overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ *is invertible.*

Remark 2.9. *If $T : \mathcal{C} \to \mathcal{D}$ is $K-$continuous we denote $T \lim K \simeq \lim TK$ to indicate that $T \overline{\lim K} \simeq \overline{\lim TK}$ and $T\overline{K} \simeq \overline{TK}$, which are understood as follows.*

By $T \overline{\lim K} \simeq \overline{\lim TK}$ we mean there is an invertible morphism $T_K : T \overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$, and by $T\overline{K} \simeq \overline{TK}$ we mean that $T\overline{K}$ and $\overline{TK}$ are related through $T\overline{K} = \overline{TK} T_{K\mathcal{B}}$.

3 Continuity for inverse systems

We recall that a partial ordered set $(\Lambda, \leq)$ may be seen as a category with the identification:

$$\text{Obj}_{\Lambda} := \Lambda \ , \qquad \text{Morf}_{\Lambda}(\alpha, \beta) := \begin{cases} \{\alpha \overset{\Lambda}{\to} \beta\} \ \text{ if } \ \alpha \leq \beta \\ \emptyset \ \text{ if } \ \alpha \not\leq \beta \ . \end{cases} \tag{7}$$

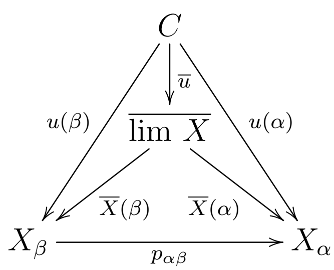
Then an inverse system in a category $\mathcal{C}$ indexed by Λ is a contravariant functor $X : \Lambda \to \mathcal{C}$.

With this interpretation, the inverse limit of the inverse system X becomes the limit of the functor $X : \Lambda \to \mathcal{C}$ i.e. $\lim X = (\overline{\lim X}, \overline{X})$ where $\overline{\lim X} \in \mathrm{Obj}_{\mathcal{C}}$ and $\overline{X} : (\overline{\lim X})_\Lambda \dot{\to} X$ is a natural transformation such that $\forall C \in \mathrm{Obj}_{\mathcal{C}}$, $\forall u : C_\Lambda \dot{\to} X$, $\exists! \overline{u} : C \stackrel{\mathcal{C}}{\to} \overline{\lim X}$ such that $\overline{X}\ \overline{u}_\Lambda = u$.

Remark 3.1. *Given an inverse system* $X : \Lambda \to \mathcal{C}$ *we denote* $X_\alpha := X_{ob}(\alpha)$ *and* $p_{\alpha\beta} := X_{mo}(\alpha \stackrel{\Lambda}{\to} \beta)$. *Then the inverse system* $X : \Lambda \to \mathcal{C}$ *may be seen as a family of objects and morphisms that we write as* $\{X_\alpha, p_{\alpha\beta}\}_\Lambda$. *If there is the inverse limit* $\lim X = (\overline{\lim X}, \overline{X})$ *we denote* $X_\infty := \overline{\lim X}$ *and* $p_\alpha := \overline{X}(\alpha) : X_\infty \stackrel{\mathcal{C}}{\to} X_\alpha$, *then we identify the inverse limit with the limit cone* $\{X_\infty, p_\alpha\}_\Lambda$. *With this notation we write* $(\overline{\lim X}, \overline{X}) = \lim X$ *as* $\{X_\infty, p_\alpha\}_\Lambda = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_\Lambda$.

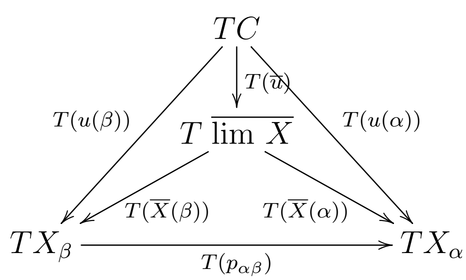
Def. 3.2. *Let* Ω *be the class of inverse systems on* $\mathcal{C}$. $T : \mathcal{C} \to \mathcal{D}$ *is continuous for inverse systems iff* T *is* $X-$*continuous for any* $X \in \Omega$.

Remark 3.3. *From definition 2.8 if* $T : \mathcal{C} \to \mathcal{D}$ *is* $X-$*continuous then it exists:* $\lim X = (\overline{\lim X}, \overline{X})$, $\lim TX = (\overline{\lim TX}, \overline{TX})$ *and an invertible morphism* $T_X : T\ \overline{\lim X} \stackrel{\mathcal{D}}{\to} \overline{\lim TX}$. *Then, existing* $\lim X$ *by definition we have the commutative diagram below* $\forall\, \alpha \stackrel{\Lambda}{\to} \beta$



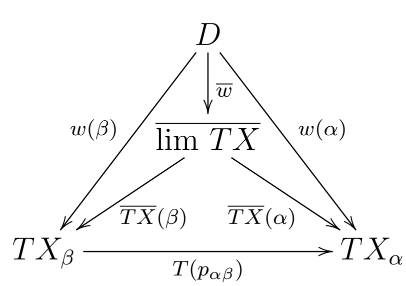
(8)

which is transformed by the functor T *into a diagram*



(9)

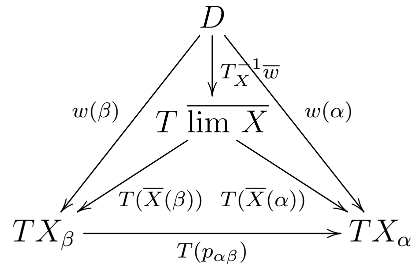
Likewise, existing $\lim TX$ *we have that for all* $w : D_\Lambda \dot{\to} TX$, $\exists! \overline{w} : D \stackrel{\mathcal{D}}{\to} \overline{\lim TX}$ *such that* $\forall \alpha \stackrel{\Lambda}{\to} \beta$ *the diagram below is commutative*



(10)

Since T_X is invertible we use $T_X^{-1} : \overline{\lim TX} \overset{\mathcal{D}}{\to} T\ \overline{\lim X}$ to insert part of the diagram (9) into the diagram (10), which gives

(11)



Comparing (10) and (11) we have that $(T\,\overline{\lim X}, T\overline{X}) \simeq (\overline{\lim TX}, \overline{TX})$*, i.e.* $T \lim X \simeq \lim TX$ *where* $\simeq$ *is the same identification established in remark 2.9 as follows:* $T\,\overline{\lim X} \simeq \overline{\lim TX}$ *is given by the functor* $T_X : T\,\overline{\lim X} \overset{\mathcal{D}}{\to} \overline{\lim TX}$ *given in theorem 2.7, and* $T\overline{X} \simeq \overline{TX}$ *is given by the relation* $T\overline{X}(\alpha) = \overline{TX}(\alpha)T_X, \forall \alpha \in \Lambda$.

Using the notation of Holsztynski introduced in remark 3.1, where inverses systems are seen as $\{X_\alpha, p_{\alpha\beta}\}_\Lambda$ and the inverse limit as $\{X_\infty, p_\alpha\}_\Lambda = \varprojlim\{X_\alpha, p_{\alpha\beta}\}_\Lambda$, we obtain an equivalent definition for the continuity for inverse systems:

Def. 3.4. *$T : \mathcal{C} \to \mathcal{D}$ is continuous for inverse systems iff*

$$T \varprojlim \{X_\alpha, p_{\alpha\beta}\}_\Lambda \simeq \varprojlim T\{X_\alpha, p_{\alpha\beta}\}_\Lambda \tag{12}$$

$\forall\{X_\alpha, p_{\alpha\beta}\}_\Lambda$ *inverse system on $\mathcal{C}$.*

Diagram (11) allow us to state condition (12) in the form:
$T : \mathcal{C} \to \mathcal{D}$ is continuous for inverse systems iff $\forall\{X_\alpha, p_{\alpha\beta}\}_\Lambda$, inverse system in $\mathcal{C}$ with $\{\overline{\lim X}, \overline{X}(\alpha)\}_\Lambda = \varprojlim\{X_\alpha, p_{\alpha\beta}\}_\Lambda$, we have that $\{TX_\alpha, T(p_{\alpha\beta})\}_\Lambda$ is an inverse system in $\mathcal{D}$ satisfying:
$\forall w_\alpha : D \overset{\mathcal{D}}{\to} TX_\alpha$ with $w_\alpha = T(p_{\alpha\beta})w_\beta, \exists! \eta : D \overset{\mathcal{D}}{\to} T\,\overline{\lim X}$ with

$$w_\alpha = T(\overline{X}(\alpha))\eta\ . \tag{13}$$

For further use we write down Holsztynski's definition of projection [H71].

Def. 3.5. *Let $\mathcal{B}$ and $\mathcal{C}$ be categories with $Obj_{\mathcal{B}} = Obj_{\mathcal{C}}$. A functor $K : \mathcal{B} \to \mathcal{C}$ is called a* projection *iff $K(B) = B, \forall B \in Obj_{\mathcal{B}}$, and $K : Morf_{\mathcal{B}}(B, B') \to Morf_{\mathcal{C}}(B, B')$ is surjective $\forall B, B' \in Obj_{\mathcal{B}}$.*

Then, when $K : \mathcal{B} \to \mathcal{C}$ is a projection it becomes implicit that we are dealing with categories $\mathcal{B}$ and $\mathcal{C}$ with $\mathrm{Obj}_{\mathcal{B}} = \mathrm{Obj}_{\mathcal{C}}$.

4 The Bacon-Cordier-Porter's (BCP) definition of continuity

We review the concept of $K-$continuity developed by BCP. First, we need to introduce some preliminary concepts (see also [CP08]).

Def. 4.1. *Let $K : \mathcal{B} \to \mathcal{C}$ be a functor and $C \in Obj_{\mathcal{C}}$. The* comma category of $K-$objects under C *is the category $C \downarrow K$ defined as follows:*

$Obj_{C\downarrow K} := \{(f,B) \mid B \in Obj_{\mathcal{B}}, f \in Morf_{\mathcal{C}}(C,KB)\}$

$Morf_{C\downarrow K}\big((f,B),(f',B')\big) := \{h : B \overset{\mathcal{B}}{\to} B' \mid f' = K(h)f\}.$

Then, $Morf_{C\downarrow K}\big((f,B),(f',B')\big) \subset Morf_{\mathcal{B}}(B,B')$ and what we concretely define as $h : (f,B) \overset{C\downarrow K}{\to} (f',B')$ is in fact a morphism $h : B \overset{\mathcal{B}}{\to} B'$.

Def. 4.2. *Let $K : \mathcal{B} \to \mathcal{C}$ be a functor, $C \in Obj_{\mathcal{C}}$ and consider $C \downarrow K$. We define the* codomain functor $\delta^{C\downarrow K} : C \downarrow K \to \mathcal{B}$ *as follows*

$$\begin{aligned}\delta^{C\downarrow K}_{ob} : Obj_{C\downarrow K} &\to Obj_{\mathcal{B}}\\ (f,B) &\to \delta^{C\downarrow K}_{ob}(f,B) := B\end{aligned}$$

$$\begin{aligned}\delta^{C\downarrow K}_{mo} : Morf_{C\downarrow K}\big((f,B),(f',B')\big) &\to Morf_{\mathcal{B}}(B,B')\\ h &\to \delta^{C\downarrow K}_{mo}(h) := h.\end{aligned}$$

Def. 4.3. *Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors and $C \in Obj_{\mathcal{C}}$, $D \in Obj_{\mathcal{D}}$. We define $Func(C \downarrow K, D \downarrow TK)$ as the class having for elements functors $V : C \downarrow K \to D \downarrow TK$ such that $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$.*

From this condition, $Func\,(C \downarrow K, D \downarrow TK)$ may be characterized in terms of a map V^* as we see in the next result.

Theorem 4.4. *The condition $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$ fixes the form of $V \in Func\,(C \downarrow K, D \downarrow TK)$ as follows*

$$\begin{aligned}V_{ob} : Obj_{C\downarrow K} &\to Obj_{D\downarrow TK}\\ (f,B) &\to V_{ob}(f,B) := (V^*(f),B)\end{aligned}$$

$$\begin{aligned}V_{mo} : Morf_{C\downarrow K}\big((f,B),(f',B')\big) &\to Morf_{D\downarrow TK}\big((V^*(f),B),(V^*(f'),B')\big)\\ h &\to V_{mo}(h) = h\end{aligned}$$

where

$$\begin{aligned}V^* : \cup_{B\in Obj_{\mathcal{B}}} Morf_{\mathcal{C}}(C,KB) &\to \cup_{B\in Obj_{\mathcal{B}}} Morf_{\mathcal{D}}(D,TKB)\\ f : C \overset{\mathcal{C}}{\to} KB &\to V^*(f) : D \overset{\mathcal{D}}{\to} TKB\end{aligned}$$

satisfies

$$V^*(f') = TK(h)V^*(f) \tag{14}$$

$\forall f : C \overset{\mathcal{C}}{\to} KB,\ \forall f' : C \overset{\mathcal{C}}{\to} KB',\ \forall h : B \overset{\mathcal{B}}{\to} B' \in Morf_{C\downarrow K}\big((f,B),(f',B')\big)$

Proof. It follows straightforwardly from the condition $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$. □

Remark 4.5. *Given f, f' the condition $f' = K(h)f$ imposes a restriction on the form of $h : B \overset{\mathcal{B}}{\to} B'$. The other condition $V^*(f') = TK(h)V^*(f)$ imposes a restriction on the form of V^*. In some cases, it may happen the latter condition follows from the former, but this is not a general case.*

Def. 4.6. *Let $K:\mathcal{B}\to\mathcal{C}$ and $T:\mathcal{C}\to\mathcal{D}$ be functors. We define the functor $\delta_T : C\downarrow K \to TC\downarrow TK$ as*

$\delta_{Tob} : Obj_{C\downarrow K} \to Obj_{TC\downarrow TK}$

$$(f,B)\to \delta_{Tob}(f,B) := (T(f),B)$$

$\delta_{Tmo} : Morf_{C\downarrow K}\big((f,B),(f',B')\big) \to Morf_{TC\downarrow TK}\big((T(f),B),(T(f'),B')\big)$

$$h \to \delta_{Tmo}(h) := h.$$

Our next definition associates to every morphism $g : D \overset{\mathcal{D}}{\to} TC$ an induced functor between comma categories.

Def. 4.7. *Let $K:\mathcal{B}\to\mathcal{C}$ and $T:\mathcal{C}\to\mathcal{D}$ be functors. Given a morphism $g : D\overset{\mathcal{D}}{\to} TC$ with $D\in Obj_{\mathcal{D}}$ and $C\in Obj_{\mathcal{C}}$, it induces a functor between comma categories $g^* : TC\downarrow TK \to D\downarrow TK$ defined as follows*

$g^*_{ob} : Obj_{TC\downarrow TK} \to Obj_{D\downarrow TK}$

$$(w,B)\to g^*_{ob}(w,B) := (wg,B)$$

$g^*_{mo} : Morf_{TC\downarrow TK}\big((w,B),(w',B')\big) \to Morf_{D\downarrow TK}\big((wg,B),(w'g,B')\big)$

$$u \to g^*_{mo}(u) := u.$$

We are now equipped to define $K-$continuity of a functor according to BCP.

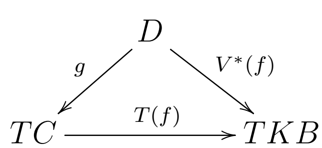
Def. 4.8. *Let $K:\mathcal{B}\to\mathcal{C}$ and $T:\mathcal{C}\to\mathcal{D}$ be functors. We say that T is $K-$*continuous *at $C\in \mathrm{Obj}_{\mathcal{C}}$ iff $\forall D\in Obj_{\mathcal{D}}$, $\forall V\in Func(C\downarrow K, D\downarrow TK)$, $\exists! g : D\overset{\mathcal{D}}{\to} TC$ such that $V = g^*\,\delta_T$.*

The condition on V is equivalent to the form given below:

$\forall D\in \mathrm{Obj}_{\mathcal{D}}$, $\forall V^* : \cup_{B\in \mathrm{Obj}_{\mathcal{B}}}\mathrm{Morf}_{\mathcal{C}}(C,KB) \to \cup_{B\in \mathrm{Obj}_{\mathcal{B}}}\mathrm{Morf}_{\mathcal{D}}(D,TKB)$
satisfying $V^*(f') = TK(h)V^*(f), \forall f : C\overset{\mathcal{C}}{\to} KB$, $\forall f' : C\overset{\mathcal{C}}{\to} KB'$, $\forall h : B\overset{\mathcal{B}}{\to} B' \in \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B')\big)$,
$\exists! g : D\overset{\mathcal{D}}{\to} TC$ such that $\forall f : C\overset{\mathcal{C}}{\to} KB$

$$V^*(f) = T(f)g, \tag{15}$$

The last relation is summarized in the commutative diagram



We say that T is $K-$*continuous* if T is $K-$continuous $\forall C\in \mathrm{Obj}_{\mathcal{C}}$.

Remark 4.9. *We observe the consistency of (15). In fact, for any $f' : C\overset{\mathcal{C}}{\to} KB'$ with $f' = K(h)f$ we expect to have $V^*(f') = T(f')g$. But $T(f') = T(K(h))T(f)$ and then $T(f')g = TK(h)T(f)g \therefore V^*(f') = TK(h)V^*(f)$, which is relation (14).*

Next we examine how the concept of $K-$continuity implies continuity relative to inverse systems as stated in definition 3.4. We use the notation for inverse system introduced in remark 3.1.

Theorem 4.10. *Let $K : \mathcal{B} \to \mathcal{C}$ be a projection satisfying the condition: $\forall \{X_\alpha, p_{\alpha\beta}\}_\Lambda$ inverse system in $\mathcal{C}$ with $\{X_\infty, p_\alpha\}_\Lambda = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_\Lambda$ we have $K\big(Morf_{C\downarrow K}\big((p_\beta, X_\beta), (p_\alpha, X_\alpha)\big)\big) = \{p_{\alpha\beta} : X_\beta \overset{\mathcal{C}}{\to} X_\alpha\}$. If $T : \mathcal{C} \to \mathcal{D}$ is $K-$continuous then T is continuous for inverse systems.*

Proof. Let $\{X_\alpha, p_{\alpha\beta}\}_\Lambda$ be an inverse system in $\mathcal{C}$ and let us assume there is defined the inverse limit $\{X_\infty, p_\alpha\}_\Lambda = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_\Lambda$. Given the covariant functor $T : \mathcal{C} \to \mathcal{D}$ we have that $\{TX_\alpha, T(p_{\alpha\beta})\}_\Lambda$ is an inverse system in $\mathcal{D}$ and $\{T\,X_\infty, T(p_\alpha)\}_\Lambda$ satisfies

$$T(p_\alpha) = T(p_{\alpha\beta})T(p_\beta)\ . \tag{16}$$

Let $\{w_\alpha : D \overset{\mathcal{D}}{\to} TX_\alpha\}_\Lambda$ be such that

$$w_\alpha = T(p_{\alpha\beta})w_\beta\ . \tag{17}$$

For $X_\infty \in \mathrm{Obj}_\mathcal{C}$ and $D \in \mathrm{Obj}_\mathcal{D}$ let us consider $Func(X_\infty \downarrow K, D \downarrow TK)$. Every $V \in Func(X_\infty \downarrow K, D \downarrow TK)$ is characterized by a map $V^* : \cup_{B\in \mathrm{Obj}_\mathcal{B}} \mathrm{Morf}_\mathcal{C}(X_\infty, B) \to \cup_{B\in \mathrm{Obj}_\mathcal{B}} \mathrm{Morf}_\mathcal{D}(D, TB)$ [1] satisfying (14), which reads as

$\forall f : X_\infty \overset{\mathcal{C}}{\to} B$, $\forall f' : X_\infty \overset{\mathcal{C}}{\to} B'$, $\forall h : B \overset{\mathcal{B}}{\to} B' \in \mathrm{Morf}_{X_\infty\downarrow K}\big((f, B), (f', B')\big)$ we have $V^*(f') = TK(h)V^*(f)$.

Consider now a particular choice for V^* such that $V^*(p_\gamma) = w_\gamma$. That this choice exists it is readily seen for if we take $f = p_\beta : X_\infty \overset{\mathcal{C}}{\to} X_\beta$, $f' = p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha$ and $h : X_\beta \overset{\mathcal{B}}{\to} X_\alpha \in \mathrm{Morf}_{X_\infty\downarrow K}\big((p_\beta, X_\beta), (p_\alpha, X_\alpha)\big)$ then for the projection we are considering we have $K(h) = p_{\alpha\beta}$ and $V^*(p_\alpha) = T(K(h))V^*(p_\beta)$, where this last condition is garanteed by (17).

Since T is continuous, from (15) we have that $\exists! g : D \overset{\mathcal{D}}{\to} TX_\infty$ such that for $p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha$ we have $V^*(p_\alpha) = T(p_\alpha)g$ i.e.

$$w_\alpha = T(p_\alpha)g\ . \tag{18}$$

From (16), (17) and (18) we have fulfilled condition (12)

$$\{TX_\infty, T(p_\alpha)\}_\Lambda = \varprojlim \{TX_\alpha, T(p_{\alpha\beta})\}_\Lambda$$

i.e. $T : \mathcal{C} \to \mathcal{D}$ is continuous for inverse systems. □

Remark 4.11. *In definition 4.8 the definition of K-continuity for $T : \mathcal{C} \to \mathcal{D}$ assumes the existence of a unique morphism g, but does not specify the conditions for this morphism to exist. However, for the projection K given in theorem 4.10, the morphism g is identified with the morphism η appearing in (13), then here we may infer that the conditions for the existence and uniqueness of η associated to the inverse limit of $\{TX_\alpha, T(p_{\alpha\beta})\}_\Lambda$ are sufficient for the existence of g.*

Theorem 4.12. *Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors such that $\exists \lim K$, $\exists \lim TK$ and T is $K-$continuous in the sense of BCP. For $V \in Func\,(\overline{\lim K} \downarrow K, \overline{\lim TK} \downarrow TK)$ with $V^*(\overline{K}(B)) = \overline{TK}(B)$ if $g : \overline{\lim TK} \overset{\mathcal{D}}{\to} T\,\overline{\lim K}$ is invertible then T is $K-$continuous in the sense of Hofmann.*

[1] Since K is a projection we have $\mathrm{Obj}_\mathcal{B} = \mathrm{Obj}_\mathcal{C}$ and $KB = B$ and $TKB = TB$.

Proof. First, we observe the consistency in considering $V^*(\overline{K}(B)) = \overline{TK}(B)$. In fact, if it exists $\lim K$ and $\lim TK$ then $\forall h : B \overset{\mathcal{B}}{\to} B'$ we have $\overline{K}(B') = K(h)\overline{K}(B)$, and $\overline{TK}(B') = TK(h)\overline{TK}(B)$. Then the identification
$V^*(\overline{K}(B)) = \overline{TK}(B)$ satisfies $V^*(\overline{K}(B')) = TK(h)V^*(\overline{K}(B))$,
$\forall h \in \mathrm{Morf}_{\overline{\lim K}\downarrow K}\big((\overline{K}(B), B), (\overline{K}(B'), B')\big)$.

Since T is $K-$continuous in the sense of BCP for this V^* there is a unique $g : \overline{\lim TK} \to T\,\overline{\lim K}$ such that $V^*(\overline{K}(B)) = T(\overline{K}(B))g$. By assumption g is invertible then we have

$$V^*(\overline{K}(B))g^{-1} = T(\overline{K}(B)) \ . \tag{19}$$

If it exists $\lim K$ and $\lim TK$ then $\exists! T_K : T\,\overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ with
$\overline{TK}(B)T_K = T(\overline{K}(B))$. Then we have

$$V^*(\overline{K}(B))T_K = T(\overline{K}(B)) \ . \tag{20}$$

Since g is the unique morphism satisfying (19) we conclude from (20) that T_K is identified with g^{-1}. Then, since g is invertible it follows that T_K is also invertible, i.e. T is $K-$continuous in the sense of Hofmann. □

5 A condition making the $K-$continuity definition of Hofmann and BCP equivalent in form

We compare the definitions of $K-$continuity given by Hofmann and by BCP noticing that we can associate to some morphisms in the Hofmann construction a counterpart in the BCP construction with the conditions satisfied by these morphisms having the same form.

Reexamining the Hofmann construction

The main elements of the definition of $K-$continuity given by Hofmann are the existence of the limits of K, TK and the existence of an isomorphism $T_K : T\,\overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$.

In what concerns the existence of $\lim K = (\overline{\lim K}, \overline{K})$, we have the conditions:

i. $\forall h : B \overset{\mathcal{B}}{\to} B'$ we have $\overline{K}(B') = K(h)\overline{K}(B)$.
ii. $\forall u : C_{\mathcal{B}} \dot{\to} K$, $\forall h : B \overset{\mathcal{B}}{\to} B'$ we have $u(B') = K(h)u(B)$.
iii. $\exists! \overline{u} : C \overset{\mathcal{C}}{\to} \overline{\lim K}$ such that $\forall B \in \mathrm{Obj}_{\mathcal{B}}$ we have $u(B) = \overline{K}(B)\overline{u}$.

In what concerns the existence of $\lim TK = (\overline{\lim TK}, \overline{TK})$, we have the following conditions:

iv. $\forall h : B \overset{\mathcal{B}}{\to} B'$ we have $\overline{TK}(B') = TK(h)\overline{TK}(B)$.
v. $\forall v : D_{\mathcal{C}} \dot{\to} TK$, $\forall h : B \overset{\mathcal{B}}{\to} B'$ we have $v(B') = TK(h)v(B)$.
vi. $\exists! \overline{v} : D \overset{\mathcal{D}}{\to} \overline{\lim TK}$ such that $\forall B \in \mathrm{Obj}_{\mathcal{B}}$ we have $v(B) = \overline{TK}(B)\overline{v}$.

And for the isomorphism $T_K : T\,\overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ we have:

vii. $\overline{TK}\,T_{K\mathcal{B}} = T\overline{K}$

From (vii) we have $\overline{TK}(B) = T(\overline{K}(B))T_K^{-1}$ and we rewrite the condition (vi) as

vi′. $\exists!\overline{v} : D \stackrel{\mathcal{D}}{\to} \overline{\lim\, TK}$ such that $\forall B \in \mathrm{Obj}_{\,\mathcal{B}}$ we have $v(B) = T(\overline{K}(B))T_K^{-1}\overline{v}$.

Then we characterize the Hofmann construction in terms of the conditions i, ii, iii, iv, v, vi′.

Reexamining the BCP construction

The main elements of the definition of $K-$continuity given by BCP are the comma categories $C \downarrow K$, $D \downarrow TK$ and the space $Func\,(C \downarrow K, D \downarrow TK)$.

As we have seen $T : \mathcal{C} \to \mathcal{D}$ is K-continuous at $C \in \mathrm{Obj}_{\,\mathcal{C}}$ iff $\forall D \in \mathrm{Obj}_{\,\mathcal{D}}$, $\forall V^* : \cup_{B\in \mathrm{Obj}_{\,\mathcal{B}}} \mathrm{Morf}_{\,\mathcal{C}}(C, KB) \to \cup_{B\in \mathrm{Obj}_{\,\mathcal{B}}} \mathrm{Morf}_{\,\mathcal{D}}(D, TKB)$ we have satisfied the following conditions:

viii. $\forall f : C \stackrel{\mathcal{C}}{\to} KB$, $\forall f' : C \stackrel{\mathcal{C}}{\to} KB'$, $\forall h : B \stackrel{\mathcal{B}}{\to} B'$ with h satisfying $f' = K(h)f$ we have $V^*(f') = TK(h)V^*(f)$.

ix. $\exists!g : D \stackrel{\mathcal{D}}{\to} TC$ such that $\forall f : C \stackrel{\mathcal{C}}{\to} KB$ we have $V^*(f) = T(f)g$.

Comparing both constructions

Examining conditions (ii), (v) and (viii) they suggest us to relate:

$$f \leftrightarrow u(B),\ f' \leftrightarrow u(B'),\ V^*(f) \leftrightarrow v(B),\ V^*(f') \leftrightarrow v(B') \tag{21}$$

and then we identify a similarity between the equations satisfied between these elements in the sense that we have

$$f' = K(h)f \quad \leftrightarrow \quad u(B') = K(h)u(B) \tag{22}$$

$$V^*(f') = TK(h)V^*(f) \quad \leftrightarrow \quad v(B') = TK(h)v(B) \tag{23}$$

Examining conditions (iii), (vi′) and (ix) it seems in the Hofmann construction we lack a morphism $D \stackrel{\mathcal{D}}{\to} TC$ that has $g : D \stackrel{\mathcal{D}}{\to} TC$ as a counterpart in the BCP construction. In order to obtain this let us conjecture:
$\forall\overline{v} : D \stackrel{\mathcal{D}}{\to} \overline{\lim\, TK}, \exists!\chi : D \stackrel{\mathcal{D}}{\to} TC$ such that $\forall\overline{u} : C \stackrel{\mathcal{C}}{\to} \overline{\lim\, K}$

$$T_K^{-1}\overline{v} = T(\overline{u})\chi\ . \tag{24}$$

Using that $T_K^{-1}\overline{v} = T(\overline{u})\chi$ we obtain from (vi′) that
$v(B) = T(\overline{K}(B))T(\overline{u})\chi = T(\overline{K}(B)\overline{u})\chi = T(u(B))\chi$ i.e. $v(B) = T(u(B))\chi$. Then since there is a bijection between u and $\overline{u}$ and between v and $\overline{v}$, we get a condition that becomes the counterpart in Hofmann's development of the condition (ix) of BCP:

$\forall v : D_{\mathcal{C}} \stackrel{\cdot}{\to} TK$, $\exists!\chi : D \stackrel{\mathcal{D}}{\to} TC$ such that $\forall u : C_{\mathcal{B}} \stackrel{\cdot}{\to} K$ we have $\forall B \in \mathrm{Obj}_{\,\mathcal{B}}$, $v(B) = T(u(B))\chi$

Then, completing the scheme shown in (21), (22) and (23) we have

$$
\begin{array}{rcl}
\underline{\textbf{BCP}} & & \underline{\textbf{Hofmann}} \\
(f,f') & \leftrightarrow & (u(B),u(B')) \\
f' = K(h)f & \leftrightarrow & u(B') = K(h)u(B) \\
(V^*(f),V^*(f')) & \leftrightarrow & (v(B),v(B')) \\
V^*(f') = TK(h)V^*(f) & \leftrightarrow & v(B') = TK(h)v(B) \\
g : D \overset{\mathcal{D}}{\rightarrow} TC & \leftrightarrow & \chi : D \overset{\mathcal{D}}{\rightarrow} TC \\
V^*(f) = T(f)g & \leftrightarrow & v(B) = T(u(B))\chi
\end{array}
\tag{25}
$$

with

$u(B) = \overline{K}(B)\overline{u},\ u(B') = \overline{K}(B')\overline{u}$

$v(B) = \overline{TK}(B)\overline{v},\ v(B') = \overline{TK}(B')\overline{v}$

$TK^{-1}\overline{v} = T(\overline{u})\chi.$

Remark 5.1. *$\forall h \in Morf_{C\downarrow K}\big((f,B),(f',B')\big)$ we have $f' = K(h)f$. Then, if we identify $u(B) = f$ since we have $u(B') = K(h)u(B)$, $\forall h : B \overset{\mathcal{B}}{\rightarrow} B'$ we have that $u(B') = f'$ and $Morf_{C\downarrow K}\big((f,B),(f',B')\big) = Morf_{\mathcal{B}}(B,B')$ that is a strong requirement. The same happens if we try to identify $v(B) = V^*(f)$. Then, the associations*

$$f \leftrightarrow u(B),\ f' \leftrightarrow u(B'),\ V^*(f) \leftrightarrow v(B),\ V^*(f') \leftrightarrow v(B'),\ g \leftrightarrow \chi$$

showed in (25) are not equalities, but work at the same level of what we see in the representation of groups where the equations these quantities satisfy have the same form despite the representation we are using.

6 A modification of the BCP construction in cases we have defined $\lim K$ and $\lim TK$

We search for an alternative formulation of the BCP construction presented in section 4 that takes into account the limits of K and TK.

Let us assume that

$\forall B \in \mathrm{Obj}_{\mathcal{B}}, \exists G_B : \overline{\lim\, K} \overset{\mathcal{C}}{\rightarrow} KB$ such that

i. $\forall h : B \overset{\mathcal{B}}{\rightarrow} B',\ T(G_{B'}) = TK(h)T(G_B)$

ii. $\forall f : C \overset{\mathcal{C}}{\rightarrow} KB, \exists! \eta_f : C \overset{\mathcal{C}}{\rightarrow} \overline{\lim\, K}$ with $f = G_B\, \eta_f$.

We start keeping definitions 4.1, 4.2, 4.3 and replacing definition 4.6 by the following one:

Def. 6.1. *Let $K : \mathcal{B} \rightarrow \mathcal{C}$ be a functor such that $\exists \lim K = (\overline{\lim\, K}, \overline{K})$. We define the functor $\delta_T : C \downarrow K \rightarrow TC \downarrow T(\overline{\lim\, K})_{\mathcal{B}}$ as*

$\delta_{Tob} : Obj_{C\downarrow K} \to Obj_{TC\downarrow T\circ(\overline{\lim K})_{\mathcal{B}}}$
$$(f,B) \to \delta_{Tob}(f,B) := (T(\eta_f), B)$$
with $f = G_B\,\eta_f : C \overset{\mathcal{C}}{\to} KB$

$\delta_{Tmo} : Morf_{C\downarrow K}((f,B),(f',B')) \to Morf_{TC\downarrow T\circ(\overline{\lim K})_{\mathcal{B}}}\big((T(\eta_f),B),(T(\eta_{f'}),B')\big)$
$$h \to \delta_{Tmo}(h) := h$$

with h, $T\eta_f$ *and* $T\eta_{f'}$ *satisfying*

$f' = K(h)f$ $\qquad\qquad T(\eta_{f'}) = T\circ(\overline{\lim K})_{\mathcal{B}}(\delta_{Tmo}(h))T(\eta_f)$

$\therefore G_{B'}\,\eta_{f'} = K(h)G_B\,\eta_f$ $\qquad\qquad \therefore T(\eta_{f'}) = T(\eta_f).$

We also replace definition 4.7 by

Def. 6.2. *Given a morphism* $g : D \overset{\mathcal{D}}{\to} TC$ *it induces a functor* $g^* : TC \downarrow T\circ(\overline{\lim K})_{\mathcal{B}} \to D\downarrow TK$ *given by*

$g^*_{ob} : Obj_{TC\downarrow T\circ(\overline{\lim K})_{\mathcal{B}}} \to Obj_{D\downarrow TK}$
$$(w,B) \to g^*_{ob}(w,B) := \big(T(G_B)wg, B\big)$$
with $w : TC \overset{\mathcal{D}}{\to} T\,\overline{\lim K}$.

$g^*_{mo} : Morf_{TC\downarrow T\circ(\overline{\lim K})_{\mathcal{B}}}\big((w,B),(w',B')\big) \to Morf_{D\downarrow TK}\Big(\big(T(G_B)wg,B\big),\big(T(G_{B'})w'g,B'\big)\Big)$
$$h \to g^*_{mo}(h) := h$$

Then given $(f,B) \in \mathrm{Obj}_{C\downarrow K}$ with $f = G_B\,\eta_f$ we obtain

$(g^*_{ob}\delta_{Tob})(f,B) = (T(f)g, B)$.

Also, given $h \in \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B')\big)$ we obtain

$(g^*_{mo}\delta_{Tmo})(h) = h$.

Therefore defining $K-$continuity in the same way as given in definition 4.8 but with δ_T and g^* replaced by definitions 6.1 and 6.2 we also obtain that $V = g^*\delta_T \Rightarrow V^*(f,B) = (T(f)g,B)$.

7 Conclusion

The $K-$continuity of a functor $T : \mathcal{C} \to \mathcal{D}$ was presented from two perspectives. One conceive $K-$continuity as a particular case of the definition of $\Omega-$continuity when we restrict $\Omega = \{K\}$, and the other as developed in the formulation of the Shape theories of Bacon, Cordier and Porter, which considers the space $Func(C\downarrow K, D\downarrow TK)$.

In the first perspective, arising from Hofmann development, it is assumed there is defined $\lim K$ and $\lim TK$. The $K-$continuity of T corresponds to have $T\lim K \simeq \lim TK$, which is established through a unique morphism $T_K : T\,\overline{\lim K} \overset{\mathcal{D}}{\to} \overline{\lim TK}$ assumed to be invertible (which means $T\,\overline{\lim K} \simeq \overline{\lim TK}$) and to satisfy $T\overline{K} = \overline{TK}T_{K\mathcal{B}}$ (which means $T\overline{K} \simeq \overline{TK}$).

In the second perspective, that is borrowed from Shape theory, there is no requirement on the existence of the limits of K and TK. Here the $K-$continuity of T is expressed in terms of functors

$V \in Func(C \downarrow K, D \downarrow TK)$ and $\delta_T : C \downarrow K \to TC \downarrow TK$ through the relation $V = g^* \delta_T$ with g^* being induced by a unique morphism $g : D \overset{\mathcal{D}}{\to} TC$. In theorem 4.12 we have shown that this $K-$continuity definition from Shape theory includes the $K-$continuity definition arising from the Hofmann development if we take $g : \overline{\lim TK} \overset{\mathcal{D}}{\to} T\, \overline{\lim K}$ to be invertible. Then we have shown that the $K-$continuity concept defined by BCP is more general than the $K-$continuity concept that arises as a particular case of the $\Omega-$continuity for $\Omega = \{K\}$.

In [B75], [CP08] there is no discussion on the conditions garanteeing the existence of the morphism g. Here, a clue is given in (24) if we consider $\chi : D \overset{\mathcal{D}}{\to} TC$ as being the morphism equivalent to g. In this case, if we can prove that the existence of χ garantees the existence of g then the condition $TK^{-1}\overline{v} = T(\overline{u})\chi$ that determines χ becomes by means of the association $g \leftrightarrow \chi$ a sufficient condition for the existence of g. However, we are not sure what restrictions on K and T will ensure the validity of this condition.

Acknowledgements

This work is done in honour of $\overline{\Pi\text{HP}}$, $\overline{\Upsilon\text{C}}$, $\overline{\Pi\text{NA}}$. The author thanks Aurelina Ferreira Lima Carvalho, José Evaristo Carvalho, Ying Chen, Teodora Pereira, and Myriam Callado de Oliveira for the constant support.

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