Credit Valuation Adjustment with Wrong Way Risk

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ABSTRACT

This paper presents a new model for calculating credit valuation adjustment and wrong way risk. Empirically, we find evidence that wrong way risk has a material effect on credit valuation adjustment. The nature and direction of effect depend on payoff, correlation, credit quality and risk mitigation. The magnitude of the impact is relatively greater in credit and equity markets. Moreover, the empirical results indicate that diversification can reduce the impact of wrong or right way risk on the risky value of a portfolio.

Key Words: credit valuation adjustment, wrong way risk, right way risk

Derivative valuation historically didn’t take counterparty risk into account. But contract parties, in reality, may have a chance of default. As a consequence, the International Accounting Standard (IAS) 39 requires banks to provide a fair-value adjustment due to counterparty risk. Now credit valuation adjustment (CVA) has become the first line of defense and the central part of counterparty risk management.

CVA not only allows institutions to move beyond the traditional control mindset of credit risk limits and to quantify counterparty risk as a single measurable P&L number, but also offers an opportunity for banks to dynamically manage, price and hedge counterparty risk.

CVA, by definition, is the difference between the risk-free portfolio value and the true (or risky or defaultable) portfolio value that takes into account the possibility of a counterparty’s default.

In general, risky valuation can be classified into two categories: the *default time approach* (DTA) and the *default probability approach* (DPA). The DTA involves the default time explicitly. Most CVA models in the literature (Brigo and Capponi (2008), Lipton and Sepp (2009), Pykhtin and Zhu (2006) and Gregory (2009), etc.) are based on this approach.

Since CVA is used for financial accounting and pricing, its accuracy is essential. Moreover, this current model is based on a well-known assumption, in which credit exposure and counterparty’s credit quality are independent. Obviously, it cannot capture wrong/right way risk properly.

In this paper, we present a framework for risky valuation and CVA. In contrast to previous studies, the model relies on the DPA rather than the DTA. Our study shows that the pricing process of a defaultable contract normally has a backward recursive nature if its payoff could be positive or negative.

An intuitive way of understanding these backward recursive behaviors is that we can think of that any contingent claim embeds two default options. In other words, when entering an OTC derivatives transaction, one party grants the other party an option to default and, at the same time, also receives an option to default itself. In theory, default may occur at any time. Therefore, the default options are American style options that normally require a backward induction valuation.

Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty. For example, in wrong way risk exposure tends to increase when counterparty credit quality worsens, while in right way risk exposure tends to decrease when counterparty credit quality declines. Wrong/right way risk, as an additional source of risk, is rightly of concern to banks and regulators. Since this new model allows us to incorporate correlated and potentially simultaneous defaults into risky valuation, it can naturally capture wrong/right way risk.

1. One-Way CVA

We consider a filtered probability space (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space; $\mathcal{F}$ denotes a σ -algebra; $\mathcal{P}$ denotes a probability measure; $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration.

The stopping (or default) time τ of a firm is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined as,

$$\tau = \inf\left\{t : \int_0^t h(s, \Phi_s)ds \geq \Delta\right\} \tag{1}$$

where $h(t)$ or $h(t, \Phi_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Γ_t, and Δ is a unit exponential random variable independent of Φ_t .

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u)du\right) \tag{2a}$$

The default probability for the period (t, s) in this framework is defined by

$$q(t,s) := P(\tau \le s \mid \tau > t, Z) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u)du\right) \quad (2b)$$

Two counterparties are denoted as A and B. Let valuation date be t. Consider a financial contract that promises to pay a $X_T > 0$ from party B to party A at maturity date T, and nothing before date T. All calculations in the paper are from the perspective of party A. The risk-free value of the financial contract is given by

$$V^F(t) = E\left[D(t,T)X_T \mid \mathcal{F}_t\right] \quad (3a)$$

where

$$D(t,T) = \exp\left[-\int_t^T r(u)du\right] \quad (3b)$$

where $E\{\bullet \mid \mathcal{F}_t\}$ denotes the expectation conditional on the $\mathcal{F}_t$, $D(t,T)$ denotes the risk-free discount factor at time t for the maturity T and $r(u)$ denotes the risk-free short rate at time u ($t \le u \le T$).

Next, we turn to risky valuation. In a unilateral credit risk case, we assume that party A is default-free and party B is defaultable. Risky valuation can be generally classified into two categories: the *default time approach* (DTA) and the *default probability (intensity) approach* (DPA).

Under a risk-neutral measure, the value of this defaultable contract is the discounted expectation of all the payoffs and is given by

$$V(t) = E\left[\left(D(t,T)\,X_T 1_{\tau > T} + D(t,\tau)\varphi V(\tau) 1_{\tau \le T}\right) \mid \mathcal{F}_t\right] \quad (4)$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise.

Although the DTA is very intuitive, it has the disadvantage that it explicitly involves the default time/jump. We are very unlikely to have complete information about a firm's default point, which is often inaccessible. Usually, valuation under the DTA is performed via Monte Carlo simulation.

The DPA relies on the probability distribution of the default time rather than the default time itself. The survival and the default probabilities for the period (t , $t+\Delta t$) are given by

$$\hat{p}(t) := p(t, t+\Delta t) = \exp\left(-h(t)\Delta t\right) \approx 1 - h(t)\Delta t \tag{5a}$$

$$\hat{q}(t) := q(t, t+\Delta t) = 1 - \exp\left(-h(t)\Delta t\right) \approx h(t)\Delta t \tag{5b}$$

The binomial default rule considers only two possible states: default or survival. For the one-period $(t, t+\Delta t)$ economy, at time $t+\Delta t$ the asset either defaults with the default probability $q(t, t+\Delta t)$ or survives with the survival probability $p(t, t+\Delta t)$. The survival payoff is equal to the market value $V(t+\Delta t)$ and the default payoff is a fraction of the market value: $\varphi(t+\Delta t)V(t+\Delta t)$. Under a risk-neutral measure, the value of the asset at *t* is the expectation of all the payoffs discounted at the risk-free rate and is given by

$$V(t) = E\left\{\exp\left(-r(t)\Delta t\right)\left[\hat{p}(t) + \varphi(t)\hat{q}(t)\right]V(t+\Delta t)\middle|\mathcal{F}_t\right\} \approx E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \tag{6}$$

where $y(t) = r(t) + h(t)\left(1-\varphi(t)\right) = r(t) + c(t)$ denotes the risky rate and $c(t) = h(t)\left(1-\varphi(t)\right)$ is called the (short) credit spread.

Similarly, we have

$$V(t+\Delta t) = E\left\{\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right\} \tag{7}$$

Note that $\exp\left(-y(t)\Delta t\right)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t+\Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left\{\exp\left(-y(t)\Delta t\right)V(t+\Delta t)\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-y(t)\Delta t\right)E\left[\exp\left(-y(t+\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right]\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\sum_{i=0}^{1} y(t+i\Delta t)\Delta t\right)V(t+2\Delta t)\middle|\mathcal{F}_t\right\} \end{aligned} \tag{8}$$

By recursively deriving from t forward over T and taking the limit as Δt approaches zero, the risky value of the asset can be expressed as

$$V(t) = E\left\{\exp\left[-\int_t^T y(u)du\right]V(T)\middle|\mathcal{F}_t\right\} \tag{9}$$

In theory, a default may happen at any time, i.e., a risky contract is continuously defaultable. This Continuous Time Risky Valuation Model is accurate but sometimes complex and expensive. For simplicity, people sometimes prefer the Discrete Time Risky Valuation Model that assumes that a default may only happen at some discrete times. A natural selection is to assume that a default may occur only on the payment dates.

For a derivative contract, usually its payoff may be either an asset or a liability to each party. Thus, we further relax the assumption and suppose that X_T may be positive or negative.

In the case of $X_T > 0$, the survival value is equal to the payoff X_T and the default payoff is a fraction of the payoff φX_T. Whereas in the case of $X_T \le 0$, the contract value is the payoff itself, because the default risk of party B is irrelevant for unilateral risky valuation in this case. Therefore, we have

Proposition 1: *The unilateral risky value of the single-payment contract in a discrete-time setting is given by*

$$V(t) = E\left[F(t,T)X_T \middle| \mathcal{F}_t\right] \tag{10a}$$

where

$$F(t,T) = D(t,T)\left[1 - 1_{X_T \ge 0}\, q(t,T)\left(1 - \varphi(T)\right)\right] \tag{10b}$$

Here $F(t,T)$ can be regarded as a risk-adjusted discount factor. Proposition 1 says that the unilateral risky valuation of the single payoff contract has a dependence on the sign of the payoff. If the payoff is positive, the risky value is equal to the risk-free value minus the discounted potential loss. Otherwise, the risky value is equal to the risk-free value.

Proposition 2: *The unilateral risky value of the multiple-payment contract is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{11a}$$

where $t = T_0$ *and*

$$F(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - 1_{(X_{j+1} + V(T_{j+1})) \geq 0}\, q(T_j, T_{j+1})\left(1 - \varphi(T_{j+1})\right)\right] \tag{11b}$$

The risky valuation in Proposition 2 has a backward nature. The intermediate values are vital to determine the final price. For a discrete time interval, the current risky value has a dependence on the future risky value. Only on the final payment date T_m, the value of the contract and the maximum amount of information needed to determine the risk-adjusted discount factor are revealed. The coupled valuation behavior allows us to capture wrong/right way risk properly where counterparty credit quality and market prices may be correlated. This type of problem can be best solved by working backwards in time, with the later risky value feeding into the earlier ones, so that the process builds on itself in a recursive fashion, which is referred to as *backward induction*.

For an intuitive explanation, we can posit that a defaultable contract under the unilateral credit risk assumption has an embedded default option (see Sorensen and Bollier (1994)). In other words, one party entering a defaultable financial transaction actually grants the other party an option to default. If we assume that a default may occur at any time, the default option is an American style option. American options normally have backward recursive natures and require backward induction valuations.

The similarity between American style financial options and American style default options is that both require a backward recursive valuation procedure. The difference between them is in the optimal strategy. The American financial option seeks an optimal value by comparing the exercise value with the continuation value, whereas the American default option seeks an optimal discount factor based on the option value in time.

The unilateral CVA, by definition, can be expressed as

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[\left(D(t, T_i) - \prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{12}$$

Proposition 2 provides a general form for pricing a unilateral defaultable contract. Applying it to a particular situation in which we assume that all the payoffs are nonnegative, we derive the following corollary:

Corollary 1*: If all the payoffs are nonnegative, the risky value of the multiple-payments contract is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} \overline{F}(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{13a}$$

where $t = T_0$ *and*

$$\overline{F}(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - q(T_j, T_{j+1})\left(1 - \varphi(T_{j+1})\right)\right] \tag{13b}$$

The proof of this corollary is easily obtained according to Proposition 2 by setting $\left(X_{j+1} + V(T_{j+1})\right) \geq 0$, since the value of the contract at any time is also nonnegative.

The CVA in this case is given by

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[D(t,T_i)\left(1 - \prod_{j=0}^{i-1}\left(1 - q(T_j, T_{j+1})(1 - \varphi(T_{j+1}))\right)\right) X_i \middle| \mathcal{F}_t\right] \tag{14}$$

2. Two-Way CVA

Two counterparties are denoted as *A* and *B*. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for party *j* (*j*=*A*, *B*) follows a Bernoulli distribution, which takes value 1 with default probability q_j and value 0 with survival probability p_j, i.e., $P\{Y_j = 0\} = p_j$ and $P\{Y_j = 1\} = q_j$. The marginal default distributions can be determined by the reduced-form models. The joint distributions of a bivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \tag{15a}$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \quad (15b)$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \quad (15c)$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \quad (15d)$$

where $E(Y_j) = q_j$, $\sigma_j^2 = p_j q_j$, $\sigma_{AB} := E[(Y_A - q_A)(Y_B - q_B)] = \rho_{AB}\sigma_A\sigma_B = \rho_{AB}\sqrt{q_A p_A q_B p_B}$ where ρ_{AB} denotes the default correlation coefficient and σ_{AB} denotes the default covariance.

Table 1. Payoffs of a bilaterally defaultable contract

State		$Y_A = 0, Y_B = 0$	$Y_A = 1, Y_B = 0$	$Y_A = 0, Y_B = 1$	$Y_A = 1, Y_B = 1$
Comments		*A* & *B* survive	*A* defaults, *B* survives	*A* survives, *B* defaults	*A* & *B* default
Probability		p_{00}	p_{10}	p_{01}	p_{11}
Payoff	$X_T > 0$	X_T	$\bar{\varphi}_B X_T$	$\varphi_B X_T$	$\varphi_{AB} X_T$
	$X_T < 0$	X_T	$\varphi_A X_T$	$\bar{\varphi}_A X_T$	$\varphi_{AB} X_T$

Suppose that a financial contract that promises to pay a X_T from party *B* to party *A* at maturity date *T*, and nothing before date *T* where $T > t$. The payoff X_T may be positive or negative, i.e. the contract may be either an asset or a liability to each party. All calculations are from the perspective of party *A*.

At time *T*, there are a total of four ($2^2 = 4$) possible states shown in Table 1. The risky value of the contract is the discounted expectation of the payoffs and is given by the following proposition.

Proposition 3: *The bilateral risky value of the single-payment contract is given by*

$$V(t) = E\left[K(t,T)X_T \middle| \mathcal{F}_t\right] = E\left[D(t,T)\left(1_{X_T \geq 0} k_B(t,T) + 1_{X_T < 0} k_A(t,T)\right)X_T \middle| \mathcal{F}_t\right] \quad (16a)$$

where

$$k_B(t,T) = p_B(t,T)p_A(t,T) + \varphi_B(T)q_B(t,T)p_A(t,T) + \overline{\varphi}_B(T)p_B(t,T)q_A(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_B(T) - \overline{\varphi}_B(T) + \varphi_{AB}(T)\right) \quad \textit{(16b)}$$

$$k_A(t,T) = p_B(t,T)p_A(t,T) + \varphi_A(T)q_A(t,T)p_B(t,T) + \overline{\varphi}_A(T)p_A(t,T)q_B(t,T) \\ + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_A(T) - \overline{\varphi}_A(T) + \varphi_{AB}(T)\right) \quad \textit{(16c)}$$

We may think of $K(t,T)$ as the risk-adjusted discount factor. Proposition 3 tells us that the bilateral risky price of a single-payment contract can be expressed as the present value of the payoff discounted by a risk-adjusted discount factor that has a switching-type dependence on the sign of the payoff.

Using a similar derivation as in Proposition 2, we can easily extend Proposition 3 from one-period to multiple-periods. Suppose that a defaultable contract has *m* cash flows. Let the *m* cash flows be represented as X_i with payment dates T_i, where $i = 1,\ldots,m$. Each cash flow may be positive or negative. The bilateral risky value of the multiple-payment contract is given by

Proposition 4: *The bilateral risky value of the multiple-payment contract is given by*

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \quad \textit{(17a)}$$

where $t = T_0$ *and*

$$K(T_j, T_{j+1}) = D(T_j, T_{j+1})\left(1_{(X_{j+1} + V(T_{j+1})) \geq 0}\, k_B(T_j, T_{j+1}) + 1_{(X_{j+1} + V(T_{j+1})) < 0}\, k_A(T_j, T_{j+1})\right) \quad \textit{(17b)}$$

where $k_A(T_j, T_{j+1})$ *and* $k_B(T_j, T_{j+1})$ *are defined in Proposition 3.*

Proposition 4 says that the pricing process of a multiple-payment contract has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of the contract and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

There is a common misconception in the market. Many people believe that the cash flows of a defaultable financial contract can be priced independently and then be summed up to give the final risky price of the contract. We emphasize here that this conclusion is only true of the financial contracts whose payoffs are always positive. In the cases where the promised payoffs could be positive or negative, the valuation requires not only a backward recursive induction procedure, but also a strategic selection of different discount factors according to the market value in time. This coupled valuation process allows us to capture correlation between counterparties and market factors.

The bilateral CVA of the multiple-payment contract can be expressed as

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} \left\{ E\left[D(t,T_i) X_i \middle| \mathcal{F}_t\right] - E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \right\} \tag{18}$$

3. Numerical Results

In this section, we present some numerical results for CVA calculation based on the theory described above. First, we study the impact of margin agreements on CVA. The testing portfolio consists of a number of interest rate and equity derivatives. The number of simulation scenarios (or paths) is 20,000. The time buckets are set weekly

For risk-neutral simulation, we use a Hull-White model for interest rate and a CIR (Cox-Ingersoll-Ross) model for hazard rate scenario generations a modified GBM (Geometric Brownian Motion) model for equity and collateral evolution. The results are presented in the following tables. Table 2 illustrates that if party *A* has an infinite collateral threshold $H_A = \infty$ i.e., no collateral requirement on *A*, the CVA value increases while the threshold H_B increases. Table 3 shows that if party *B* has an infinite collateral threshold $H_B = \infty$, the CVA value actually decreases while the threshold H_A increases. This reflects the bilateral impact of the collaterals on the CVA. The impact is mixed in Table 4 when both parties have finite collateral thresholds.

Table 2. The impact of collateral threshold H_B on the CVA

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	19,550.91	20,528.65	21,368.44	22,059.30

Table 3. The impact of collateral threshold H_A on the CVA

Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	28,283.64	25,608.92	23,979.11	22,059.30

Table 4. The impact of the both collateral thresholds on the CVA

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	25,752.98	22,448.45	23,288.24	22,059.30

Next, we examine the impact of wrong way risk. Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty. Wrong/right way risk, as an additional source of risk, is rightly of concern to banks and regulators.

Some financial markets are closely interlinked, while others are not. For example, CDS price movements have a feedback effect on the equity market, as a trading strategy commonly employed by banks and other market participants consists of selling a CDS on a reference entity and hedging the resulting credit exposure by shorting the stock. On the other hand, Moody's Investor's Service (2000) presents statistics that suggest that the correlations between interest rates and CDS spreads are very small.

To capture wrong/right way risk, we need to determine the dependency between counterparties and to correlate the credit spreads or hazard rates with the other market risk factors, e.g. equities, commodities, etc., in the scenario generation.

We use an equity swap as an example. Assume the correlation between the underlying equity price and the credit quality (hazard rate) of party B is ρ . The impact of the correlation on the CVA is show in Table 5. The results say that the CVA increases when the absolute value of the negative correlation increases.

Table 5. The impact of wrong way risk on the CVA

Correlation ρ	0	-50%	-100%
CVA	165.15	205.95	236.99

4. Conclusion

This article presents a framework for pricing risky contracts and their CVAs. The model relies on the probability distribution of the default jump rather than the default jump itself, because the default jump is normally inaccessible. We find that the valuation of risky assets and their CVAs, in most situations, has a backward recursive nature and requires a backward induction valuation.

An intuitive explanation is that two counterparties implicitly sell each other an option to default when entering into an OTC derivative transaction. If we assume that a default may occur at any time, the default options are American style options. If we assume that a default may only happen on the payment dates, the default options are Bermudan style options. Both Bermudan and American options require backward induction valuations.

Based on our theory, we propose a novel cash-flow-based framework (see appendix) for calculating bilateral CVA at the counterparty portfolio level. This framework can easily incorporate various credit mitigation techniques, such as netting agreements and margin agreements, and can capture wrong/right way risk. Numerical results show that these credit mitigation techniques and wrong/right way risk have significant impacts on CVA.

Appendix

A. Proofs

Proof of Proposition 1: Under the unilateral credit risk assumption, we only consider the default risk when the asset is in the money. Assume that a default may only occur on the payment date. Therefore, the risky value of the asset at *t* is the discounted expectation of all possible payoffs and is given by

$$\begin{aligned} V(t) &= E\left\{D(t,T)\left[1_{X_T\geq 0}\left(p(t,T)+\varphi(T)q(t,T)\right)+1_{X_T<0}\right]X_T\middle|\mathcal{F}_t\right\} \\ &= E\left\{D(t,T)\left[1-1_{X_T\geq 0}\left(1-\varphi(T)\right)q(t,T)\right]X_T\middle|\mathcal{F}_t\right\}=E\left[F(t,T)X_T\middle|\mathcal{F}_t\right] \end{aligned} \quad \text{(A1a)}$$

where

$$F(t,T)=D(t,T)\left[1-1_{X_T\geq 0}\,q(t,T)\left(1-\varphi(T)\right)\right] \quad \text{(A1b)}$$

Proof of Proposition 2: Let $t=T_0$. On the first payment day, let $V(T_1)$ denote the risky value of the asset excluding the current cash flow X_1. According to Proposition 1, the risky value of the asset at *t* is given by

$$V(t)=E\left[F(T_0,T_1)\left(X_1+V(T_1)\right)\middle|\mathcal{F}_t\right] \quad \text{(A2a)}$$

where

$$F(T_0,T_1)=D(T_0,T_1)\left[1-1_{(V(T_1)+X_1)\geq 0}\,q(t,T)\left(1-\varphi(T)\right)\right] \quad \text{(A2b)}$$

Similarly, we have

$$V(T_1)=E\left[F(T_1,T_2)\left(X_2+V(T_2)\right)\middle|\mathcal{F}_{T_1}\right] \quad \text{(A3)}$$

Note that $F(T_0,T_1)$ is $\mathcal{F}_{T_1}$ -measurable. According to the *taking out what is known* and *towe*r properties of conditional expectation, we have

$$V(t)=E\left[F(T_0,T_1)\left(X_1+V(T_1)\right)|\mathcal{F}_t\right]=E\left[F(T_0,T_1)X_1|\mathcal{F}_t\right]$$
$$+E\left\{F(T_0,T_1)\left[E\left(F(T_1,T_2)X_2\middle|\mathcal{F}_{T_1}\right)+E\left(F(T_1,T_2)V(T_2)\middle|\mathcal{F}_{T_1}\right)\right]|\mathcal{F}_t\right\} \quad \text{(A4)}$$
$$=\sum_{i=1}^{2}E\left[\left(\prod_{j=0}^{i-1}F(T_j,T_{j+1})\right)X_i|\mathcal{F}_t\right]+E\left[\left(\prod_{j=0}^{1}F(T_j,T_{j+1})\right)V(T_2)|\mathcal{F}_t\right]$$

By recursively deriving from T_2 forward over T_m, where $V(T_m)=X_m$, we have

$$V(t)=\sum_{i=1}^{m}E\left[\left(\prod_{j=0}^{i-1}F(T_j,T_{j+1})\right)X_i|\mathcal{F}_t\right] \quad \text{(A5)}$$

Proof of Proposition 3: We assume that a default may only occur on the payment date. At time *T*, there are four possible states: 1) both *A* and *B* survive, 2) *A* defaults but *B* survives, 3) *A* survives but *B* defaults, and 4) both *A* and *B* default. The joint distributions of *A* and *B* are given by (15). Depending on whether the payoff is in the money or out of the money at *T*, we have

$$V(t)=E\left\{D(t,T)\left[1_{X_T\ge 0}\left\langle p_{00}(t,T)+\varphi_B(T)p_{01}(t,T)+\overline{\varphi}_B(T)p_{10}(t,T)+\varphi_{AB}(T)p_{11}(t,T)\right\rangle X_T|\mathcal{F}_t\right.\right.$$
$$\left.\left.+1_{X_T<0}\left\langle p_{00}(t,T)+\overline{\varphi}_A(T)p_{01}(t,T)+\varphi_A(T)p_{10}(t,T)+\varphi_{AB}(T)p_{11}(t,T)\right\rangle X_T|\mathcal{F}_t\right]\right\} \quad \text{(A6a)}$$
$$=E\left(K(t,T)X_T|\mathcal{F}_t\right)=E\left[D(t,T)\left(1_{X_T\ge 0}k_B(t,T)+1_{X_T<0}k_A(t,T)\right)X_T|\mathcal{F}_t\right]$$

where

$$k_B(t,T)=p_B(t,T)p_A(t,T)+\varphi_B(T)q_B(t,T)p_A(t,T)+\overline{\varphi}_B(T)p_B(t,T)q_A(t,T)$$
$$+\varphi_{AB}(T)q_B(t,T)q_A(t,T)+\sigma_{AB}(t,T)\left(1-\varphi_B(T)-\overline{\varphi}_B(T)+\varphi_{AB}(T)\right) \quad \text{(A6b)}$$

$$k_A(t,T)=p_B(t,T)p_A(t,T)+\varphi_A(T)q_A(t,T)p_B(t,T)+\overline{\varphi}_A(T)p_A(t,T)q_B(t,T)$$
$$+\varphi_{AB}(T)q_B(t,T)q_A(t,T)+\sigma_{AB}(t,T)\left(1-\varphi_A(T)-\overline{\varphi}_A(T)+\varphi_{AB}(T)\right) \quad \text{(A6c)}$$

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