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RESEARCH PAPER

On Factorization of Multivectors in $Cl(2,1)$, by Exponentials and Idempotents

Eckhard Hitzer

[1]College of Liberal Arts, International Christian University, Tokyo, Japan

Correspondence
Eckhard Hitzer Email: hitzer@icu.ac.jp

Present Address
College of Liberal Arts, International Christian University, Tokyo, Japan

Summary

In this paper we consider general multivector elements of Clifford algebras $Cl(2,1)$, and look for possibilities to factorize multivectors into products of blades, idempotents and exponentials, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar). We will succeed mostly, with a minor open case remaining.

KEYWORDS:
Clifford algebra, factorization, idempotents

1 | INTRODUCTION

The importance of the *polar representation* of complex numbers and quaternions is widely known. Here we endeavor to *extend* this approach to the higher dimensional associative Clifford geometric algebra $Cl(2,1)$, which plays an important role in geometry, physics and computer science[3,4,14,31,20,38]. Namely, it is the physical algrebra of 2+1 *space-time*, and the *conformal geometric algebra* $Cl(1+1,1)$ of one-dimensional Euclidean space $\mathbb{R}^1$. Our results may therefore be of special interest in the special theory of relativity, and for the conformal geometry of a Euclidean line. Moreover, this algebraic study will also help to further elucidate the structure of Clifford algebras.

Exponentials of hyper complex elements and blades also appear as *kernels* in complex, quaternionic and Clifford Fourier and wavelet transforms[21,29]. Important *related questions* are the computation of logarithms of multivectors[8,5], square roots[8,17,19,22,32], inverses[6,1,23,36], transformation rotors[15,30,7,2,37,35], and polar decompositions[8,34], etc. Concrete applications may therefore be to forward and reverse kinematic motions of robot arms, where such factorizations could be useful, or in drone controls.[2]

In earlier work the question of factorization into exponential factors, blades and idempotents for Clifford algebras $Cl(p,q)$, $n = p + q = 1, 2$[26] has been studied, as well as for $Cl(3,0)$, $Cl(1,2)$, and $Cl(0,3)$ in[27]. This motivates us to progress by extending[26] and[27] to the relatively more involved case $Cl(2,1)$.

Because *subalgebras isomorphic* to the algebra of *hyperbolic numbers* appear frequently, we include the description of hyperbolic planes of[26] again, also in order to introduce important notation. Furthermore, the subalgebra structure, in particular that of even subalgebras, is seen to play an essential role, therefore we also study the even subalgebra of $Cl(2,1)$, isomorphic to $Cl(2,0)$, i.e. split-quaternions or coquaternions. As far as possible we aim at explicit, step by step verifiable proofs. An introduction to Clifford geometric algebras is contained in[18], a concise mathematical definition in[10], and a comprehensive study relevant for mathematics and physics in[14].

[1]The use of this paper is subject to the *Creative Peace License*[16]. We dedicate this paper to the *truth* (Jesus: *I am the way and the truth and the life. No one comes to the Father except through me*, see John 14:6, NIV). *Soli Deo Gloria.*

[2]Private communcation with R. Abłamowicz.

Because $Cl(2,1)$ is not a division algebra, we necessarily have non-invertible multivectors and their factorizations are found to include non-invertible idempotents as factors or even their linear combinations. Note that we also include the representation (2.11) for elements of a hyperbolic plane in our wider notion of exponential factors.

The paper is *structured* as follows. Section 2 reviews[26] hyperbolic numbers and their factorization in terms of *exponentials* and *idempotents*, and invertibility. Section 3 studies the important *even subalgebra* of $Cl(2,1)$, providing essential results for the full blown study of $Cl(2,1)$ following later. The elaborate *direct* factorization in $Cl(2,1)$ of Section 4 has results summarized in Section 5, which in some sense also shows the limitations of our approach, and the emerging complexity, mainly due to the intricate idempotent structure. The paper concludes with Section 6, followed by acknowledgments and references.

2 | HYPERBOLIC PLANES

Since subalgebras isomorphic to the algebra of a hyperbolic plane[3] will occur repeatedly in our analysis, and to establish notation for later use in this paper, we reproduce this short study of hyperbolic planes from[26]. An element $u \neq 1$ that squares to $u^2 = +1$ generates a *hyperbolic plane* $\{b + au\}$, $a, b \in \mathbb{R}$ with basis $\{1, u\}$. A relevant alternative basis $\{id_-, id_+\}$ is given by two not invertible idempotents

$$\begin{aligned} &id_+ = \frac{1+u}{2}, \quad id_- = \frac{1-u}{2}, \quad id_+ + id_- = 1, \quad id_+ - id_- = u, \\ &id_+^2 = id_+, \quad id_-^2 = id_-, \quad id_+ id_- = id_- id_+ = 0. \end{aligned} \tag{2.1}$$

Adopting the definitions

$$x^0 = 1, \qquad 0! = 1, \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{2.2}$$

for powers of a general element x and its exponential[4], we obtain for $a \in \mathbb{R}$

$$e^{a\, id_\pm} = 1 + (e^a - 1) id_\pm, \qquad e^{au} = \cosh a + u \sinh a. \tag{2.3}$$

General *nonzero* elements $m = b + au$ of the hyperbolic plane can be classified by whether $|a| = |b|$ (m is not invertible), or $|a| \neq |b|$ (m is invertible). For $|a| = |b|$ we have the four subcases

$$\begin{aligned} &b = a > 0, \quad m = 2b\, id_+, \\ &b = a < 0, \quad m = 2b\, id_+ = -2|b|\, id_+, \\ &b = -a > 0, \; m = 2b\, id_-, \\ &b = -a < 0, \; m = 2b\, id_- = -2|b|\, id_-. \end{aligned} \tag{2.4}$$

Examples are for each line of (2.4): $1 + u = 2(1+u)/2 = 2id_+, -2 - 2u = -4(1+u)/2 = 4(-id_+), 3 - 3u = 6(1-u)/2 = 6id_-, -4 + 4u = -8(1-u)/2 = 8(-id_-)$. Thus according to (2.4) for $|a| = |b| \neq 0$ we can always represent m as[5]

$$m = 2|b| h^{id}(u), \quad \text{with} \quad h^{id}(u) \in \{\pm id_+, \pm id_-\}, \tag{2.5}$$

and therefore as

$$m = e^{\alpha_0} h^{id}(u), \qquad \alpha_0 = \ln(2|b|). \tag{2.6}$$

Note that $h^{id}(u)^2 = id_\pm$. Geometrically, the four values of $h^{id}(u)$ specify *four bisector* directions, one in each quadrant of the hyperbolic plane. Because idempotents $id_\pm$ are not invertible, all hyperbolic numbers with $|a| = |b|$ cannot be inverted.

For general (evidently *nonzero*) elements $m = b + au$ with $|a| \neq |b|$ we can distinguish four subcases

$$\begin{aligned} &b > |a| \geq 0, \quad m = b + au, \\ &a > |b| \geq 0, \quad m = (a + bu)u, \\ &b < -|a| \leq 0, \; m = -(-b - au), \\ &a < -|b| \leq 0, \; m = -(-a - bu)u. \end{aligned} \tag{2.7}$$

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[3]The Clifford algebra $Cl(1,0)$ and the even subalgebra $Cl_2(1,1)$ (itself a subalgebra of $Cl(2,1)$) of the two-dimensional space-time algebra are both isomorphic to the hyperbolic plane. Invertible elements of $Cl_2(1,1)$ represent boosts (changes of velocity), of elementary importance in special relativity.

[4]In this paper we do not make further use of $e^{a\, id_\pm}$. But we note that even though $id_\pm$ is not invertible, $e^{a\, id_\pm}$ has inverse $e^{-a\, id_\pm}$, similar to null-vectors not being invertible, but their exponential functions have a multiplicative inverse.

[5]Note that (2.5) together with (2.4) provides a unique specification for the assignment of $h^{id}(u)$ from the set $\{\pm id_+, \pm id_-\}$, thus effectively *defining* the four-valued function $h^{id}(u)$. Similarly (2.8) together with (2.7) effectively *defines* $h(u)$ uniquely.

TABLE 1 Multiplication table of $Cl_2(2,1)$.

	1	e_{12}	e_{23}	e_{31}
1	1	e_{12}	e_{23}	e_{31}
e_{12}	e_{12}	-1	$-e_{31}$	e_{23}
e_{23}	e_{23}	e_{31}	$+1$	e_{12}
e_{31}	e_{31}	$-e_{23}$	$-e_{12}$	$+1$

Examples for (2.7) are line by line: $4 \pm u, \pm 1 + 4u = (4 \pm u)u, -4 \mp u = -(4 \pm u), \mp 1 - 4u = -(4 \pm u)u$. Thus according to (2.7) for $|a| \neq |b|$ we can always represent any m as

$$m = (\beta + \alpha u)h(u), \quad \text{with} \quad h(u) \in \{\pm 1, \pm u\}, \tag{2.8}$$

such that $\beta > |\alpha| \geq 0$, and therefore m can be factored as

$$m = e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_u u} h(u), \quad \alpha_0 = \tfrac{1}{2}\ln(\beta^2 - \alpha^2), \quad \alpha_u = \operatorname{atanh}(\alpha/\beta). \tag{2.9}$$

In the examples for (2.7) we have $\alpha = \pm 1, \beta = 4, \alpha_0 \approx 1.35, \alpha_u \approx \pm 0.255$. Note that $h(u)^2 = 1$ and therefore $h(u)^{-1} = h(u)$. Geometrically, the four possible values of $h(u)$ uniquely specify the four quadrants in the hyperbolic plane, delimited by two straight lines (bisectors) with directions $id_{\pm}$. The inverse of hyperbolic numbers with $|a| \neq |b|$ can *always* be easily computed as

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_u u} h(u). \tag{2.10}$$

In *summary*, any $m = b + au \neq 0$ in the hyperbolic plane can be factorized as

$$m = E(m) = E(a,b,u) = e^{\alpha_0} \begin{cases} h^{id}(u) & \text{for} \quad |a| = |b|, \\ e^{\alpha_u u} h(u) & \text{for} \quad |a| \neq |b|. \end{cases} \tag{2.11}$$

Equation (2.11) provides a first example of what we mean by *exponential factorization*. Note that we introduce the *new notation* $E(m) = E(a,b,u)$ to indicate the factorization (2.11) in terms of one or two exponential functions and eight possible values. The computation of the factorization (2.11) is based on (2.4) to (2.6) for the first four cases involving idempotents, i.e. $h^{id}(u) \in \{+id_+, -id_+, +id_-, -id_i\}$, and on (2.7) to (2.9) for the remaining four cases involving the hyperbolic exponential factor and $h(u) \in \{+1, -1, +u, -u\}$. The hyperbolic number m is invertible if and only if $|a| \neq |b|$.

3 | THE EVEN SUBALGEBRA OF $CL(2,1)$

3.1 | Isomorphism of even subalgebra of $Cl(2,1)$

We can expect that the even subalgebra $Cl_2(2,1)$ with basis $\{1, e_{12}, e_{23}, e_{31}\}$ of $Cl(2,1)$ might be of high relevance for factorization. It has the following multiplication table: Table 1 .

Furthermore, the table is isomorphic to $Cl(2,0)$ by identifying $e_1 = e'_{23}$, $e_2 = e'_{31}$, $e_{12} = e'_{12}$, where $\{e_1, e_2, e_{12}\} \subset Cl(2,0)$ and $\{e'_{12}, e'_{23}, e'_{31}\} \subset Cl_2(2,1)$.

The isomorphism with $Cl(2,0)$ does allow to utilize the factorization of $Cl(2,0)$ derived in Section 5 of[26]. We recapitulate the result here[6]

$$\begin{aligned} m &= m_1 e_1 + m_2 e_2 + m_0 + m_{12} e_{12} \\ &= \begin{cases} e^{\alpha_0} e^{\alpha_2 e_{12}}, \ \ \alpha_0 = \ln(b), \ \ \alpha_2 = \operatorname{atan2}(m_{12}, m_0) & \text{for } m_1 = m_2 = 0, \\ e^{\alpha'_0} u', \quad \alpha'_0 = \ln(a), & \text{for } m_0 = m_{12} = 0, \\ (b + au)\, e^{\alpha_2 e_{12}} = E(a,b,u)\, e^{\alpha_2 e_{12}}, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a &= \sqrt{(m_1 e_1 + m_2 e_2)^2} = \sqrt{m_1^2 + m_2^2}, \qquad b = \sqrt{m_0^2 + m_{12}^2}, \\ u' &= (m_1 e_1 + m_2 e_2)/a, \qquad u = e^{\alpha_2 e_{12}}\, u', \end{aligned} \tag{3.2}$$

[6] Note that the meaning of $\operatorname{atan2}(y,x)$ is the mathematically positive angle of the vector $xe_1 + ye_2$ with the x-axis in the Euclidean plane, if the vector is attached to the origin.

and $E(a, b, u)$ has been defined in (2.11). Because a and b are positive, the eight possible values of $E(a, b, u)$ reduce to only three, i.e. only the first line of (2.4) and the first two lines of (2.7) are relevant. We further observe about (3.1) that the third line subsumes the first for $a = 0$, and the third line subsumes the second for $b = \alpha_2 = 0$. This means that $m \in Cl(2, 0)$, can *always* be factored in the form

$$m = (b + au)e^{\alpha_2 e_{12}}, \tag{3.3}$$

with $a \geq 0$ and $b \geq 0$. And m is always invertible, except when $a = b$. In (3.3) u is a vector with positive unit square and e_{12} is a bivector with negative unit square. In the next Section 3.2, we discuss an interesting alternative factorization which aims at a single exponential factor with bivector exponent, and explain why we still prefer (3.3) in the rest of this paper.

3.2 | Alternative factorization of $Cl_2(2, 1)$

An alternative factorization of $Cl_2(2, 1)$ can be obtained in the following way.

$$m = m_0 + m_{23}e_{23} + m_{31}e_{31} + m_{12}e_{12}. \tag{3.4}$$

We distinguish five cases. First $m_0 \neq 0$, $\langle m \rangle_2 = m_{23}e_{23} + m_{31}e_{31} + m_{12}e_{12} = 0$:

$$m = m_0 = \frac{m_0}{|m_0|}e^{\alpha_0} = \pm\, e^{\alpha_0}, \qquad \alpha_0 = \ln(|m_0|). \tag{3.5}$$

Second, $\langle m \rangle_2^2 < 0$:

$$\begin{aligned} m &= m_0 + |\langle m \rangle_2| \frac{\langle m \rangle_2}{|\langle m \rangle_2|} = a_m e^{\alpha_2 i_2} = e^{\alpha_0} e^{\alpha_2 i_2}, \qquad |\langle m \rangle_2| = \sqrt{-\langle m \rangle_2^2}, \\ i_2 &= \frac{\langle m \rangle_2}{|\langle m \rangle_2|}, \qquad i_2^2 = -1, \qquad \alpha_2 = \mathrm{atan2}(|\langle m \rangle_2|, m_0), \\ a_m &= \sqrt{m_0^2 + |\langle m \rangle_2|^2} = \sqrt{m_0^2 - \langle m \rangle_2^2}, \qquad \alpha_0 = \ln(a_m). \end{aligned} \tag{3.6}$$

We observe that the second case subsumes the first case for $\alpha_2 \in \{0, \pi\}$. Third, $m_0 = 0$, $m = \langle m \rangle_2 \neq 0$, $m^2 = \langle m \rangle_2^2 = 0$:

$$m = \langle m \rangle_2 = e^{\alpha_0} i_2, \quad \alpha_0 = \ln(\sqrt{2}|m_{12}|), \quad i_2 = \frac{\langle m \rangle_2}{\sqrt{2}|m_{12}|}, \quad m^2 = i_2^2 = 0. \tag{3.7}$$

We observe that in the third case m is a not invertible null-bivector. As an *example*[7] for the third case we consider the following example.

Example.

$$\begin{aligned} m &= \langle m \rangle_2 = 3e_{12} + 3e_{23} \approx e^{1.45}\, \frac{e_{12} + e_{23}}{\sqrt{2}}, \quad m_{12} = |m_{12}| = m_{23} = 3, \\ \alpha_0 &= \ln(\sqrt{2}\, 3) \approx 1.45, \quad i_2 = \frac{e_{12} + e_{23}}{\sqrt{2}}. \end{aligned} \tag{3.8}$$

Fourth, $m_0 \neq 0$, $\langle m \rangle_2 \neq 0$, $\langle m \rangle_2^2 = 0$:

$$\begin{aligned} m &= m_0 + \langle m \rangle_2 = m_0(1 + \frac{1}{m_0}\langle m \rangle_2) = \frac{m_0}{|m_0|}e^{\alpha_0}(1 + \alpha_2 i_2) = \pm e^{\alpha_0} e^{\alpha_2 i_2}, \\ \alpha_0 &= \ln(|m_0|), \quad i_2 = \frac{\langle m \rangle_2}{\sqrt{2}|m_{12}|}, \quad \alpha_2 = \frac{\sqrt{2}|m_{12}|}{m_0}, \end{aligned} \tag{3.9}$$

where the sign factor is determined by $\frac{m_0}{|m_0|} = \pm 1$. Fifth, $\langle m \rangle_2^2 > 0$:

$$\begin{aligned} m &= m_0 + \langle m \rangle_2 = m_0 + |\langle m \rangle_2| i_2 = E(|\langle m \rangle_2|, m_0, i_2), \\ |\langle m \rangle_2| &= \sqrt{\langle m \rangle_2^2}, \qquad i_2 = \frac{\langle m \rangle_2}{|\langle m \rangle_2|}, \qquad i_2^2 = +1, \end{aligned} \tag{3.10}$$

[7] In conformal geometric algebra $Cl(4, 1)$ two null-vectors are defined for the origin and for infinity. Conventionally they are $e_0 = (e_5 - e_4)/2$, $e_\infty = e_5 + e_4$, such that $e_0 \cdot e_\infty = -1$. In certain contexts it has proven to be of advantage to instead choose a symmetric definition $e_0 = (e_5 - e_4)/\sqrt{2}$, $e_\infty = (e_5 + e_4)/\sqrt{2}$, see e.g.[25]. By analogy, this motivates our introduction of $\sqrt{2}$ in the denominator of the null bivector i_2 above.

where $E(|\langle m\rangle_2|, m_0, i_2)$ is determined by (2.11), with $a = |\langle m\rangle_2|$, $b = m_0$, $u = i_2$. In the fifth case m is not invertible for $|m_0| = |\langle m\rangle_2|$. We finally summarize all five cases[8]

$$m = \begin{cases} \pm e^{\alpha_0} & \text{for} \quad \langle m\rangle_2 = 0, \\ e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad \langle m\rangle_2^2 < 0, \\ e^{\alpha_0} i_2 & \text{for} \quad m_0 = 0, \quad \langle m\rangle_2 \neq 0, \quad \langle m\rangle_2^2 = 0, \\ \pm e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad m_0 \neq 0, \quad \langle m\rangle_2 \neq 0, \quad \langle m\rangle_2^2 = 0, \\ E(|\langle m\rangle_2|, m_0, i_2) & \text{for} \quad \langle m\rangle_2^2 > 0. \end{cases} \tag{3.11}$$

Let us compare the factorizations (3.3) and (3.11): (3.11) always has only one bivector exponential (except for the third line $e^{\alpha_0} i_2$), but it is *more complicated* (more case distinctions) than (3.3). Because following (3.3) all cases can be accommodated in the *single* expression $m = (b + au)e^{\alpha_2 e_{12}}$, with $a \geq 0$ and $b \geq 0$, which is always invertible except when $a = b$ (presence of an idempotent factor for $a = b \neq 0$). The inverse is given by

$$m^{-1} = e^{-\alpha_2 e_{12}}(b + au)^{-1} = e^{-\alpha_2 e_{12}} \frac{b - au}{b^2 - a^2}, \tag{3.12}$$

whenever $a \neq b$, compare (3.2). By these reasons, we prefer to use (3.3) in the rest of the paper.

4 | DIRECT FACTORIZATION OF $CL(2, 1)$

Because the unit pseudoscalar i in $Cl(2, 1)$ squares to $i^2 = +1$ the idempotent structure becomes even more complex than e.g. in $Cl(1, 2)$[27].

4.1 | The product $m\overline{m}$

In $Cl(2, 1)$ the central pseudoscalar squares to $i^2 = +1$ and

$$\begin{aligned} &e_1^2 = e_2^2 = -e_3^2 = -e_{12}^2 = e_{31}^2 = e_{23}^2 = 1, \\ &e_1 = ie_{23}, \quad e_2 = ie_{31}, \quad e_3 = -ie_{12}. \end{aligned} \tag{4.1}$$

This allows us to rewrite a general multivector as

$$\begin{aligned} m &= m_0 + m_1 e_1 + m_2 e_2 + m_3 e_3 + m_{12} e_{12} + m_{31} e_{31} + m_{23} e_{23} + m_{123} i \\ &= m_0 + m_{23} e_{23} + m_{12} e_{12} + m_{31} e_{31} + i(m_{123} + m_1 e_{23} + m_2 e_{31} - m_3 e_{12}) \\ &= p_0 + p_{12} e_{12} + p_{23} e_{23} + p_{31} e_{31} + i(q_0 + q_{12} e_{12} + q_{23} e_{23} + q_{31} e_{31}) \\ &= p + iq \end{aligned} \tag{4.2}$$

with suitable identifications of the eight coefficients of m with four coefficients of p and four coefficients of q, where both $p, q \in Cl_2(2, 1) \cong Cl(2, 0)$. We can therefore represent both p and q as

$$\begin{aligned} p &= (b_p + a_p u_p)e^{\alpha_{2p} e_{12}}, \quad b_p = \sqrt{p_0^2 + p_{12}^2}, \quad a_p = \sqrt{p_{23}^2 + p_{31}^2}, \quad u_p^2 = 1, \\ q &= (b_q + a_q u_q)e^{\alpha_{2q} e_{12}}, \quad b_q = \sqrt{q_0^2 + q_{12}^2}, \quad a_q = \sqrt{q_{23}^2 + q_{31}^2}, \quad u_q^2 = 1, \end{aligned} \tag{4.3}$$

following (3.1). The unit bivectors u_p, u_q, with positive square, are linear combinations of e_{23} and e_{31}. We now give an example.

Example. Note that in this example we always round after the fourth nonzero digit. Assume a multivector $m \in Cl(2, 1)$, $i = e_{123}$, with value

$$\begin{aligned} m &= 6\, e_1 + 38\, e_2 + 28\, e_3 + 24\, e_{123} = i\,(24 - 28\, e_{12} + 6\, e_{23} + 38\, e_{31}) \\ &= i\,(36.88 + 38.47\, u_q)\, e^{-0.8622\, e_{12}}, \end{aligned} \tag{4.4}$$

[8]Note that in lines two to five of (3.11) the bivectors i_2 are specific to each line, as defined in (3.6), (3.7), (3.9), and (3.10), respectively.

with $a_q = 38.47$, $b_q = 36.88$, $\alpha_{2q} = -0.8622$, and

$$\begin{aligned}
u'_q &= \frac{6\,e_{23} + 38\,e_{31}}{38.47} = 0.1560\,e_{23} + 0.9878\,e_{31},\\
u_q &= e^{-0.8622 e_{12}} u'_q = (\cos 0.8622 - e_{12}\sin 0.8622)(0.1560\,e_{23} + 0.9878\,e_{31})\\
&= (0.6508 - e_{12}\,0.7593)(0.1560\,e_{23} + 0.9878\,e_{31})\\
&= 0.1015\,e_{23} + 0.6429\,e_{31} - 0.1185\,e_{12}e_{23} - 0.7500\,e_{12}e_{31}\\
&= (0.1015 - 0.7500)\,e_{23} + (0.6429 + 0.1185)\,e_{31} = -0.6485\,e_{23} + 0.7614\,e_{31},\\
u_q^2 &= 0.6485^2 + 0.7614^2 = 1.000.
\end{aligned} \tag{4.5}$$

The factorization of $b_q + a_q u_q = 36.88 + 38.47(-0.6485\,e_{23} + 0.7614\,e_{31}) = E(a_q, b_q, u_q)$, which has $a_q > b_q$, hence $h(u_q) = u_q$, gives by (2.9)

$$E(a_q, b_q, u_q) = b_q + a_q u_q = (a_q + b_q u_q)\,u_q, = (38.47 + 36.88 u_q)\,u_q = e^{2.393} e^{1.930\,u_q}\,u_q. \tag{4.6}$$

So the full factorization of $m = 6\,e_1 + 38\,e_2 + 28\,e_3 + 24\,e_{123}$ becomes

$$m = e^{2.393}\; e^{1.930\,u_q}\, u_q\, e^{-0.8622\,e_{12}}\, i, \tag{4.7}$$

where u_q is defined in (4.5). This ends the example.

If $a_p = b_p = 0$ (compare e.g. the above example) or $a_q = b_q = 0$, then the final factorization is given by

$$m = iq = i(b_q + a_q u_q)e^{\alpha_{2q} e_{12}} \tag{4.8}$$

or by

$$m = p = (b_p + a_p u_p)e^{\alpha_{2p} e_{12}}, \tag{4.9}$$

respectively. In the rest of this section we can therefore assume that both $p \neq 0$ and $q \neq 0$.

p is proportional to an idempotent $(1 + u_p)/2$ and not invertible for $a_p = b_p$, and likewise q is is proportional to an idempotent $(1 + u_q)/2$ and not invertible for $a_q = b_q$. For later use we compute

$$\begin{aligned}
p\overline{p} &= b_p^2 - a_p^2, \quad q\overline{q} = b_q^2 - a_q^2,\\
\frac{1}{2}(q\overline{p} + p\overline{q}) &= p_0 q_0 + p_{12} q_{12} - (p_{23} q_{23} + p_{31} q_{31}).
\end{aligned} \tag{4.10}$$

Let us also compute

$$m\overline{m} = (p + iq)(\overline{p} + i\overline{q}) = p\overline{p} + i^2 q\overline{q} + i(q\overline{p} + p\overline{q}) = p\overline{p} + q\overline{q} + i(q\overline{p} + p\overline{q}). \tag{4.11}$$

4.2 | Discussion for non-invertible $m\overline{m}$

$m\overline{m}$ is zero if (1) the following sum is zero

$$p\overline{p} + q\overline{q} = b_p^2 - a_p^2 + (b_q^2 - a_q^2) = b_p^2 + b_q^2 - (a_p^2 + a_q^2) = 0, \tag{4.12}$$

and (2) a sort of four-dimensional hyperbolic orthogonality condition is met

$$\frac{1}{2}(q\overline{p} + p\overline{q}) = p_0 q_0 + p_{12} q_{12} - (p_{23} q_{23} + p_{31} q_{31}) = 0. \tag{4.13}$$

In our context ($i^2 = +1$), a non-zero $m\overline{m}$ is also not invertible when the scalar part equals the trivector part in magnitude, i.e. if

$$|p\overline{p} + q\overline{q}| = |q\overline{p} + p\overline{q}| \;\Leftrightarrow\; p\overline{p} + q\overline{q} = (q\overline{p} + p\overline{q}) \;\text{ or }\; p\overline{p} + q\overline{q} = -(q\overline{p} + p\overline{q}), \tag{4.14}$$

because according to Section 6 of[23], $(m\overline{m})(m\overline{m})^{\sim} = (p\overline{p}+q\overline{q})^2 - (q\overline{p}+p\overline{q})^2 = 0$, iff m is not invertible. This leads to the following proposition.

Proposition 4.1. A non-zero multivector $m = p + iq \in Cl(2,1)$, $p, q \in Cl_2(2,1)$, $i = e_1e_2e_3$, is not invertible, iff its two even subalgebra components p, q fulfill

$$(p - q)\overline{(p - q)} = 0 \quad \text{or} \quad (p + q)\overline{(p + q)} = 0. \tag{4.15}$$

Proof. We assume $m = p + iq \in Cl(2,1)$, $p, q \in Cl_2(2,1)$, $i = e_1e_2e_3$, and compute

$$\begin{aligned}(p \pm q)\overline{(p \pm q)} &= (p_0 \pm q_0)^2 + (p_{23} \pm q_{23})^2 - (p_{12} \pm q_{12})^2 - (p_{31} \pm q_{31})^2 \\ &= p_0^2 + p_{23}^2 + q_0^2 + q_{23}^2 - p_{12}^2 - q_{12}^2 - p_{31}^2 - q_{31}^2 \\ &\quad \pm 2(p_0q_0 + p_{23}q_{23} - p_{12}q_{12} - p_{31}q_{31}) \\ &= p\overline{p} + q\overline{q} \pm (p\overline{q} + q\overline{p}). \end{aligned} \tag{4.16}$$

This means $(p \pm q)\overline{(p \pm q)} = 0$, iff

$$\begin{aligned} & p\overline{p} + q\overline{q} = \mp(p\overline{q} + q\overline{p}) \\ \Leftrightarrow \quad & |p\overline{p} + q\overline{q}| = |p\overline{q} + q\overline{p}| \\ \Leftrightarrow \quad & (p\overline{p} + q\overline{q})^2 = (p\overline{q} + q\overline{p})^2 \\ \Leftrightarrow \quad & (p\overline{p} + q\overline{q})^2 - (p\overline{q} + q\overline{p})^2 = 0 \\ \Leftrightarrow \quad & (m\overline{m})(m\overline{m})^\sim = 0. \end{aligned} \tag{4.17}$$

If $m = 0$, and therefore $p = q = 0$, the argument is trivial. If $m \neq 0$ then we have shown

$$(p \pm q)\overline{(p \pm q)} = 0 \quad \Leftrightarrow \quad (m\overline{m})(m\overline{m})^\sim = 0. \tag{4.18}$$

□

Every element of the even subalgebra $x \in Cl_2(2,1) \cong Cl(2,0)$ can be represented as ($a_x, b_x \in \mathbb{R}$, unit bivector u_x: $u_x^2 = +1$, $0 \leq \alpha_{2x} < 2\pi$)

$$x = (b_x + a_x u_x)e^{\alpha_{2x}e_{12}}, \tag{4.19}$$

and iff x is not invertible, then $x\overline{x} = 0$ (see Section 5 of[23]), which means that $b_x = a_x$. If m is not invertible, we can therefore represent $p + q$ or $p - q$ as

$$p + q = 2a_x\frac{1 + u_x}{2}e^{\alpha_{2x}e_{12}} \quad \text{or} \quad p - q = 2a_y\frac{1 + u_y}{2}e^{\alpha_{2y}e_{12}}. \tag{4.20}$$

This means a non-invertible m can be written as

$$m = p + iq = p + i(p + q) - ip = 2p\tfrac{1-i}{2} + i2a_x\tfrac{1+u_x}{2}e^{\alpha_{2x}e_{12}}, \tag{4.21}$$

or as

$$m = p + iq = p - q + q + iq = 2q\tfrac{1+i}{2} + 2a_y\tfrac{1+u_y}{2}e^{\alpha_{2y}e_{12}}, \tag{4.22}$$

with central idempotent $\frac{1\pm i}{2}$, and idempotents $\frac{1+u_x}{2}$ or $\frac{1+u_y}{2}$. Easy special cases are, e.g., $q = \pm p$, then $m = 2p\frac{1\pm i}{2}$ is not invertible because of the central idempotent factor $\frac{1\pm i}{2}$.

4.2.1 | Case of non-invertible components p, q of $m = p + iq$

If p is not invertible it can be written as

$$p = 2a_p\frac{1 + u_p}{2}e^{\alpha_{2p}e_{12}}, \tag{4.23}$$

where $\frac{1+u_p}{2}$ is an idempotent. Similarly, if q is not invertible it can be written as

$$q = 2a_q\frac{1 + u_q}{2}e^{\alpha_{2q}e_{12}}, \tag{4.24}$$

where $\frac{1+u_q}{2}$ is an idempotent.

Therefore if both p and q are not invertible, then m takes the form

$$m = 2a_p\frac{1 + u_p}{2}e^{\alpha_{2p}e_{12}} + i2a_q\frac{1 + u_q}{2}e^{\alpha_{2q}e_{12}}. \tag{4.25}$$

In this case we can compute

$$\begin{aligned} m\overline{m} &= 0 + i^2 0 + i(q\overline{p} + p\overline{q}) \\ &= ia_pa_q[(1 + u_p)e^{\alpha_{2p}e_{12}}e^{-\alpha_{2q}e_{12}}(1 - u_q) \\ &\qquad + (1 + u_q)e^{\alpha_{2q}e_{12}}e^{-\alpha_{2p}e_{12}}(1 - u_p)], \end{aligned} \tag{4.26}$$

with $\Delta = \alpha_{2p} - \alpha_{2q}$ and $e^{\pm\Delta e_{12}} = \cos\Delta \pm e_{12}\sin\Delta$, this becomes

$$\begin{aligned} m\overline{m} &= ia_p a_q\{(1+u_p)\cos\Delta(1-u_q) + (1+u_q)\cos\Delta(1-u_p) \\ &\quad + \sin\Delta[(1+u_p)e_{12}(1-u_q) - (1+u_q)e_{12}(1-u_p)]\} \\ &= ia_p a_q\{\cos\Delta[(1+u_p)(1-u_q) + (1+u_q)(1-u_p)] \\ &\quad + \sin\Delta e_{12}[(1-u_p)(1-u_q) - (1-u_q)(1-u_p)]\} \\ &= ia_p a_q\{\cos\Delta[2 - u_p u_q - u_q u_p] + \sin\Delta e_{12}[u_p u_q - u_q u_p]\}. \end{aligned} \tag{4.27}$$

Now the product of the unit bivectors equals the product of the two positive definite vectors $\vec{u}_p, \vec{u}_q$ in the e_{12}-plane with mutual angle ϑ

$$u_p u_q = e_3\vec{u}_p e_3\vec{u}_q = \vec{u}_p\vec{u}_q. \tag{4.28}$$

Therefore

$$\begin{aligned} u_p u_q + u_q u_p &= 2\vec{u}_p \cdot \vec{u}_q = 2\cos\vartheta, \\ u_p u_q - u_q u_p &= 2\vec{u}_p \wedge \vec{u}_q = 2e_{12}\sin\vartheta. \end{aligned} \tag{4.29}$$

Hence

$$\begin{aligned} m\overline{m} &= ia_p a_q\{\cos\Delta[2 - 2\cos\vartheta] + 2\sin\Delta\sin\vartheta(e_{12}^2)\} \\ &= 2ia_p a_q\{\cos\Delta - \cos\Delta\cos\vartheta - \sin\Delta\sin\vartheta\} \\ &= 2ia_p a_q\{\cos\Delta - \cos(\Delta - \vartheta)\} \end{aligned} \tag{4.30}$$

So the product $m\overline{m} = 0$ for the following combinations of Δ and ϑ

$$\begin{array}{ll} \vartheta = 0, & \text{any} \quad 0 \le \Delta < 2\pi, \\ \vartheta = \pi, & \Delta = \frac{\pi}{2}, \frac{3\pi}{2}, \\ 0 \le \vartheta < 2\pi, & \Delta = \pi + \frac{\vartheta}{2}. \end{array} \tag{4.31}$$

Note that the second line is a special case of the third line for $\vartheta = \pm\pi$. In all other cases $m\overline{m} \neq 0$ and m will be invertible, even under the assumption that p and q are not invertible. This means that for $m\overline{m} = 0$ the non-invertible multivector m will take one of these three forms

$$m = \left\{ \begin{array}{l} (1+u_p)[a_p e^{\Delta e_{23}} + ia_q] \\ \left[a_p(1+u_p)(\pm e_{23}) + ia_q(1-u_p)\right] \\ \left[a_p(1+u_p)e^{(\pi+\vartheta/2)e_{23}} + ia_q(1+u_q)\right] \end{array} \right\} e^{\alpha_{2q}e_{23}}. \tag{4.32}$$

Note that in the third line the angle ϑ is the angle between u_p and u_q, as determined above by (4.29).

A potentially useful factorization of the non-zero $m\overline{m}$ of (4.30) when both p and q are not invertible, can be given by

$$m\overline{m} = \pm ie^{2\alpha_0}, \qquad \alpha_0 = \frac{1}{2}\ln(|2a_p a_q\{\cos\Delta - \cos(\Delta - \vartheta)\}|), \tag{4.33}$$

and the leading sign would be identical to that of $2a_p a_q\{\cos\Delta - \cos(\Delta - \vartheta)\}$.

4.2.2 | Case of invertible component p of $m = p + iq$

We now first assume that p is invertible, i.e. $a_p \neq b_p$, without assuming q to be invertible. After discussing this case, we will discuss the analogous case for which q is assumed to be invertible, but not p. We can therefore compute the product

$$s = p^{-1}q = (b_s + a_s u_s)e^{\alpha_{2s}e_{12}}, \tag{4.34}$$

which must also be an element of the subalgebra $Cl_2(2,1)$ and can therefore be represented in this form, where a_s, b_s are real non-negative numbers, bivector u_s has square $u_s^2 = +1$, and $0 \le \alpha_{2s} < 2\pi$. This allows us to rewrite m as

$$m = p(1 + ip^{-1}q) = p(1 + is). \tag{4.35}$$

For this form of m we compute

$$\begin{aligned} m\overline{m} &= p(1+is)(1-i\overline{s})\overline{p} = p[1+i^2 s\overline{s} + i(s+\overline{s})]\overline{p} \\ &= p[1+i^2(b_s^2-a_s^2) + i(2b_s\cos\alpha_{2s} + a_s\sin\alpha_{2s}u_s e_{12} + a_s\sin\alpha_{2s}e_{12}u_s)]\overline{p} \\ &= p\overline{p}[1+i^2(b_s^2-a_s^2)+i2b_s\cos\alpha_{2s}] \\ &= (b_p^2-a_p^2)[1+(b_s^2-a_s^2)+i2b_s\cos\alpha_{2s}], \end{aligned} \tag{4.36}$$

where we have used for the fourth equality that $u_s e_{12} = -e_{12}u_s$. By assumption the factor $(b_p^2 - a_p^2) \neq 0$, so for $m\overline{m}$ to be zero we must have

$$b_s^2 - a_s^2 = -1 \Leftrightarrow a_s^2 - b_s^2 = 1, \tag{4.37}$$

and we must have

$$b_s \cos\alpha_{2s} = 0, \tag{4.38}$$

i.e. $b_s = 0$ or $\alpha_{2s} = \frac{\pi}{2}, \frac{3\pi}{2}$. If $b_s = 0$, then $a_s^2 = 1$, i.e. $a_s = \pm 1$ but without restriction on α_{2s}. If $b_s \neq 0$, then $\alpha_{2s} = \frac{\pi}{2}, \frac{3\pi}{2}$, i.e. $e^{\alpha_{2s}e_{12}} = \pm e_{12}$ and $a_s^2 - b_s^2 = 1$. The relationship $a_s^2 - b_s^2 = 1$ is that of hyperbolic cosine and sine for some angle φ_s. Hence $m\overline{m}$ will be zero for either this form of quotient s

$$s = (b_s + a_s u_s)e^{\alpha_{2s}e_{12}} = (\sinh\varphi_s + \cosh\varphi_s u_s)(\pm e_{12}) = \pm e^{\varphi_s u_s}u_s e_{12}, \tag{4.39}$$

or for

$$s = \pm u_s e^{\alpha_{2s}e_{12}}. \tag{4.40}$$

Therefore

$$q = ps = \pm p\, e^{\varphi_s u_s}u_s e_{12} \quad \text{or} \quad q = \pm p\, u_s e^{\alpha_{2s}e_{12}}. \tag{4.41}$$

and

$$m = p + iq = 2p\frac{1+is}{2}. \tag{4.42}$$

We compute the square of s as either

$$\begin{aligned} s^2 &= (\pm e^{\varphi_s u_s}u_s e_{12})^2 = e^{\varphi_s u_s}u_s e_{12} e^{\varphi_s u_s}u_s e_{12} = e^{\varphi_s u_s}u_s e^{-\varphi_s u_s}e_{12}u_s e_{12} \\ &= u_s e^{\varphi_s u_s}e^{-\varphi_s u_s}(-u_s)e_{12}e_{12} = -u_s^2 e_{12}^2 = (-1)^2 = 1, \end{aligned} \tag{4.43}$$

or as

$$s^2 = (\pm u_s e^{\alpha_{2s}e_{12}})^2 = u_s e^{\alpha_{2s}e_{12}}u_s e^{\alpha_{2s}e_{12}} = u_s^2 e^{-\alpha_{2s}e_{12}}e^{\alpha_{2s}e_{12}} = 1, \tag{4.44}$$

where we used repeatedly that $e_{12}u_s = -e_{12}u_s$. This means that for both forms of s

$$(is)^2 = i^2 s^2 = +1, \tag{4.45}$$

and the factor $\frac{1+is}{2}$ is therefore an idempotent. So assuming that p is invertible and m is not invertible we obtain the factorization of m as

$$m = 2p\frac{1+is}{2} = 2(b_p + a_p u_p)e^{\alpha_{2p}e_{12}}\frac{1+is}{2}, \tag{4.46}$$

where the factor $2(b_p + a_p u_p)$ can further be put into exponential form using $E(2a_p, 2b_p, u_p)$ in (2.11). The idempotent factor $\frac{1+is}{2}$ means that m is manifestly (obviously) not invertible.

Furthermore, $m\overline{m}$ will also not be invertible for

$$|1 + (b_s^2 - a_s^2)| = |2b_s\cos\alpha_{2s}|, \tag{4.47}$$

which is a special case of the above analysis that followed immediately after Proposition 4.1.

4.2.3 | Case of invertible component q of $m = p + iq$

Now let us instead assume, that q is invertible. We can multiply m with i

$$\begin{aligned} m' &= im = ip + iiq = q + ip = p' + iq', \\ p' &= q, \qquad q' = p. \end{aligned} \tag{4.48}$$

We can now apply the above analysis of m with p invertible to m' with p' invertible, and in the end multiply the result again with i to get the expression for $m = im' = iim$. We also notice that

$$m'\overline{m'} = i^2 m\overline{m} = m\overline{m}, \tag{4.49}$$

which means that $m'\overline{m'} = 0$, iff $m\overline{m} = 0$, and if we factorize $m'\overline{m'} \neq 0$ and compute its square root $\sqrt{m'\overline{m'}}$, then also $\sqrt{m\overline{m}} = \sqrt{m'\overline{m'}}$.

Doing this we get that

$$s' = p'^{-1}q' = q^{-1}p. \tag{4.50}$$

Following the analogous steps above we obtain, if we assume that p' is invertible and m' (and therefore m) is not invertible, then the factorization of m' (and m) will be

$$\begin{aligned} m' &= 2p'\frac{1+is'}{2} = 2(b_{p'} + a_{p'}u_{p'})e^{\alpha_{2p'}e_{12}}\frac{1+is'}{2} \\ &\overset{p'=q}{=} 2q\frac{1+is'}{2} = 2(b_q + a_q u_q)e^{\alpha_{2q}e_{12}}\frac{1+is'}{2} \\ m &= im' = i2(b_q + a_q u_q)e^{\alpha_{2q}e_{12}}\frac{1+is'}{2}, \end{aligned} \tag{4.51}$$

where the factor $2(b_q + a_q u_q)$ can further be put into exponential form using $E(2a_q, 2b_q, u_q)$ in (2.11). The idempotent factor $\frac{1+is'}{2}$ means that m' (and therefore m) is again manifestly not invertible.

4.3 | Case of invertible $m\overline{m}$ and factorization of normed M with $M\overline{M} = h(i)$

If the central value $m\overline{m} \neq 0$ and not proportional to an idempotent, then m is invertible (compare Section 6 of[23]) as

$$m^{-1} = \frac{\overline{m}}{m\overline{m}}. \tag{4.52}$$

Because $m\overline{m} = r_0 + ir_3$ is then given as a non-zero sum of scalar and trivector, and $i^2 = +1$, we can always represent it as

$$m\overline{m} = e^{2\alpha_0}e^{2\alpha_3 i}h(i), \tag{4.53}$$

and we define the invertible central root-like multivector as

$$m_r = e^{\alpha_0}e^{\alpha_3 i} \quad \text{such that} \quad m_r^2 h(i) = m\overline{m}, \tag{4.54}$$

and we can divide m by this m_r to get a new normed multivector

$$M = mm_r^{-1} = me^{-\alpha_0}e^{-\alpha_3 i}, \qquad M\overline{M} = h(i). \tag{4.55}$$

We represent M again as a sum of two elements from the even subalgebra $Cl_2(1,2)$

$$\begin{aligned} M &= P + iQ, \quad P = \langle M\rangle_{even} = (b_P + a_P)e^{\alpha_{2P}e_{23}}, \\ Q &= \langle M\rangle_{odd}i^{-1} = (b_Q + a_Q)e^{\alpha_{2Q}e_{23}}. \end{aligned} \tag{4.56}$$

and compute

$$M\overline{M} = (P + iQ)(\overline{P} + i\overline{Q}) = P\overline{P} + i^2 Q\overline{Q} + i(Q\overline{P} + P\overline{Q}) = h(i). \tag{4.57}$$

4.3.1 | Case of $h(i) = \pm 1$ in $M\overline{M} = h(i)$

Hence for $h(i) = \pm 1$ we must have

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow Q\overline{P} = -P\overline{Q}. \tag{4.58}$$

If P is not invertible, then we have $b_P = a_P$, and if Q is not invertible we have $b_Q = a_Q$. If we assume both P and Q not invertible then we have $P\overline{P} = Q\overline{Q} = 0$ and consequently

$$M\overline{M} = P\overline{P} + i^2 Q\overline{Q} = 0 + i^2 0 = 0 \neq \pm 1, \tag{4.59}$$

which is a contradiction. Therefore either P or Q or both must be invertible.

We first assume P to be invertible, which allows us to compute

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow P(P^{-1}Q + \overline{Q}\,\overline{P}^{-1})\overline{P} = 0 \Leftrightarrow P^{-1}Q + \overline{Q}\,\overline{P}^{-1} = 0. \tag{4.60}$$

Then M can be rewritten as

$$\begin{aligned} M &= P + iQ = P(1 + iP^{-1}Q) = P(1 + i(P^{-1}Q - 0)) \\ &= P(1 + i(P^{-1}Q - \frac{1}{2}P^{-1}Q - \frac{1}{2}\overline{Q}\,\overline{P}^{-1}) \\ &= P(1 + i\frac{1}{2}(P^{-1}Q - \overline{Q}\,\overline{P}^{-1})), \end{aligned} \tag{4.61}$$

where

$$\frac{1}{2}(P^{-1}Q - \overline{Q}\,\overline{P}^{-1}) = \langle P^{-1}Q\rangle_2 \tag{4.62}$$

is a pure bivector and therefore

$$i\langle P^{-1}Q\rangle_2 = \vec{\omega} \tag{4.63}$$

a vector. Therefore

$$\begin{aligned} M &= P(1 + \vec{\omega}), \\ M\overline{M} &= P(1 + \vec{\omega})(1 - \vec{\omega})\overline{P} = P(1 - \vec{\omega}^2)\overline{P} = P\overline{P} - P\overline{P}\vec{\omega}^2 \\ &= P\overline{P} + i^2 Q\overline{Q}. \end{aligned} \tag{4.64}$$

Hence

$$-P\overline{P}\vec{\omega}^2 = +i^2 Q\overline{Q}, \tag{4.65}$$

that is

$$\vec{\omega}^2 = \frac{-i^2 Q\overline{Q}}{P\overline{P}} = -\frac{Q\overline{Q}}{P\overline{P}} \begin{cases} < 0 & \text{for } \frac{Q\overline{Q}}{P\overline{P}} > 0, \\ = 0 & \text{for } Q\overline{Q} = 0, \\ > 0 & \text{for } \frac{Q\overline{Q}}{P\overline{P}} < 0. \end{cases} \tag{4.66}$$

This leads to the following factorization of M

$$M = (b_P + a_P u_P)e^{\alpha_{2P}e_{23}} \begin{cases} e^{\alpha_0'}e^{\alpha_1\frac{\vec{\omega}}{\omega}}, & \omega = \sqrt{-\vec{\omega}^2}, \\ 1 + \vec{\omega} = e^{\vec{\omega}}, & \vec{\omega}^2 = 0, \\ E(\omega, 1, \frac{\vec{\omega}}{\omega}), & \omega = \sqrt{\vec{\omega}^2}, \end{cases} \tag{4.67}$$

with

$$\alpha_1 = \operatorname{atan2}(\omega, 1), \qquad \alpha_0' = \ln(\sqrt{1 + \omega^2}). \tag{4.68}$$

Now let us instead assume that Q is invertible (and therefore $\overline{Q}$ as well), without specifying the invertibility of P.

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow Q(\overline{P}\,\overline{Q}^{-1} + Q^{-1}P)\overline{Q} \Leftrightarrow \overline{P}\,\overline{Q}^{-1} + Q^{-1}P = 0. \tag{4.69}$$

We can therefore express

$$\begin{aligned} \overline{M} &= \overline{P} + i\overline{Q} = (\overline{P}\,\overline{Q}^{-1} + i)\overline{Q} = (\overline{P}\,\overline{Q}^{-1} - 0 + i)\overline{Q} \\ &= (\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P + i)\overline{Q} \\ &= (\tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P + i)\overline{Q} \end{aligned} \tag{4.70}$$

with pure bivector

$$\overline{B} = \tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P = -\tfrac{1}{2}\langle Q^{-1}P\rangle_2 = \tfrac{1}{2}\langle\overline{P}\,\overline{Q}^{-1}\rangle_2 = \tfrac{1}{2}\langle\overline{Q^{-1}P}\rangle_2. \tag{4.71}$$

So we get

$$\overline{M} = (i + \overline{B})\overline{Q}, \tag{4.72}$$

and hence

$$M = Q(i + B) = iQ(1 + i^{-1}B) = iQ(1 + \vec{\mu}), \quad \vec{\mu} = i^{-1}B. \tag{4.73}$$

We further compute

$$\begin{aligned} M\overline{M} &= iQ(1 + \vec{\mu})i(1 - \vec{\mu})\overline{Q} = i^2 Q(1 - \vec{\mu}^2)\overline{Q} = i^2 Q\overline{Q} - i^2\vec{\mu}^2 Q\overline{Q} \\ &= P\overline{P} + i^2 Q\overline{Q} \end{aligned} \tag{4.74}$$

which implies that for $i^2 = -1$

$$P\overline{P} = -i^2\vec{\mu}^2 Q\overline{Q} \Leftrightarrow \vec{\mu}^2 = -i^2\frac{P\overline{P}}{Q\overline{Q}} = -\frac{P\overline{P}}{Q\overline{Q}}\begin{cases} > 0 & \text{for } \frac{P\overline{P}}{Q\overline{Q}} < 0. \\ = 0 & \text{for } P\overline{P} = 0, \\ < 0 & \text{for } \frac{P\overline{P}}{Q\overline{Q}} > 0. \end{cases} \tag{4.75}$$

This leads to the following factorization of M

$$M = i(b_Q + a_Q u_Q)e^{\alpha_{2Q}e_{23}}\begin{cases} E(\boldsymbol{\mu}, 1, \frac{\vec{\mu}}{\mu}), & \boldsymbol{\mu} = \sqrt{\vec{\mu}^2}, \\ 1 + \vec{\mu} = e^{\vec{\mu}}, & \vec{\mu}^2 = 0, \\ e^{\alpha_0''}e^{\alpha_1\frac{\vec{\mu}}{\mu}}, & \boldsymbol{\mu} = \sqrt{-\vec{\mu}^2}, \end{cases} \tag{4.76}$$

with

$$\alpha_1 = \text{atan2}(\boldsymbol{\mu}, 1), \qquad \alpha_0'' = \ln(\sqrt{1 + \boldsymbol{\mu}^2}). \tag{4.77}$$

We note that the two factorizations (4.67) or (4.76) have a nearly identical form. We obtain (4.76) by exchanging P and Q in (4.67) and by multiplying with i.

Finally, for either P invertible or Q invertible we obtain

$$m = m_r M = e^{\alpha_0}e^{\alpha_3 i}M, \tag{4.78}$$

assuming the factorized forms (4.67) or (4.76) for M.

4.3.2 | Case of $h(i) = \pm i$ in $M\overline{M} = h(i)$

Now let us assume, that $h(i) = \pm i$ in $M\overline{M}$. Then we have $P\overline{P} + i^2 Q\overline{Q} = P\overline{P} + Q\overline{Q} = 0$, and $Q\overline{P} + P\overline{Q} = \pm 1$, respectively.

Additionally assuming P invertible ($P\overline{P} = b_P^2 - a_P^2 \neq 0$), then by $P\overline{P} + Q\overline{Q} = 0$, we have $Q\overline{Q} = -P\overline{P}$, and therefore Q will also be invertible. Obviously, if we assume instead first Q invertible ($Q\overline{Q} \neq 0$), then by the same argument $P\overline{P} = -Q\overline{Q}$, and P will also be invertible. Similar to (4.36) we first define $S = P^{-1}Q = (b_S + a_S u_S)e^{\alpha_{2S}e_{12}}$ and obtain the condition

$$M\overline{M} = (b_P^2 - a_P^2)[1 + (b_S^2 - a_S^2) + i2b_S\cos\alpha_{2S}] = \pm i. \tag{4.79}$$

We must therefore have zero scalar part, i.e.

$$1 + (b_S^2 - a_S^2) = 0 \quad \Leftrightarrow \quad a_S^2 - b_S^2 = 1, \tag{4.80}$$

and can therefore represent with some angle α_S

$$a_S = \cosh\alpha_S, \qquad b_S = \sinh\alpha_S. \tag{4.81}$$

The condition for the trivector part gives

$$2(b_P^2 - a_P^2)\, b_S\cos\alpha_{2S} = \pm 1, \tag{4.82}$$

which means that $b_S \neq 0$, and therefore $\alpha_S \neq 0$. Then we can compute α_{2S} dependent on $P\overline{P}$ and α_S as:

$$\cos\alpha_{2S} = \frac{\pm 1}{2(b_P^2 - a_P^2)b_S} = \frac{\pm 1}{2(b_P^2 - a_P^2)\sinh\alpha_S}. \tag{4.83}$$

Since the range of the cosine function is $[-1, +1]$, the equation for $\cos\alpha_{2S}$ imposes further restrictions on the product $2(b_P^2 - a_P^2)\sinh\alpha_S$, i.e. $|2(b_P^2 - a_P^2)\sinh\alpha_S| \geq 1$. So only if $a_S^2 - b_S^2 = 1$, $b_S = \sinh\alpha_S \neq 0$, and $|2(b_P^2 - a_P^2)b_S| = |2(b_P^2 - a_P^2)\sinh\alpha_S| \geq 1$, leads the combination of P and Q in M to the result $M\overline{M} = \pm i$.

Would it be possible to obtain $M\overline{M} = \pm i$ for both P and Q non-zero but not invertible, i.e. $P\overline{P} = Q\overline{Q} = 0$? This would be possible, and the analysis would work similar to (4.23) to (4.30) and the result would then be

$$M\overline{M} = 2ia_P a_Q\{\cos\Delta - \cos(\Delta - \vartheta)\} = \pm i, \tag{4.84}$$

where $\Delta = \alpha_{2P} - \alpha_{2Q}$, ϑ the dihedral angle between the unit bivectors u_P and u_Q, when we parametrize

$$P = 2a_P\frac{1 + u_P}{2}e^{\alpha_{2P}}, \qquad Q = 2a_Q\frac{1 + u_Q}{2}e^{\alpha_{2Q}}. \tag{4.85}$$

So the parameters of P, Q would need to satisfy

$$a_P a_Q = \frac{\pm 1}{2\{\cos\Delta - \cos(\Delta - \vartheta)\}}, \tag{4.86}$$

with necessary non-zero condition for $\{\cos\Delta - \cos(\Delta - \vartheta)\}$, otherwise $M\overline{M} = 0$ and not $\pm i$.

But whether P and Q are both invertible or both not invertible, the equations obtained seem not to suggest a meaningful factorization for M if $M\overline{M} = \pm i$. We therefore do not pursue the factorization question for M in the case of $M\overline{M} = \pm i$ any further in this work, but it certainly remains an interesting *open question for further research*.

5 | RESULTS FOR DIRECT FACTORIZATION IN $CL(2,1)$

Summarizing the results obtained in Section 4, we find the following.

If only one of the two even subalgebra (isomorphic to $Cl(2,0)$) components p, q of $m = p + iq \in Cl(2,1)$ is non-zero, then the final factorizations are directly given by the factorization of iq in (4.8) or p in (4.9).

Proposition 4.1 states a necessary and sufficient condition for the even subalgebra components p, q of $m = p + iq$ so that the central multivector $m\overline{m}$ will not be invertible. Equations (4.21) and (4.22) give the explicit forms obtained for m in this situation in terms of the components p or q, idempotents and exponentials with e_{12} in the exponent.

An explicit form of m is given when both components p and q are not invertible in (4.25). Even in this case $m\overline{m}$ has in general a non-zero trivector component (4.30), which vanishes only if special conditions for the relative parameters of p and q are met as specified in (4.31). Explicit simplified forms of m for non-invertible p and q with vanishing $m\overline{m}$ are given in (4.32). A factorization of non-vanishing $m\overline{m}$ for non-invertible p and q is given in (4.33).

For invertible component p a factorization of non-invertible multivectors m is given in (4.46). For invertible component q a factorization of non-invertible multivectors m is given in (4.51).

Factorizations of m with normed multivector factors $M = P + iQ$, $P, Q \in Cl_2(2,1)$, when $M\overline{M} = \pm 1$ are given in (4.78), based on the factorizations of M in (4.67) for invertible component P of M, and in (4.76) for invertible component Q of M.

The case of $M\overline{M} = \pm i$ is discussed in Section 4.3.2, but seems not to lead to meaningful factorizations of M, which therefore currently poses some restriction to factorization in $Cl(2,1)$ not encountered in this way in the other three algebras $Cl(3,0)$, $Cl(0,3)$ and $Cl(1,2)$ that have already been studied in[27]. It may therefore be an interesting case for further research.

6 | CONCLUSION

In this paper we have considered general elements of the Clifford algebra $Cl(2,1)$, and studied multivector factorization into products of exponentials, idempotents and blades, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar). We used methods of direct computation or applied several isomorphisms, to simplify the computation at hand or make use of known results in isomorphic representations. Our approach turned out to become relatively complex in the case of $Cl(2,1)$, compared to the three algebras $Cl(3,0)$, $Cl(0,3)$ and $Cl(1,2)$ that have already been studied in[27]. As indicated further research could be done in the special case $M\overline{M} = \pm i$. Furthermore, all results of this work could be implemented in Clifford algebra software like[33].

It may be possible in the future to extend this approach to even higher dimensional Clifford algebras, but simple products of exponentials and idempotents may, due to the dimensionality of the k-vector spaces, have to include multiple non-commuting exponential factors with k-vectors of the same grade in the exponents. Of particular interest would be to apply our methods to conformal geometric algebra $Cl(4,1)$ widely used in computer graphics and robotics[20,9]. Furthermore a complete factorization study of $Cl(1,3)$ and $Cl(3,1)$ that are both of great importance in special relativity and relativistic physics[14,15,7,24] may be of considerable interest. The present work can e.g. be applied in the study of Lipschitz versors, see e.g. E.4.2 in[38], pinor and spinor groups, and in the development of Clifford Fourier and wavelet transformations[21,24], compare also the motivation for this research in the introduction Section 1.

It might also be of interest to represent the Clifford algebra $Cl(2,1)$ in terms of tensor products of quaternions and their subalgebras, and reexpress the results we have obtained above, or even further develop them, compare[12,13]. Finally, in recent work it appears that, different from all other Clifford algebras over real quadratic three-dimensional vector spaces, a minimal embedding of octonions in $Cl(2,1)$ may possibly not exist[28]. One wonders, if this could be related to the higher complexity of the factorizations studied in the current paper, compared to the case of all other Clifford algebras over real quadratic three-dimensional vector spaces[27].

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AUTHOR BIOGRAPHY

Eckhard Hitzer. Eckhard Hitzer holds a PhD in theoretical physics from the University of Konstanz in Konstanz, Germany. He is Senior Associate Professor at College of Liberal Arts of International Christian University in Tokyo, Japan. His main research interests are theory and application of Clifford geometric algebras.

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