Ampere's Circuital Law

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Abstract

Ampere's circuital law states that the tangential line integral of a magnetic field around a closed loop is equal to the product of current enclosed by the loop and magnetic permeability of the medium. In this paper a rigorous proof of the Ampere's circuital law has been presented.

Keywords : Tangential line integral, Closed loop.

1 DERIVATION

Let the origin of a cylindrical coordinate system be situated at the center of a circular loop L_1 of radius ρ. Consider a straight current carrying wire of infinite length which passes through the origin and perpendicular to the plane of the loop L_1 . It can be obtained from Biot-Savart law that

$$\mathbf{B} = \frac{\mu}{2\pi}\frac{I}{r}\hat{\boldsymbol{\theta}} = C\,\mathbf{F}$$

where

$$C = \frac{\mu I}{2\pi}$$

and

$$\mathbf{F} = \frac{1}{r}\hat{\boldsymbol{\theta}}$$

$$\Rightarrow \oint_{L_1} \mathbf{B.dl} = \int_0^{2\pi}\left(\frac{\mu}{2\pi}\frac{I}{\rho}\hat{\boldsymbol{\theta}}\right).\left(\rho\, d\theta\,\hat{\boldsymbol{\theta}}\right) = \mu I \qquad [r = \rho]$$

$$\Rightarrow \oint_{L_1} \mathbf{B.dl} = \mu I \qquad \text{(i)}$$

let

$$\mathbf{F} = f_r\hat{\mathbf{r}} + f_\theta\hat{\boldsymbol{\theta}} + f_z\hat{\mathbf{k}}$$

$$\Rightarrow f_r = f_z = 0 \quad and \quad f_\theta = \frac{1}{r}$$

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now

$\nabla \times \mathbf{F}$

$$= \begin{vmatrix} \frac{\hat{\mathbf{r}}}{r} & \hat{\boldsymbol{\theta}} & \frac{\hat{\mathbf{k}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f_r & r f_\theta & f_z \end{vmatrix} = \begin{vmatrix} \frac{\hat{\mathbf{r}}}{r} & \hat{\boldsymbol{\theta}} & \frac{\hat{\mathbf{k}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r \times \frac{1}{r} & 0 \end{vmatrix} = 0$$

$$\Rightarrow \nabla \times \mathbf{B} = C \nabla \times \mathbf{F} = 0$$

Part A :

now consider an arbitrary shaped closed loop L_2 that lies inside the circular loop L_1 and encloses the origin

$$\int_A (\nabla \times \mathbf{B}).\mathbf{dA} = 0$$

where A is the area between the circular loop L_1 and the arbitrary closed loop L_2

$$\Rightarrow \oint_{L_1} \mathbf{B.dl} + \oint_{L_2} \mathbf{B.dl} = 0 \qquad \text{[using Stokes' theorem]}$$

$$\Rightarrow \mu I + \oint_{L_2} \mathbf{B.dl} = 0 \qquad \text{[using (i)]}$$

$$\Rightarrow \oint_{L_2} \mathbf{B.dl} = -\mu I$$

$$\Rightarrow \oint_{-L_2} \mathbf{B.dl} = \mu I = \mu I_{enc} \qquad [\, I_{enc} = I \,]$$

Part B :

now consider an arbitrary shaped closed loop L_3 that lies completely outside the current carrying wire

$$\oint_{L_3} \mathbf{B.dl} = \int_A (\nabla \times \mathbf{B}).\mathbf{dA} = 0 \qquad \text{[using Stokes' theorem]}$$

where A is the area inside the loop L_3 .

$$\Rightarrow \oint_{L_3} \mathbf{B.dl} = 0 = \mu I_{enc} \qquad [\, I_{enc} = 0 \,]$$

Combining the results of part A and part B, we obtain

$$\Rightarrow \oint_L \mathbf{B.dl} = \mu I_{enc}$$

2 CONCLUSION

Thus the tangential line integral of the magnetic field around an arbitrary closed loop is equal to the product of current enclosed by the loop and magnetic permeability of the medium. Since all infinite and straight current carrying structures can be assumed to be composed of infinite and straight current carrying wires therefore by superposition principle, Ampere's circuital law holds for all such infinite and straight current carrying structures.

References

1. Hugh D. Young, Roger A. Freedman, Albert Lewis Ford, *“Sears’ and Zemansky’s University Physics with Modern Physics 13th edition.”*