Rotating Frame Paradox in Quantum Mechanics

Karl De Paepe*

Abstract

We consider a one particle quantum rotating system. We expect the probability densities at a point to be the same for an inertial and rotating frames of reference. We show this is not the case. We also show given the wave function in the inertial frame of reference and angular velocity we can't obtain the wave function in the rotating frame of reference.

1 Introduction

Consider a frame of reference $\mathcal{F}'$ with coordinates $\mathbf{r}', t'$ rotating with constant angular velocity ω about the z axis of an inertial frame of reference $\mathcal{F}$ with coordinates $\mathbf{r}, t$. The coordinates being related by

$$x' = x\cos\omega t + y\sin\omega t \qquad y' = -x\sin\omega t + y\cos\omega t \qquad z' = z \qquad t' = t \tag{1}$$

Define $\rho', \rho, \varphi', \varphi$ by

$$\rho' = \sqrt{x'^2 + y'^2} \qquad \rho = \sqrt{x^2 + y^2} \qquad \tan\varphi' = \frac{y'}{x'} \qquad \tan\varphi = \frac{y}{x} \tag{2}$$

With respect to $\mathcal{F}$ let there be a quantum system of a particle with mass m in a potential $V(\rho)$. Let $\psi(\mathbf{r}, t)$ be the wave function with respect to $\mathcal{F}$ and $\psi'(\mathbf{r}', t')$ the wave function with respect to $\mathcal{F}'$. We expect the probability densities at a point in the two frames to be equal hence

$$|\psi'(\mathbf{r}', t')|^2 = |\psi(\mathbf{r}, t)|^2 \tag{3}$$

By (3) there is then a real valued function $\beta(\mathbf{r}, t)$ such that

$$\psi'(\mathbf{r}', t') = e^{-\frac{i}{\hbar}\beta(\mathbf{r},t)}\psi(\mathbf{r}, t) \tag{4}$$

2 Schrödinger equations

With respect to $\mathcal{F}$ the wave function $\psi(\mathbf{r}, t)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\rho)\psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) \tag{5}$$

The Lagrangian with respect to $\mathcal{F}'$ is

$$L' = \frac{1}{2}m\mathbf{v}'^2 + m\mathbf{v}' \cdot \boldsymbol{\omega} \times \mathbf{r}' + \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r}')^2 - V'(\rho') \tag{6}$$

*k.depaepe@alumni.utoronto.ca

From this construct the Hamiltonian. The wave function $\psi'(\mathbf{r}', t')$ then satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla'^2\psi'(\mathbf{r}', t') + V'(\rho')\psi'(\mathbf{r}', t') - \frac{1}{2}m\omega^2\rho'^2\psi'(\mathbf{r}', t') = i\hbar\frac{\partial\psi'}{\partial t'}(\mathbf{r}', t') \tag{7}$$

Choose $V(\rho)$ so that

$$V'(\rho') = V(\rho) \qquad \nabla'^2 = \nabla^2 \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\varphi} \tag{8}$$

On substituting (4) in (7) and using (1), (2), (5), (8) we have

$$\left[\frac{i\hbar}{2m}\nabla^2\beta + \frac{1}{2m}(\nabla\beta)^2 - \omega\frac{\partial\beta}{\partial\varphi} - \frac{1}{2}m\omega^2\rho^2 - \frac{\partial\beta}{\partial t}\right]\psi - i\hbar\omega\frac{\partial\psi}{\partial\varphi} + \frac{i\hbar}{m}\nabla\beta\cdot\nabla\psi = 0 \tag{9}$$

Adding and subtracting (9), multiplied by ψ^*, and its complex conjugate gives the two equations

$$\begin{aligned} 2\left[\frac{1}{2m}(\nabla\beta)^2 - \omega\frac{\partial\beta}{\partial\varphi} - \frac{1}{2}m\omega^2\rho^2 - \frac{\partial\beta}{\partial t}\right]\psi\psi^* &+ \frac{i\hbar}{m}\nabla\beta\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) \\ &- i\hbar\omega\left[\psi^*\frac{\partial\psi}{\partial\varphi} - \psi\frac{\partial\psi^*}{\partial\varphi}\right] = 0 \end{aligned} \tag{10}$$

$$\nabla\cdot(\psi\psi^*\nabla\beta) = m\omega\frac{\partial(\psi\psi^*)}{\partial\varphi} \tag{11}$$

3 No solution for β

Choose a $\psi(\mathbf{r}, t)$ that satisfies (5) and does not depend on φ. By (11) we then have since $\frac{\partial\psi}{\partial\varphi} = 0$ that

$$\nabla\cdot(\psi\psi^*\nabla\beta) = 0 \tag{12}$$

It is shown in the appendix that $\nabla\beta = 0$. There is then a function $F(t)$ such that $\beta(\mathbf{r}, t) = F(t)$. By (10) we have

$$-\frac{1}{2}m\omega^2\rho^2 - \frac{dF}{dt} = 0 \tag{13}$$

which does not hold. This ψ has then no solution for β.

4 Conclusion

No solution implies that (3) does not hold. There are then points where the probability densities at a point are not the same for the stationary and rotating frames of reference. Also since we do not have a solution for β we do not have a relation of wave functions of form (4). Consequently given the wave function $\psi(\mathbf{r}, t)$ with respect to $\mathcal{F}$ and given ω how then can we determine the wave function $\psi'(\mathbf{r}', t')$ with respect to $\mathcal{F}'$?

References

[1] Physics Essays, September 2008

5 Appendix

Let B be a ball of radius R centred at the origin. Let $f(\mathbf{r})$ be a real valued smooth function such that $f(\mathbf{r}) > 0$ for interior points of B and $f(\mathbf{r}) = 0$ for all other points. Choose a $\psi(\mathbf{r}, 0)$ so that $\psi(\mathbf{r}, 0)\psi^*(\mathbf{r}, 0) = f(\mathbf{r})$. From $\psi(\mathbf{r}, 0)$ we can determine $\psi(\mathbf{r}, t)$. Assuming there is a solution we have by (10) and (11) there is a $\beta(\mathbf{r}, t)$. Using (12) we have

$$\nabla \cdot \Big(f(\mathbf{r}) \nabla F(\mathbf{r}) \Big) = 0 \tag{14}$$

where $F(\mathbf{r}) = \beta(\mathbf{r}, 0)$. Assume there is an interior point $\mathbf{r}_1$ of B such that $\nabla F(\mathbf{r}_1) \neq 0$. There is then a curve with tangent vector $\nabla F(\mathbf{r})$ and containing $\mathbf{r}_1$. Following the curve in the direction of $\nabla F(\mathbf{r}, t)$ from $\mathbf{r}_1$ we will reach a point $\mathbf{r}_2$ where $F(\mathbf{r}_2)$ is a maximum or $\mathbf{r}_2$ is a point on the boundary of B. In either case $f(\mathbf{r}_2)\nabla F(\mathbf{r}_2) = 0$. Using $f(\mathbf{r}_1)\nabla F(\mathbf{r}_1) \neq 0$ and (14) we have $f(\mathbf{r}_2)\nabla F(\mathbf{r}_2) \neq 0$. This is a contradiction hence $\nabla F(\mathbf{r}_1) = 0$. On all of B we then have $\nabla F(\mathbf{r}) = 0$.

Let $g(\mathbf{r})$ be a smooth real valued function such that $g(\mathbf{r}) \geq 0$ for points of B and is zero outside B. There is a sequence $f_n(\mathbf{r})$ be a real valued smooth functions such that $f_n(\mathbf{r}) > 0$ for $\mathbf{r}$ an interior point of B and $f_n(\mathbf{r}) = 0$ at other points and $f_n(\mathbf{r})$ approaches $g(\mathbf{r})$. Let $F_n(\mathbf{r})$ be the solution of (14) with $f(\mathbf{r})$ replaced by $f_n(\mathbf{r})$. Require also of the sequence $F_n(\mathbf{r})$ that there is a function $G(\mathbf{r})$ that is the limit of the $F_n(\mathbf{r})$. We then have since $\nabla F_n(\mathbf{r}) = 0$ that $\nabla G(\mathbf{r}) = 0$ in the interior of B. We can use balls of larger and larger R to conclude $\nabla G(\mathbf{r}) = 0$ for all points of $\mathbb{R}^3$. Instead of beginning at $t = 0$ we could of began at any t. We can conclude $\nabla \beta(\mathbf{r}, t) = 0$.