A Rotating Frame Paradox in Quantum Mechanics

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Abstract

We consider a one particle quantum rotating system. We expect the probability densities at a point to be the same for a stationary and rotating frames of reference. We show this is not the case.

1 Introduction

Consider a frame of reference $\mathcal{F}'$ with coordinates $\mathbf{r}', t'$ rotating with constant angular velocity ω about the z axis of a frame of reference $\mathcal{F}$ with coordinates $\mathbf{r}, t$. The coordinates are related by

$$\rho' = \rho \qquad \varphi' = \varphi - \omega t \qquad z' = z \qquad t' = t \tag{1}$$

With respect to $\mathcal{F}$ let there be a quantum system of a particle with mass m in a potential $V(\mathbf{r})$. For the wave function $\psi(\mathbf{r}, t)$ with respect to $\mathcal{F}$ let $\psi'(\mathbf{r}', t')$ be the corresponding wave function with respect to $\mathcal{F}'$. We expect the probability densities in the two frames are equal hence [1]

$$|\psi'(\mathbf{r}', t')|^2 = |\psi(\mathbf{r}, t)|^2 \tag{2}$$

Consequently there is a real valued function $\beta(\mathbf{r}, t)$ such that

$$\psi'(\mathbf{r}', t') = e^{-\frac{i}{\hbar}\beta(\mathbf{r},t)}\psi(\mathbf{r}, t) \tag{3}$$

2 Schrödinger Equations

With respect to $\mathcal{F}$ the wave function $\psi(\mathbf{r}, t)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r})\psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r}', t') \tag{4}$$

The Lagrangian with respect to $\mathcal{F}'$ is

$$L' = \frac{1}{2}m\mathbf{v}'^2 + m\mathbf{v}' \cdot \boldsymbol{\omega} \times \mathbf{r}' + \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r}')^2 - V'(\mathbf{r}') \tag{5}$$

Construct the Hamiltonian from L'. The wave function $\psi'(\mathbf{r}', t')$ then satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla'^2\psi'(\mathbf{r}', t') - \frac{1}{2}m\omega^2\rho'^2\psi'(\mathbf{r}', t') + V'(\mathbf{r}')\psi'(\mathbf{r}', t') = i\hbar\frac{\partial\psi'}{\partial t'}(\mathbf{r}', t') \tag{6}$$

Now

$$V'(\mathbf{r}') = V(\mathbf{r}) \qquad \nabla' = \nabla \qquad \frac{\partial}{\partial\varphi'} = \frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\varphi} \tag{7}$$

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On substituting (3) in (6) and using (1), (4), and (7) we have

$$\left[\frac{i\hbar}{2m}\nabla^2\beta + \frac{1}{2m}(\nabla\beta)^2 - \omega\frac{\partial\beta}{\partial\varphi} - \frac{1}{2}m\omega^2\rho^2 - \frac{\partial\beta}{\partial t}\right]\psi - i\hbar\omega\frac{\partial\psi}{\partial\varphi} + \frac{i\hbar}{m}\nabla\beta\cdot\nabla\psi = 0 \tag{8}$$

Adding and subtracting (8), multiplied by ψ^*, and its complex conjugate gives the two equations

$$\begin{aligned} 2\left[\frac{1}{2m}(\nabla\beta)^2 - \omega\frac{\partial\beta}{\partial\varphi} - \frac{1}{2}m\omega^2\rho^2 - \frac{\partial\beta}{\partial t}\right]\psi\psi^* &+ \frac{i\hbar}{m}\nabla\beta\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) \\ &- i\hbar\omega\left(\psi^*\frac{\partial\psi}{\partial\varphi} - \psi\frac{\partial\psi^*}{\partial\varphi}\right) = 0 \qquad (9) \\ \nabla\cdot(\psi\psi^*\nabla\beta) &= m\omega\frac{\partial(\psi\psi^*)}{\partial\varphi} \qquad (10) \end{aligned}$$

3 No Solution to Equations

Choose V and ψ so that ψ has form $\psi(\rho, z, t)$ and at z and t if $\psi(\rho, z, t)$ is zero it is zero at a decrete set of ρ. Assume there is a point $p_0 = (\rho_0, z_0, t_0)$ such that $\nabla\beta(p_0) \neq 0$. We can also choose p_0 so that also $\psi(p_0) \neq 0$. We then have $\psi(p_0)\psi^*(p_0)\nabla\beta(p_0) \neq 0$. There is a curve with tangent vector $\nabla\beta$ and containing p_0. Since the system is symmetric about the z axis following this curve from p_0 along the direction of $\nabla\beta$ or in the opposite direction we will reach a point p_1 such that $\nabla\beta(p_1) = 0$.

From (10) and $\partial\psi/\partial\varphi = 0$ we have

$$\nabla\cdot(\psi\psi^*\nabla\beta) = 0 \tag{11}$$

hence

$$\frac{\partial}{\partial s}\left[\psi(s)\psi^*(s)\frac{\partial\beta}{\partial s}(s)\right] = 0 \tag{12}$$

where s is the coordinate along $\nabla\beta$. This implies

$$\psi(p_0)\psi^*(p_0)\nabla\beta(p_0) = \psi(p_1)\psi(p_1)\nabla\beta(p_1) = 0 \tag{13}$$

This is a contradiction hence $\nabla\beta = 0$. There is then a function $f(t)$ such that $\beta(\mathbf{r}, t) = f(t)$. By (9) and form of ψ

$$-\frac{1}{2}m\omega^2\rho^2 - \dot{f} = 0 \tag{14}$$

which does not hold. This ψ has then no solution for β.

4 Conclusion

No solution implies that (2) does not hold. Consequently measuring position of the mass can give the mass is at a point in the stationary frame but is not at that point in the rotating frame of reference.

References

[1] Physics Essays, September 2008