Generalized Branes in Noncommutative Clifford Spaces

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Abstract

Starting with a brief review of our prior construction of n-ary algebras, based on the relation among the **n**-ary commutators of noncommuting spacetime coordinates $[X^1, X^2, ..., X^n]$ with the polyvector valued coordinates $X^{123...n}$ in noncommutative Clifford spaces, $[X^1, X^2, ..., X^n] = n!\ X^{123...n}$, we proceed to construct generalized brane actions in noncommutative matrix coordinates backgrounds in Clifford-spaces (C-spaces). An instrumental role is played by the Clifford-valued field $\Phi(\sigma^A) = \Phi^M(\sigma^A)\Gamma_M$ which allows to construct a matrix realization of the n-ary algebra of the form $\mathbf{X}^M \equiv \Phi^{-1}(\sigma^A)\Gamma^M\Phi(\sigma^A)$, and that is given in terms of the world manifold's σ^A polyvector-valued coordinates of the generalized brane, and which by construction, $satisfy$ the n-ary algebra. One then learns that is the presence of matter which *endows* the spacetime points with a noncommutative algebraic structure. We finalize with an extension of coherent states in C-spaces and provide a preliminary study of strings in target C-space backgrounds.

Keywords : Strings; Branes; Clifford algebras; n-ary algebras; Noncommutative Geometry.

1 Introduction : Noncommutative Clifford Space Coordinates and n-ary Algebras

After decades of string theory research its physical foundation is still unknown and the question *what* is string theory remains unanswered. General relativity is based on the principle of equivalence and general coordinate covariance. It is desirable to decipher the principle governing string theory. We have learned that string theory not only involves one-dimensional extended objects but higher dimensional ones, p and D-branes. Furthemore, the quantization of membranes and higher dimensional extended objects has been extremely difficult due to the intrinsic nonlinearity. The aim of this work is an attempt to bridge these conceptual obstacles by introducing Clifford spaces (C-spaces) [1].

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [1] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds. Namely, C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing [1].

Given $\mathbf{X} = X_M \Gamma^M$, a Clifford-valued coordinate associated to Clifford space (C-space), it admits the following expansion in terms of the Clifford algebra generators in D-dimensions : $\mathbf{1}, \gamma^\mu, \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \cdots, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_D}$

$$\mathbf{X} \; = \; s \; \mathbf{1} \; + \; x_\mu \; \gamma^\mu \; + \; x_{\mu_1 \mu_2} \; \gamma^{\mu_1} \wedge \gamma^{\mu_2} \; + \; x_{\mu_1 \mu_2 \mu_3} \; \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \; + \; ...... \; + \\ x_{\mu_1 \mu_2 \mu_3 ...... \mu_D} \; \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \; \gamma^{\mu_3} .... \; \wedge \gamma^{\mu_D} \tag{1.1}$$

The numerical combinatorial factors can be omitted by imposing the ordering prescription $\mu_1 < \mu_2 < \mu_3 \cdots < \mu_D$. In order to match physical units in each term of (1.1) a length scale parameter must be suitably introduced in the expansion in eq-(1.1). In [1] we introduced the Planck scale as the expansion parameter in (1.1), and which was set to unity, when one adopts the units $\hbar = c = G = 1$.

The commuting scalar, vectorial, antisymmetric coordinates $s, x_\mu, x_{\mu_1 \mu_2} = -x_{\mu_2 \mu_1}, \cdots, x_{\mu_1 \mu_2 \cdots \mu_D}$ are the scalar, vector, bivector, trivector, $\cdots$ components of the polyvector-valued coordinates in C-space. A *noncommutative* extension of these polyvector-valued coordinates was developed in [3]. In this introduction, we briefly review such construction to prepare the groundwork for the study of branes in noncommutative flat target C-space backgrounds.

We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [2]

$$[\; \gamma_{b_1 b_2 .....b_p}, \; \gamma^{a_1 a_2 ......a_q} \;] \; = \; 2 \gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \; - \\ \frac{2p!q!}{2!(p-2)!(q-2)!} \; \delta^{[a_1 a_2}_{[b_1 b_2} \; \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \; + \; \frac{2p!q!}{4!(p-4)!(q-4)!} \; \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \; \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \; - \; ...... \tag{1.2}$$

for $pq = even$ one has

$$[\; \gamma_{b_1 b_2 .....b_p}, \; \gamma^{a_1 a_2 ......a_q} \;] \; = \; - \; \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \; \delta^{[a_1}_{[b_1} \; \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \; - \\ \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \; \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \; \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \; + \; ...... \tag{1.3}$$

The anti-commutators for $pq = even$ are

$$\{ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q} \} = 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!} \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} - ...... \tag{1.4}$$

and the anti-commutators for $pq = odd$ are

$$\{ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q} \} = - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} + ... \tag{1.5}$$

The second step is to write down the *noncommutative* algebra associated with the noncommuting polyvector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad .... \gamma^{\mu_1 \mu_2 ....\mu_n} \leftrightarrow X^{\mu_1 \mu_2 ....\mu_n}. \tag{1.6}$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (1.6), and using the Clifford algebraic relations (1.2-1.5) (after one relabels indices), one can then construct the following *noncommutative* algebra among the polyvector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[X^{\mu_1}, X^{\mu_2}] = X^{\mu_1} X^{\mu_2} - X^{\mu_2} X^{\mu_1} = 2 X^{\mu_1 \mu_2}. \tag{1.7}$$

As mentioned above, in most of the remaining commutators a suitable length scale parameter must be introduced in order to match units. We shall set this length scale (let us say the Planck scale) to *unity*. Secondly, by choosing the C-space coordinates to behave like anti-Hermitian operators we avoid the need to introduce i factors in the right hand side of (1.7), since the commutator of two anti-Hermitian operators is anti-Hermitian.

The other commutators are

$$[X^{\mu_1 \mu_2}, X^\nu] = 4 (g^{\mu_2 \nu} X^{\mu_1} - g^{\mu_1 \nu} X^{\mu_2}). \tag{1.8}$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^\nu] = 2 X^{\mu_1 \mu_2 \mu_3 \nu}, \quad [X^{\mu_1 \mu_2 \mu_3 \mu_4}, X^\nu] = -8 g^{\mu_1 \nu} X^{\mu_2 \mu_3 \mu_4} \pm ...... \tag{1.9}$$

$$[X^{\mu_1 \mu_2}, X^{\nu_1 \nu_2}] = -8 g^{\mu_1 \nu_1} X^{\mu_2 \nu_2} + 8 g^{\mu_1 \nu_2} X^{\mu_2 \nu_1} +$$

$$8 g^{\mu_2 \nu_1} X^{\mu_1 \nu_2} - 8 g^{\mu_2 \nu_2} X^{\mu_1 \nu_1}. \tag{1.10}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2}\ ]\ =\ 12\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\nu_2}\ \pm\ .... \quad (1.11)$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}\ ]\ =\ -36\ G^{\mu_1\mu_2\ \nu_1\nu_2}\ X^{\mu_3\nu_3}\ \pm\ ...... \quad (1.12)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]\ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \quad (1.13)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]\ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \quad (1.14)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\ ]\ =\ 48\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4} - 48\ G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3} + ... \quad (1.15)$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}\ ]\ =\ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \quad (1.16)$$

where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}\ =\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n}\ +\ \textit{signed permutations} \quad (1.17a)$$

etc......The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n \times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I\nu_J}$

$$det\ \mathbf{G}_{n\times n}\ =\ \frac{1}{n!}\ \epsilon_{i_1 i_2.....i_n}\ \epsilon_{j_1 j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \quad (1.17b)$$

$i_1, i_2, ..., i_n \subset I = 1, 2, ..., D$ and $j_1, j_2, ..., j_n \subset J = 1, 2, ..., D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the polyvector-valued coordinates in $D = 4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$.

N-ary algebras have been known for some time [8] since Nambu introduced his bracket (a Jacobian) in the study of branes and the generalizations of Hamiltonian mechanics based on Poisson brackets. In this section we shall show how polyvector valued coordinates admit a very natural interpretation in terms of $n-$ary commutators of vector-valued coordinates.

The ternary commutator for noncommuting coordinates is defined as

$$[X^1, X^2, X^3]\ =\ X^1\ [X^2, X^3]\ +\ X^2\ [X^3, X^1]\ +\ X^3\ [X^1, X^2]\ =$$

$$\frac{1}{2}\ \{\ X^1,\ [X^2, X^3]\ \}\ +\ \frac{1}{2}\ [\ X^1,\ [X^2, X^3]\ ]\ +\ \textit{cyclic permutations} \quad (1.18)$$

Due to the Jacobi identities, the terms

$$\frac{1}{2} [\ X^1, \ [X^2, X^3] \ ] \ + \ cyclic\ permutations \ = \ 0. \tag{1.19}$$

so that the ternary commutators become

$$[X^1, X^2, X^3] \ = \ \frac{1}{2} \{ \ X^1, \ [X^2, X^3] \ \} \ + \ cyclic\ permutations. \tag{1.20}$$

After using the relations

$$[X^2, X^3] \ = \ 2 \ X^{23}, \quad \{ \ X^1, \ X^{23} \ \} \ = \ 2 \ X^{123}. \tag{1.21}$$

one gets finally

$$[X^1, X^2, X^3] \ = \ 2 \ X^{123} \ + \ cyclic\ permutations \ = \ 6 \ X^{123}. \tag{1.22}$$

since $X^{123} = X^{231} = X^{312} = -X^{132} = ...$

After using the above noncommutative algebraic relations, after some laborious but straightforward algebra, one arrives by recursion at the most general n-ary commutator given by

$$[\ X^1, \ X^2, \ ...., \ X^n \ ] \ = \ n! \ X^{123.....n}. \tag{1.23}$$

for all $n = 2, 3, \cdots, D$ [3].

The immediate consequence of the n-ary algebra of the noncommutative polyvector-valued coordinates, associated with a quantum extension of the classical C-space, is that one must extend the usual formulation of Quantum Mechanics involving ordinary commutators of operators to one requiring n-ary commutators. In other words, quantizing the classical Nambu-Poisson mechanics [8]. The findings of this introductory section will allow to construct generalized p-brane actions in noncommutative matrix coordinates backgrounds in Clifford-spaces (C-spaces) in section **2**. We then proceed with an analysis of the deformation quantization of p-branes in C-spaces. And in section **3** we describe an extension of coherent states in C-spaces, and provide a study of strings in target C-space backgrounds. We conclude with some final remarks.

2 Generalized Branes in Noncommutative C-spaces

2.1 Matrix Coordinates in C-space

Given a fermionic field $\Psi = \Psi(x^\mu)$, one could interpret the informal "inverse" operation $x^\mu = x^\mu(\Psi)$, relating x^μ to the value of the fermionic field at that point, from the correspondence given by $x^\mu \leftrightarrow \bar{\Psi}\gamma^\mu\Psi$. Based on this correspondence we shall define the following *matrices*

$$\mathbf{X}, \mathbf{X}^\mu, \mathbf{X}^{\mu_1\mu_2}, \cdots, \mathbf{X}^{\mu_1\mu_2\cdots\mu_n} \qquad (2.1)$$

that have a one-to-one correspondence with the polyvector-valued coordinates $x, x^\mu, x^{\mu_1\mu_2}, \cdots, x^{\mu_1\mu_2\cdots\mu_n}$, in terms of Φ, as follows

$$\mathbf{X}^\mu \;=\; \Phi^{-1}\;\gamma^\mu\;\Phi, \quad \mathbf{X}^{\mu_1\mu_2} \;=\; \Phi^{-1}\;\gamma^{\mu_1\mu_2}\;\Phi$$
$$\mathbf{X}^{\mu_1\mu_2\mu_3} \;=\; \Phi^{-1}\;\gamma^{\mu_1\mu_2\mu_3}\;\Phi, \quad \cdots \quad \mathbf{X}^{\mu_1\mu_2\cdots\mu_D} \;=\; \Phi^{-1}\;\gamma^{\mu_1\mu_2\cdots\mu_D}\;\Phi \qquad (2.2)$$

where $\Phi = \Phi(x, x^\mu, x^{\mu_1\mu_2}, \cdots, x^{\mu_1\mu_2\cdots\mu_D})$ is a Clifford-valued auxiliary field

$$\Phi \;\equiv\; \Phi^M\Gamma_M \;=\; \phi + \phi^\mu\gamma_\mu + \frac{1}{2!}\;\phi^{\mu\nu}\gamma_{\mu\nu} \;+\; \cdots \;+ \frac{1}{D!}\;\phi^{\mu_1\mu_2\cdots\mu_D}\gamma_{\mu_1\mu_2\cdots\mu_D} \qquad (2.3)$$

living in the $flat$ C-space associated to the Clifford algebra in D-dim.[1] In $D = 4$, the Clifford algebra is $2^4 = 16$ dimensional and Φ can be represented in terms of the entries of a 4×4 matrix. Φ^{-1} is the inverse 4×4 matrix-valued field (assuming $det(\Phi) \neq 0$) and such that all the matrix coordinates displayed in eq-(2.2) obey the previous n-ary commutation relations found in section **1** due to the $\Phi\Phi^{-1} = \Phi^{-1}\Phi = \mathbf{1}$ condition. In D-dim the Clifford-valued field Φ is represented by a $2^{[\frac{D}{2}]} \times 2^{[\frac{D}{2}]}$ matrix where $[\frac{D}{2}]$ is the integer part of $\frac{D}{2}$. We shall take D even for simplicity.

Therefore, the construction of the matrices in eqs-(2.2) in terms of the auxiliary field Φ will automatically obey the n-ary commutators

$$[\;\mathbf{X}^\mu,\;\mathbf{X}^\nu\;] \;\sim\; \mathbf{X}^{\mu\nu}, \quad [\;\mathbf{X}^{\mu_1},\;\mathbf{X}^{\mu_2},\;\mathbf{X}^{\mu_3}\;] \;\sim\; \mathbf{X}^{\mu_1\mu_2\mu_3}$$
$$[\;\mathbf{X}^{\mu_1},\;\mathbf{X}^{\mu_2},\;.......,\;\mathbf{X}^{\mu_n}\;] \;\sim\; \mathbf{X}^{\mu_1\mu_2\cdots\mu_n} \qquad (2.4)$$

Hence, one has constructed an explicit matrix realization in (2.2) of the n-ary commutation relations in terms of $\Phi = \Phi^M\Gamma_M$.

For example, if the $\Phi = \Phi^M(x, x^\mu, x^{\mu_1\mu_2}, \cdots)\Gamma_M$ Clifford-valued field is designed to obey a generalized version of the massless Klein-Gordon equation

$$\partial_N\partial^N(\Phi^M\Gamma_M) \;=\; 0; \quad \partial_N = \{\partial_x, \partial_{x^\mu}, \partial_{x^{\mu_1\mu_2}}, \cdots\} \qquad (2.5)$$

then any solution of eq-(2.5) will provide an explicit construction (realization) of the family of matrices in eqs-(2.2) obeying the n-ary algebra, and which are given in terms of $\Phi = \Phi^M\Gamma_M$. Each one of the components Φ^M are functions of the polyvector-valued variables $x, x^\mu, x^{\mu_1\mu_2}, \cdots, x^{\mu_1\mu_2\cdots\mu_D}$.

If one constrains the solutions of (2.5) to obey $\partial_N\partial^N(\Phi^M) = 0$, for *all* values of the polyvector-valued index M ranging from 1 all the way to 2^D (the dimension of the Clifford algebra in D-dim), one will have 2^D *uncoupled* equations for the Φ^M components. Whereas if one has *one* single equation of

[1] In eq-(2.3) we introduced the combinatorial numerical factors

the form displayed by eq-(2.5) one will have an equation which will *couple* all of the Φ^M components.

In a nut-shell, the central idea that can inferred from eq-(2.2) is that the noncommutativity of the polyvector matrix coordinates stems from postulating the existence of a Clifford-valued field $\Phi = \Phi(x, x^\mu, x^{\mu_1\mu_2}, \cdots, x^{\mu_1\mu_2\cdots\mu_D})$ in C-space; i.e. it is the presence of matter which *endows* the spacetime points with a noncommutative algebraic structure.

In string theory, the target spacetime coordinates $X^\mu(\sigma,\tau), \mu = 0,1,2,\cdots,D-1$ are the components of spacetime *vectors*, but they are D scalar fields $X^0(\sigma,\tau)$, $X^1(\sigma,\tau),\cdots,X^{D-1}(\sigma,\tau)$ from the two-dim worldsheet point of view. Thus one has a spacetime coordinate/world sheet scalar field correspondence. In a similar fashion, one has in eq-(2.2) a matrix coordinate/field correspondence given by $\mathbf{X} = \mathbf{X}^M\Gamma_M = (\Phi^{-1}\Gamma^M\Phi)\Gamma_M$.

Instead of imposing the generalized Klein-Gordon equation for Φ another route one can take is in the study of p-branes moving in *Noncommutative* target C-space backgrounds. A p-brane action associated with the commutative embedding functions $X^\mu(\sigma^a), a = 1,2,\cdots,p+1$, from the $p+1$-dim world-manifold into a target background can be generalized to C-spaces [4] by embedding a Clifford world-manifold of dimension 2^d into a target Clifford space of dimension 2^D with $d \leq D$ via means of the commutative embedding polyvector-valued functions

$$X^M(\sigma^A) \;=\; X(\sigma,\sigma^a,\sigma^{a_1a_2},\cdots,\sigma^{a_1a_2\cdots a_d}),\;\; X^\mu(\sigma,\sigma^a,\sigma^{a_1a_2},\cdots,\sigma^{a_1a_2\cdots a_d}),$$
$$X^{\mu_1\mu_2}(\sigma,\sigma^a,\sigma^{a_1a_2},\cdots,\sigma^{a_1a_2\cdots a_d}),\;\; X^{\mu_1\mu_2\mu_3}(\sigma,\sigma^a,\sigma^{a_1a_2},\cdots,\sigma^{a_1a_2\cdots a_d}),\;\;\cdots$$
$$X^{\mu_1\mu_2\cdots\mu_D}(\sigma,\sigma^a,\sigma^{a_1a_2},\cdots,\sigma^{a_1a_2\cdots a_d}) \tag{2.6}$$

The C-space version of a p-brane action is [1]

$$S \;=\; -\frac{T}{2}\int d\sigma d\sigma^a d\sigma^{a_1a_2}\cdots d\sigma^{a_1a_2\cdots a_d}\;\sqrt{|H|}\;\left(H^{AB}\;\partial_A X^M\;\partial_B X^N\;G_{MN}\;-\;(2^d-2)\right) \tag{2.7}$$

with $\partial_A = \partial_{\sigma^A} = \partial_\sigma, \partial_{\sigma^a}, \partial_{\sigma^{a_1a_2}},\cdots,\partial_{\sigma^{a_1a_2\cdots a_d}}$, and $H = det(H_{AB})$ is the determinant of the $2^d\times 2^d$ auxiliary metric H_{AB} on the Clifford world-manifold of dimension 2^d. The components of such metric have a similar form to the metric in eqs-(1.17), and such that its determinant is given in terms of the sums of antisymmetrized products of block determinants. For example, if one has a 2×2 block matrix comprised of entries $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ the determinant would be $det(\mathbf{A})\;det(\mathbf{D}) - det(\mathbf{B})\;det(\mathbf{C})$. G_{MN} is the $2^D\times 2^D$ metric on the target Clifford space background of dimension $2^D \geq 2^d$. To simplify matters we shall work on a *flat* C-space background. T is the tension of the generalized brane in Clifford space and whose units must be such to render the action (2.7) dimensionless.

One can proceed next to construct the *noncommutative* C-space generalization of the above action (2.7) by promoting the C-space commuting polyvector coordinates $X^M(\sigma^A)$ to matrix-valued *noncommuting* polyvector coordinates (denoted by a bold face font) $\mathbf{X}^M(\sigma^A), M = 1,2,\cdots,2^D$, and defined by

$\mathbf{X}^M(\sigma^A) = \Phi^{-1}(\sigma^A)\Gamma^M\Phi(\sigma^A)$, with $\Gamma^M = \mathbf{1}, \gamma^\mu, \gamma^{\mu_1\gamma_2}, \cdots, \gamma^{\mu_1\mu_2\cdots\mu_D}$. All the derivatives of the matrices $\mathbf{X}^M(\sigma^A)$ with respect to σ^A can be written in terms of the derivatives with respect to the Clifford-valued field $\Phi(\sigma^A) \equiv \Phi^N(\sigma^A)\Gamma_N$, as follows

$$\partial_A \mathbf{X}^M \;=\; -\frac{1}{2}(\Phi^{-2}\,\partial_A\Phi \;+\; \partial_A\Phi\;\Phi^{-2})\;\Gamma^M\;\Phi \;+\; \Phi^{-1}\;\Gamma^M\;\partial_A\Phi \qquad (2.8)$$

where one has taken into consideration the ordering due to the noncommutative nature of the matrix representation of the Clifford-valued field Φ.

For example, in a D target spacetime background, one has a total number of 2^D field components in the definition of $\Phi(\sigma^A) = \Phi^N(\sigma^A)\Gamma_N$ given by $\phi, \phi^\mu, \phi^{\mu_1\mu_2}, \phi^{\mu_1\mu_2\mu_3}, \cdots, \phi^{\mu_1\mu_2\cdots\mu_D}$, and where all of the latter field components are themselves functions of the 2^d polyvector-valued coordinates σ^A associated with the Clifford world-manifold of the generalized brane of dimension $2^d \leq 2^D \Rightarrow \sigma^A = \sigma, \sigma^a, \sigma^{a_1a_2}, \sigma^{a_1a_2a_3}, \cdots, \sigma^{a_1a_2\cdots a_d}$.

The action in a *flat* C-space background involving the matrices $\mathbf{X}^M$ is given by

$$S \;=\; -\;\frac{T}{2}\int [D\Omega]\;\sqrt{|H|}\;\Big(\;H^{AB}\;Trace\;(\;\partial_A\mathbf{X}^M\;\partial_B\mathbf{X}^N\;G_{MN}\;)\;-\;(2^d-2)\;\Big) \qquad (2.9)$$

where all the derivatives of the matrix coordinates $\partial_A\mathbf{X}^M$ can be rewritten in terms of the derivatives of Φ via eq-(2.8). The measure $[D\Omega]$ is defined as

$$[D\Omega] \;\equiv\; \prod d\sigma^A \;=\; d\sigma\;d\sigma^a\;d\sigma^{a_1a_2}\;\cdots\;d\sigma^{a_1a_2\cdots a_d},\;\; d \leq D \qquad (2.10)$$

and $G_{MN} = \eta_{MN}$ is the flat C-space metric for the target background.

Such an action (2.9) will provide the sought-after 2^D equations of motion $\frac{\delta S}{\delta X^M} = 0$ for the 2^D matrices $\mathbf{X}^M(\sigma^A), M = 1, 2, \cdots, 2^D$, and which in turn due to eq-(2.8), can be recast in terms of the equations of motion for $\Phi(\sigma^A)$ so that one will be able to determine (in principle) the 2^D field components of $\Phi(\sigma^A) \equiv \Phi^N(\sigma^A)\Gamma_N$. Once the functional expression for the $2^{[\frac{D}{2}]} \times 2^{[\frac{D}{2}]}$ *matrix* $\Phi = \Phi^N(\sigma^A)\Gamma_N$ is known one can read-off the expressions for the matrix coordinates $\mathbf{X}^M(\sigma^A)$ directly from the definition $\mathbf{X}^M \equiv \Phi^{-1}(\sigma^A)\Gamma^M\Phi(\sigma^A)$ by performing a simple multiplication of matrices, with the bonus that the matrix coordinates $\mathbf{X}^M(\sigma^A)$ will also *satisfy* the n-ary algebra (1.23), by construction.

One must note that *not* all of the solutions $\mathbf{X}^M$ to the equations of motion are independent due to the fact that one must obey the n-ary commutation relations. In *flat* C-space backgrounds, the matrices $\mathbf{X}^\mu, \mu = 1, 2, 3, \cdots, D$ reproduce all of the n-ary algebra elements since the bivectors $\mathbf{X}^{\mu_1\mu_2}$, trivectors $\mathbf{X}^{\mu_1\mu_2\mu_3}, \cdots$ are generated by simply performing the n-ary commutation relations (1.23) involving the matrices $\mathbf{X}^\mu$'s. In a sense, the branes living in Noncommutative C-space are condensates of lower dimensional branes since the bivectors, trivectors, $\cdots$ are composites of the $\mathbf{X}^\mu$ elements.

2.2 Deformation Quantization of Branes in C-spaces

The Moyal noncommutative but associative star product in ordinary $2d$-dim phase space comprised of coordinates $q^a, p_a; a = 1, 2, \cdots, d$ is given by [5]

$$X * Y \equiv e^{\frac{i\hbar}{2}\Omega^{ij}\partial_i \wedge \partial_j} X(q^a, p_a) Y(q^a, p_a) =$$

$$\sum_{n=0}^{\infty} \frac{(i\hbar/2)^n}{n!} \Omega^{i_1 j_1}\Omega^{i_2 j_2} \cdots \Omega^{i_n j_n} \partial^n_{i_1 i_2 \cdots i_n} X \partial^n_{j_1 j_2 \cdots j_n} Y \quad (2.11)$$

where $\Omega^{ij} = -\Omega^{ji}$ is the inverse of the symplectic antisymmetric $2d \times 2d$ matrix Ω_{ij} in the $2d$-dim phase space and $\partial_i \equiv (\partial_{q^a}, \partial_{p_a}), a = 1, 2, \cdots, d$ are the phase space derivatives. The Poisson bivector is defined as $\mathbf{\Pi} = \Omega^{ij}\partial_i \wedge \partial_j$. Noncommutative and nonassociative star products have been studied by many authors, see [13] and references therein.

A C-space generalization of the star product (2.11), when there is *no* mixing of the different *grades* in the derivatives with respect to the polyvector coordinates, is of the form

$$X * Y \equiv e^{\frac{i\hbar}{2}\Omega\partial_q \wedge \partial p} e^{\frac{i\hbar}{2}\Omega^{ij}\partial_i \wedge \partial_j} e^{\frac{i\hbar^2}{4}\Omega^{i_1 i_2 | j_1 j_2}\partial_{i_1 i_2} \wedge \partial_{j_1 j_2}} \cdots$$

$$e^{\frac{i\hbar^n}{2n!}\Omega^{i_1 i_2 \cdots i_n | j_1 j_2 \cdots j_n}\partial_{i_1 i_2 \cdots i_n} \wedge \partial_{j_1 j_2 \cdots j_n}} X(q^A, p_A) Y(q^A, P_A) \quad (2.12)$$

The derivatives $\partial_I \equiv (\partial_{q^A}, \partial_{p_A}), A = 1, 2, \cdots, 2^n$ are the Clifford phase space derivatives. $\Omega^{IJ} = -\Omega^{JI}$ is the inverse of the symplectic matrix in the Clifford phase space of dimensions 2^{n+1} and is comprised of blocks of different sizes depending on the grade of the polyvector-valued coordinates

$$q^A = (q, q^a, q^{a_1 a_2} \cdots q^{a_1 a_2 \cdots a_n}), \quad p_A = (p, p_a, p_{a_1 a_2} \cdots p_{a_1 a_2 \cdots a_n}) \quad (2.13)$$

The powers of $\hbar$ in (2.12) are required to compensate for the units of the cells "areas" in Clifford phase space. For example, the cells "areas" of the form $dq^{a_1 a_2} \wedge dp_{a_1 a_2}$ have dimensions of $\hbar^2$. The star product (2.12) is very different from the more general star product described at the end of this section [3].

Inspired by the Weyl-Wigner-Moyal-Groenewold (WWMG) deformation quantization procedure [5], one may find the correspondence between *operators* $\hat{X}^M(\hat{q}^A, \hat{p}_A)$ in the Hilbert space, which depend on the position $\hat{q}^A$ and momentum operators $\hat{p}_A$, and the functions $X^M(q^A, p_A)$ of the Clifford phase space coordinates $q^A = (q, q^a, q^{a_1 a_2}, \cdots)$; $p_A = (p, p_a, p_{a_1 a_2}, \cdots)$. The C-space extension of the WWMG map is given by[2]

$$X^M(q^A, p_A) \sim \int \langle q^A - q'^A | \hat{X}^M | q^A + q'^A \rangle e^{2ip_A q'^A/\hbar^{|A|}} \prod dq'^A \quad (2.14)$$

where $|A|$ denotes the grade of the polyvector-valued coordinates. $|A| = 0, 1, 2, 3, \cdots, D$.

[2]We omit factors involving powers of (2π) in front of the integrals

Such mapping is the C-space extension of the Weyl-Wigner-Moyal-Groenewold (WWMG) correspondence [5] between operators $\hat{A}, \hat{B}$ in a Hilbert space and functions in phase space $A(q^a, p_a), B(q^a, p_a)$, such that $W[A] = \hat{A}; W[B] = \hat{B} \Rightarrow W[A]W[B] = \hat{A}\hat{B} = W[A * B]$, and leading to $W^{-1}[\hat{A}\hat{B}] = A * B$. Therefore, the star product obeys similar conditions so that

$$(X^M * X^N)(q^A, p_A) \ \sim \ \int \ \langle q^A - q'^A | \ \hat{X}^M \hat{X}^N \ | q^A + q'^A \rangle \ e^{2ip_A q'_A/\hbar^{|A|}} \ \prod dq'^A \tag{2.15}$$

The WWMG quantum map of the operator $\hat{X}^M$ can also be rewritten as

$$X^M(q^A, p_A) \sim \int \sum_{m,n} \psi_m(q^A - q'^A) \ \langle \psi_m | \hat{X}^M | \psi_n \rangle \ \psi_n^*(q^A + q'^A) \ e^{ip_A q'^A/\hbar^{|A|}} \ \prod dq'^A \tag{2.16}$$

after inserting $1 = \sum_m |\psi_m\rangle\langle\psi_m|$ in eq-(2.14). In order to evaluate the quantities $\langle \psi_m | \hat{X}^M | \psi_n \rangle$ one needs to know what are the quantum states $|\psi_n\rangle$ of the generalized brane in flat C-spaces, and in order to attain that, one has to quantize the ordinary brane in the first place which is notoriously difficult due to the nonlinearity of the equations of motion.

A different generalized Wigner function ansatz than the one displayed by eq-(2.14) was proposed by [7]. The C-space extension of the generalized Wigner ansatz provided by [7] in ordinary spaces is given by

$$Y^M(q^A, p_A) \ \sim \ \int \ \Psi^\dagger_\alpha(q^A - q'^A) \ \Gamma^M_{\alpha\beta} \ \Psi_\beta(q^A + q'^A) \ e^{2ip_A q'^A/\hbar^{|A|}} \ \prod dq'^A \tag{2.17}$$

where Ψ is a spinor with $2^{[\frac{D}{2}]}$ components, and one has written the explicit matrix (spinorial) indices of the gamma matrices $\Gamma^M_{\alpha\beta} = (\mathbf{1}_{\alpha\beta}, \gamma^\mu_{\alpha\beta}, \gamma^{\mu_1\mu_2}_{\alpha\beta}, \cdots, \gamma^{\mu_1\mu_2\cdots\mu_D}_{\alpha\beta})$. In essence, eq-(2.17) states that the bosonic fields in the Clifford phase space $Y^M(q^A, p_A)$'s are nonlocal composites of fermionic bilinears.

The star product resulting from eq-(2.17) turns out to be

$$(Y^M * Y^N)(q^A, p_A) \ \sim \ \int \ \Psi^\dagger_\alpha(q^A - w^A) \ (\Gamma^M \Gamma^N)_{\alpha\beta} \ \Psi_\beta(q^A + w^A) \ e^{2ip_A w_A/\hbar^{|A|}} \ \prod dw^A, \tag{2.18}$$

with $w^A = q'^A + q''^A$. The key condition $W^{-1}[\Gamma^M \Gamma^N] = Y^M * Y^N$ will impose strong constraints on the above spinorial fields $\Psi_\alpha(q^A)$ in C-space.

Extending the numerical calculations of [7] to C-spaces one finds the required conditions on the Ψ's to be given by

$$\int \ \Psi^\dagger_\alpha(q^A - v^A) \ \Psi_\beta(q^A - v^A) \ \prod d(q^A - v^A) \ = \ \delta_{\alpha\beta}, \ \ v^A = q'^A - q''^A \tag{2.19}$$

in order for eq-(2.18) to hold.

Comparing eq-(2.17) with eq-(2.14) is tantamount of establishing the correspondence $X^M(q^A, p_A) \leftrightarrow Y^M(q^A, p_A)$, and $\hat{X}^M \leftrightarrow \Gamma^M$. The latter was precisely the same required correspondence at the beginning of this work in order to derive the n-ary algebra (1.23) of the noncommutative polyvector coordinates associated with the noncommutative C-space.

The Moyal bracket of two functions in phase space is defined by

$$\{A(q^a,p_a), B(q^a,p_a)\}_{MB} \equiv A(q^a,p_a)*B(q^a,p_a) \; - \; B(q^a,p_a)*A(q^a,p_a) \quad (2.20)$$

and vanishes in the classical limit. Consequently, the $\hbar \to 0$ limit involving the commutator of two operators $\hat{A}, \hat{B}$ in a Hilbert space as follows

$$lim_{\hbar\to 0} \; \frac{1}{i\hbar} \; [\hat{A},\hat{B}] \; = \; lim_{\hbar\to 0} \; \frac{1}{i\hbar} \; \{A,B\}_{MB} \; = \; \{A,B\}_{PB} \quad (2.21)$$

yields the classical Poisson bracket.

Finaly, we arrive at one of the main results of this section. Since the star product is associative $W^{-1}[\hat{A}\hat{B}\hat{C}] = A * B * C$, the integral representation of eqs-(2.17) can be extended to

$$(Y^{M_1} * Y^{M_2} * Y^{M_3})(q^A, p_A) \; \sim$$

$$\int \; \Psi^\dagger_\alpha(q^A - w^A) \; (\Gamma^{M_1}\Gamma^{M_2}\Gamma^{M_3})_{\alpha\beta} \; \Psi_\beta(q^A + w^A) \; e^{2ip_A w_A/\hbar^{|A|}} \; \prod dw^A, \quad (2.22)$$

and so forth for multiple star products, such that

$$[\; Y^{\mu_1}, \; Y^{\mu_2} \;]_* \; = \; Y^{\mu_1} \; * \; Y^{\mu_2} \; - \; Y^{\mu_2} \; * \; Y^{\mu_1} \; = \; \{\; Y^{\mu_1}, \; Y^{\mu_2} \;\}_{MB} \; = \; 2Y^{\mu_1\mu_2} \quad (2.23)$$

and after more laborious algebra one can show that

$$[\; Y^{\mu_1}, \; Y^{\mu_2} \;, Y^{\mu_3} \;]_* \; = \; 3! Y^{\mu_1\mu_2\mu_3} \quad (2.24)$$

$$[\; Y^{\mu_1}, \; Y^{\mu_2}, \; \cdots, \; Y^{\mu_n} \;]_* \; = \; n! Y^{\mu_1\mu_2\cdots\mu_n} \quad (2.25)$$

Consequently, one recovers in this way via the Moyal deformation quantization, an n-ary algebra which is *isomorphic* to the n-ary algebra displayed in section **1**, and involving the *noncommutative* coordinates of Clifford space.

When $p+1 = 2n$, the $p+1$ coordinates of the $p+1$-dim world volume of the p-brane have a one-to-one correspondence with the $q^1, p^1, q^2, p^2, \cdots, q^n, p^n$ phase space coordinates of a $2n$-dim phase space. In this way the star product deformation of an ordinary p-brane action in flat target Minkowsky backgrounds, when $p+1 = 2n$ is even, could be given by [3]

$$S_p \; = \; \frac{T}{(i\hbar)^{(p+1)/2}} \; \int \; d^{p+1}\sigma \; \sqrt{(\{X^{\mu_1}, X^{\mu_2}, \cdot, X^{p+1}\})^2_{MNPB}} \; \rightarrow$$

$$T \int \; d^{p+1}\sigma \; \sqrt{(\{X^{\mu_1}, X^{\mu_2}, \cdots, X^{p+1}\})^2_{NPB}} \quad (2.26)$$

and such that in the classical $\hbar = 0$ limit, the Moyal defomed Nambu Poisson brackets (MNPB) divided by $(i\hbar)^{(p+1)/2}$ lead to the Nambu Poisson Brackets (NPB) . In order to show this, one requires to decompose the MNPB into sums of products of Moyal brackets, when $p+1 = d = 2n = even$, as follows [8]

$$\{ X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_{p+1}} \}_{MNPB} \ = \ \{ X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_{p+1}} \}_* =$$

$$\{ X_{\mu_1}, X_{\mu_2} \}_* \ * \ \{ X_{\mu_3}, X_{\mu_4} \}_* \ * \ .....* \ \{ X_{\mu_p}, X_{\mu_{p+1}} \}_* \ \pm \ .... \qquad (2.27)$$

where the ellipsis denotes signed permutations; i.e. the star-product deformations of the Nambu-Poisson-Brackets can be decomposed as a suitable anti-symmetrized sum of the star products of the Moyal brackets among *pairs* of variables. For instance

$$\{A, B, C, D\}_* \ = \ \{A, B\}_* * \{C, D\}_* + \{C, D\}_* * \{A, B\}_* + \{C, A\}_* * \{B, D\}_* +$$

$$\{B, D\}_* \ * \ \{C, A\}_* \ + \ \{D, A\}_* \ * \ \{C, B\}_* \ + \ \{C, B\}_* \ * \ \{D, A\}_* \qquad (2.28)$$

Each term in (2.28) splits into 4 terms giving a total of $4 \times 6 = 24 = 4!$ terms out of which 12 have a positive sign and 12 have a negative sign.

When $p+1 = odd$, attempts have been made to introduce quantum deformations based on the Zariski star product deformations of the Nambu Poisson Brackets (NPB), but unfortunately these deformed brackets failed to obey all the required algebraic properties of a (quantum) bracket [8]. Therefore, to our knowledge, only when $p+1 = 2n$ is even one can perform a suitable star product deformations of the Nambu-Poisson Brackets (NPB).

The Moyal deformations of the generalized brane actions in flat target C-spaces given by eq-(2.7) can be obtained by replacing ordinary products in eq-(2.7) for star products in C-space. This procedure is much simpler than trying to construct the C-space extension of eq-(2.26). However, one can *no* longer use the star product involving the phase space variables (2.12) but a different one. The correct noncommutative and *associative* star product [9],[10],[11] corresponding to a Lie-algebraic-like structure of the noncommutative polyvector-valued coordinates σ^A of the 2^d-dim world manifold, and associated with the motion of a generalized brane in target flat C-space backgrounds described by the functions $X^M(\sigma^A)$, is given by

$$(X^{M_1} * X^{M_2})(\sigma^A) \ = \ exp \left(\ \frac{i}{2} \ \sigma^A \Lambda_A[\ i \ \partial_{\sigma'^A}, \ i \ \partial_{\sigma''^A} \ ] \ \right) \ X^{M_1}(\sigma'^A) \ X^{M_2}(\sigma''^A)|_{\sigma'^A = \sigma''^A = \sigma^A}. \qquad (2.29)$$

where the expression for the bilinear differential polynomial $\Lambda_A[i\partial_{\sigma'^A}, i\partial_{\sigma''^A}]$ appearing in the kernel of the exponential (2.29), and derived from the Baker-Campbell-Hausdorff formula, has the following form

$$\Lambda_A[k, p] \ = \ i \ k_B \ p_C \ f_A^{BC} \ + \frac{i^2}{6} \ k_{B_1} \ p_{C_1} \ (p_{B_2} - k_{B_2}) \ f_D^{B_1 C_1} \ f_A^{D B_2} \ +$$

$$\frac{i^3}{24}\ (p_{B_2}\ k_{C_2}\ +\ k_{B_2}\ p_{C_2})\ k_{B_1}\ k_{C_1}\ f_{D_1}^{B_1C_1}\ f_{D_2}^{D_1B_2}\ f_A^{D_2C_2}\ +\ .... \qquad (2.30)$$

The above kernel is given in terms of the structure constants $[\sigma^B, \sigma^C] = f_A^{BC}\sigma^A$ of the polyvector coordinates algebra displayed below, after setting $k_B = i\ \partial_{\sigma'^B}$, $p_C = i\ \partial_{\sigma''^C}$.

The commutators $[\sigma^B, \sigma^C] = f_A^{BC}\sigma^A$ are defined in the same manner as the noncommutative polyvector coordinates algebra in section **1** as follows

$$[\ \sigma^{a_1},\ \sigma^{a_2}\ ]\ =\ \sigma^{a_1}\ \sigma^{a_2}\ -\ \sigma^{a_2}\ \sigma^{a_1}\ =\ 2\ \sigma^{a_1a_2}. \qquad (2.31a)$$

$$[\ \sigma^{a_1a_2},\ \sigma^b\ ]\ =\ \sigma^{a_1a_2}\ \sigma^b\ -\ \sigma^b\ \sigma^{a_1a_2}\ = \\ 4\ \left(\ \eta^{a_2b}\ \sigma^{a_1}\ -\ \eta^{a_1b}\ \sigma^{a_2}\ \right). \qquad (2.31b)$$

$$[\ \sigma^{a_1a_2a_3},\ \sigma^b\ ]\ =\ \sigma^{a_1a_2a_3}\ \sigma^b\ -\ \sigma^b\ \sigma^{a_1a_2a_3}\ =\ 2\ \sigma^{a_1a_2a_3b}. \qquad (2.31c)$$

$$[\ \sigma^{a_1a_2a_3a_4},\ \sigma^b\ ]\ =\ \sigma^{a_1a_2a_3a_4}\ \sigma^b\ -\ \sigma^b\ \sigma^{a_1a_2a_3a_4}\ =\ -\ 8\ \eta^{a_1b}\ \sigma^{a_2a_3a_4}\ \pm ...... \qquad (2.31d),\ \cdots$$

The metric η^{AB} appearing in the above polyvector-coordinate algebra (2.31) is a flat world manifold metric which is required in order for the algebra to obey the Jacobi identities. Therefore, one must *not* confuse the flat η^{AB} metric appearing in the algebra (2.31) with the auxiliary world manifold metric H^{AB} appearing in the action (2.7).

Because the star product (2.29,2.30) is very *elaborate*, the star product deformation of the action (2.7) is far more *complicated* than the mere expression of the action (2.9) involving directly the noncommutative matrix coordinates $\mathbf{X}^M$ in C-space. This was one of main purposes of this section : to construct generalized brane actions in *noncommutative* matrix coordinates backgrounds in C-space rather than perform a deformation quantization procedure. A key instrumental role was played by the auxiliary Clifford-valued field, $\Phi(\sigma^A) = \Phi^N(\sigma^A)\Gamma_N$, which provided the functional form of the noncommutative matrix coordinates in C-space given by : $\mathbf{X}^M = \Phi^{-1}(\sigma^A)\Gamma^M\Phi(\sigma^A)$, and which by construction, *satisfy* the n-ary algebra (1.23). The Clifford-valued field Φ might have a connection to dark matter but it is too early at this stage to speculate.

The n-ary algebra found in section **1** is an example of L_∞-structures in noncommutative field theories which have recently captured a lot of interest. Such noncommutative field theories are based on homotopy algebras (n-ary algebras). A recent review of L_∞-structures in noncommutative gravity can be found in [14].

3 Coherent States and Strings in Clifford space

In this last section we briefly discuss the extension of coherent states in C-spaces and provide a preliminary study of strings in target C-space backgrounds. Guided by the definition of a coherent state associated with a quantum harmonic oscillator as a displacement of the ground state (vacuum)

$$|z\rangle = D(z)|0\rangle > \; = \; e^{z a^\dagger \; - \; \bar{z}^\mu a} \; |0\rangle \; = \; e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \; |n\rangle \tag{3.1}$$

the generalized coherent states in Clifford (phase) space are defined as

$$|Z, Z^\mu, Z^{\mu\nu}, \cdots, Z^{\mu_1\mu_2\cdots,\mu_n}\rangle \; = \; e^{Z a^\dagger + Z^\mu a^\dagger_\mu + Z^{\mu\nu} a^\dagger_{\mu\nu} + \; \cdots \; - \; \bar{Z} a - \bar{Z}^\mu a_\mu - \bar{Z}^{\mu\nu} a_{\mu\nu} \cdots} \; |0, 0, \cdots, 0\rangle \tag{3.2}$$

One can perform the power series expansion after recurring to the Baker-Campbell-Hausdorff formula in order to generate the C-space version of the infinite sum in (3.1). This is attained via the use of the generalized bosonic creation and annihilation operators (bosonic oscillators) in C-space which obey the following non-zero commutation relations

$$[a, a^\dagger] \; = \; 1, \quad [a_\mu, \; a^\dagger_\nu] \; = \; \eta_{\mu\nu}, \quad [a_{\mu_1\mu_2}, \; a^\dagger_{\nu_1\nu_2}] \; = \; \eta_{\mu_1\mu_2|\nu_1\nu_2}$$
$$[a_{\mu_1\mu_2\cdots\mu_n}, \; a^\dagger_{\nu_1\nu_2\cdots\nu_n}] \; = \; \eta_{\mu_1\mu_2\cdots\mu_n|\nu_1\nu_2\cdots\nu_n} \tag{3.3}$$

while the other commutators are *zero*.

The action of the creation operators on the vacuum is

$$|n_\mu\rangle = \frac{(a^\dagger_\mu)^{n_\mu}}{\sqrt{n_\mu!}} \; |0\rangle, \quad \textit{no sum over } \mu \tag{3.4}$$

$$|n_{\mu\nu}\rangle = \frac{(a^\dagger_{\mu\nu})^{n_{\mu\nu}}}{\sqrt{n_{\mu\nu}!}} \; |0\rangle, \quad \textit{no sum over } \mu, \nu \tag{3.5}$$

$$|n_{\mu\nu\rho}\rangle = \frac{(a^\dagger_{\mu\nu\rho})^{n_{\mu\nu\rho}}}{\sqrt{n_{\mu\nu\rho}!}} \; |0\rangle, \quad \textit{no sum over } \mu, \nu, \rho \tag{3.6}$$

etc $\cdots$. When one performs the power series sum over all the mode numbers $n, n_\mu, n_{\mu\nu}, \cdots$ in (3.2) one recovers the generalized coherent state in C-space indicated by the left hand side of (3.2).

Let us shift the focus now from coherent states to the study of strings in C-spaces. Adopting the units $\hbar = c = G = 1$, an open string with wordsheet (dimensionless) coordinates σ, τ, moving in a flat C-space target background $X^A = X^A(\sigma, \tau)$ admits the solutions to the equations of motion $(\partial^2_\sigma - \partial^2_\tau) X^A = 0$

given by the following open string mode expansion[3]

$$X^A \;=\; X_0^A \;+\; (l_s)^{2|A|} P^A \tau \;+ i l_s^{|A|} \sum_{n \neq 0} \frac{1}{n}\; \alpha_n^A \; e^{-in\tau} \; cos(n\sigma) \tag{3.7}$$

where $|A|$ is the grade of the polyvector coordinate X^A. X_0^A is the center of mass position and P^A the total string momentum describing the center of mass motion of the string. l_s is the string length, and the string tension is $T \sim l_s^{-2}$. The closed string mode expansion is split into left and right movers modes α_n^A and $\tilde{\alpha}_n^A$ as follows

$$X_R^A \;=\; \frac{1}{2} X_0^A \;+ \frac{1}{2} (l_s)^{2|A|} \; P^A (\tau - \sigma) \;+ i l_s^{|A|} \sum_{n \neq 0} \frac{1}{n} \; \alpha_n^A \; e^{2in(\tau-\sigma)} \tag{3.8a}$$

$$X_L^A \;=\; X_0^A \;+ \frac{1}{2} (l_s)^{2|A|} P^A (\tau + \sigma) \;+ i l_s^{|A|} \sum_{n \neq 0} \frac{1}{n} \; \tilde{\alpha}_n^A \; e^{-2in(\tau+\sigma)} \tag{3.8b}$$

In a canonical quantization procedure the mode coefficients become the creation and annihilation operators associated with the oscillator modes, and one can perform the following rescaling

$$a_n^A \;\equiv\; \frac{1}{\sqrt{n}} \; \alpha_n^A, \quad a_n^{A\dagger} \;=\; a_{-n}^A, \quad n > 0 \tag{3.9}$$

leading to the following non-vanishing commutators

$$[a_m, a_n^\dagger] \;=\; \delta_{m,n}, \quad [a_m^\mu, \; a_n^{\nu\dagger}] \;=\; \delta_{m,n} \; \eta^{\mu\nu}, \quad m, n > 0 \tag{3.10}$$

$$[a_m^{\mu_1\mu_2}, \; a_n^{\nu_1\nu_2\dagger}] \;=\; \delta_{m,n} \; \eta^{\mu_1\mu_2|\nu_1\nu_2}, \quad m, n > 0 \tag{3.11}$$

$$[a_m^{\mu_1\mu_2\cdots\mu_n}, \; a_n^{\nu_1\nu_2\cdots\nu_n\dagger}] \;=\; \delta_{m,n} \; \eta^{\mu_1\mu_2\cdots\mu_n|\nu_1\nu_2\cdots\nu_n}, \quad m, n > 0 \tag{3.12}$$

Similar results follow for the closed string modes where the right moving and left moving oscillators commute.

In the ordinary string moving in flat target Minkowski backgrounds, states with an even number of temporal creation operators acting on the ground state

$$|\phi\rangle \;=\; a_{m_1}^{\mu_1\dagger} a_{m_2}^{\mu_2\dagger} \; \cdots \; a_{m_n}^{\mu_n\dagger} |0, k\rangle, \quad \hat{P}^\mu |\phi\rangle \;=\; k^\mu |\phi\rangle \tag{3.13}$$

have a positive norm, while those which can be constructed with an odd number of temporal creation operators have negative norm (ghosts) [2]. For example, the state $|\phi\rangle = a_m^{0\dagger}|0\rangle$ has $\langle\phi|\phi\rangle = -1$. The common lore is that negative-norm states lead to violations of causality and unitary. The bosonic string theory is

[3]Since we are dealing with a *flat* C-space background we may use now the index A instead of M for X^A

free of negative-norm states, in $D = 26$ [2], and when the Regge intercept (due to normal orderings) is $a = 1$.

In C-space the situation is far more complex. Firstly, the effective 2^D dimensions of the C-space corresponding to a Clifford algebra in a D-dim Minkowski spacetime is a space of *split* signature. For instance, in $D = 3 + 1$ spacetime, the $2^4 = 16$ dim C-space interval

$$(d\omega)^2 \ = \ (dx)^2 + dx_\mu dx^\mu + dx_{\mu_1\mu_2} dx^{\mu_1\mu_2} + dx_{\mu_1\mu_2\mu_3} dx^{\mu_1\mu_2\mu_3} + dx_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1\mu_2\mu_3\mu_4} \tag{3.14}$$

has a split signature $(8, 8)$ [12]. The terms containing the temporal variable $x^0, x^{0\mu}, x^{0\mu_1\mu_2}, x^{0\mu_1\mu_2\mu_3}$ appear with a minus sign, and there are $1+3+3+1 = 8$ of them in (3.14). Therefore, having a split signature is more problematic since there will be a proliferation of negative-norm and null sates. But according to an alternative quantization [15] a split-signature leads to negative energies which is consistent with the correspondence principle, while the former one (negative norms) is incorrect. In particular, recent findings by [16] and others [17] reveal that negative energies do not lead to instabilities in *bounded* regions.

Nevertheless, the formulation of the no-ghost theorem of a bosonic string living in target flat C-space backgrounds is more complicated than in Minkowski backgrounds. Among other problems is that $SO(8, 8)$ is not the Lorentz group in a $15 + 1$-dim Minkowski spacetime. A particle moving in a spacetime of split signature $(8, 8)$ does not have transverse degrees of freedom to the light-cone directions since the number of light-like directions is 16.

Therefore, more work remains in order to study the spectrum of strings moving in flat C-space backgrounds. In particular, one will have states like

$$|\Omega\rangle \ = \ a^{\mu_1\nu_1 \dagger}_{m_1} a^{\mu_2\nu_2 \dagger}_{m_2} \ \cdots \ a^{\mu_n\nu_n \dagger}_{m_n} |\mathbf{0}, \mathbf{k}\rangle, \ \ \hat{P}^{\mu\nu} |\Omega\rangle \ = \ k^{\mu\nu} |\Omega\rangle, \ \ etc, \cdots \tag{3.14}$$

which are not conventional antisymmetric tensor fields. The use of Young tableaux will be essential to describe the symmetry/antisymmetry structure of the indices of the fields.

To sum up : our prior construction of n-ary algebras, based on the relation among the $\mathbf{n}$-ary commutators of noncommuting spacetime coordinates $[X^1, X^2, ..., X^n]$ with the polyvector valued coordinates $X^{123...n}$ in noncommutative Clifford spaces, $[X^1, X^2, ..., X^n] = n! \ X^{123...n}$, allowed to construct generalized brane actions in noncommutative matrix coordinates backgrounds in Clifford-spaces (C-spaces) given by eq-(2.9). An instrumental role is placed by the *auxiliary* Clifford-valued field $\Phi(\sigma^A) = \Phi^N(\sigma^A)\Gamma_N$, living on the world manifold of the generalized brane, which allows to construct a matrix realization of the n-ary algebra given by $\mathbf{X}^M \equiv \Phi^{-1}(\sigma^A)\Gamma^M\Phi(\sigma^A)$. There was no need to introduce quantum groups nor Clifford-Hopf algebras. We then proceeded with a discussion and an analysis on the deformation quantization of branes in C-spaces. We finalized by describing an extension of coherent states in C-spaces and provided a preliminary study of strings in target C-space backgrounds. It remains to analyze in further detail the spectrum, the analog of the no-ghost theorem, and the critical dimension.

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