On critical line of nontrivial zeros of Riemann zeta function $\zeta(s)$

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Abstract

In this paper, we find a curious and simple possible solution to the critical line of nontrivial zeros in the strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ of Riemann zeta function $\zeta(s)$. We show that exists $s_\sigma \in \mathbb{C}$ where $\{s_\sigma = \sigma + it : (\sigma \in \mathbb{R}, 0 < \sigma < 1); \forall t \in \mathbb{R}\}$ with i as the imaginary unit, such that satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0 \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

1 Introduction.

There is a large and extensive bibliography on the Riemann zeta function and its zeros. Basically, Riemann zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by the absolutely convergent infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Leonhard Euler already considered this series for real values of s. He also proved that it equals the Euler product:

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

where the infinite product extends over all prime numbers p. However, we can also define the Riemann zeta function Eq.(1) as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad \Rightarrow \quad \zeta(s) = \frac{1}{2^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \right)$$

Which can also be expressed as:

$$\zeta(s) = \frac{1}{2^s} \left[\zeta(s) + B(s) \right] \iff B(s) = \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \tag{2}$$

Thus, by Eq.(2) we can definitely express the Riemann zeta function as:

$$\zeta(s) = (2^s - 1)^{-1} B(s) \tag{3}$$

As is well known, the Riemann zeta function $\zeta(s)$ and the Dirichlet eta function $\eta(s)$ satisfy the relation:

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s) \tag{4}$$

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Thus, by Eq.(3) we can now express the Dirichlet eta function as:

$$\eta(s) = \left(\frac{1-2^{1-s}}{2^s-1}\right)B(s) \tag{5}$$

2 Proof.

By Eq.(2), Eq.(4) and Eq.(5) we can obtain:

$$2^{1-s} = 1 - \frac{\eta(s)}{\zeta(s)} = 2^s \cdot \frac{\zeta(s)-\eta(s)}{\zeta(s)+B(s)} \quad \Rightarrow \quad 2^{1-2s} = \frac{\zeta(s) + \left(\frac{2^{1-s}-1}{2^s-1}\right)B(s)}{\zeta(s)+B(s)}$$

Which can also be expressed as:

$$2^{1-2s} = \left(\frac{2^{1-s}-1}{2^s-1}\right)A(s) \quad \Longleftrightarrow \quad A(s) = \frac{\left(\frac{2^s-1}{2^{1-s}-1}\right)\zeta(s)+B(s)}{\zeta(s)+B(s)} \tag{6}$$

However, exists $s_\sigma \in \mathbb{C}$ where $\{s_\sigma = \sigma + it : (\sigma \in \mathbb{R}, 0 < \sigma < 1); \forall t \in \mathbb{R}\}$ with i as the imaginary unit, such that satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0$$

Therefore, calculating $(\lim_{s\to s_\sigma})$ in Eq.(6), we have:

$$\lim_{s\to s_\sigma}\left[2^{1-2s}\cdot\left(\frac{2^s-1}{2^{1-s}-1}\right)\right] = \lim_{s\to s_\sigma} A(s) \quad \Rightarrow \quad 2^{1-2s_\sigma}\cdot\left(\frac{2^{s_\sigma}-1}{2^{1-s_\sigma}-1}\right) = A(s_\sigma)$$

However, by Eq.(3) we have that:

$$\zeta(s_\sigma) = 0 \iff B(s_\sigma) = 0$$

Then by Eq.(6) we obtain for $A(s_\sigma)$ an indeterminacy of the type $\frac{0}{0}$. Thus, by successive applications of the L'hôpital rule until any nth and mth derivatives for $\zeta(s)$ and $B(s)$ respectively, by which $A(s_\sigma)$ is not an indeterminacy, that is:

$$\left[\zeta^{(n)}(s_\sigma) \neq 0 \vee B^{(m)}(s_\sigma) \neq 0\right] \iff \left[\left(\forall j < n : \zeta^{(j)}(s_\sigma) = 0\right) \wedge \left(\forall k < m : B^{(k)}(s_\sigma) = 0\right)\right]$$

we obtain definitively:

$$2^{1-2s_\sigma} = \left(\frac{2^{1-s_\sigma}-1}{2^{s_\sigma}-1}\right)A(s_\sigma) \tag{7}$$

where $A(s_\sigma) \neq 0$ since we would obtain: $2^{1-2s_\sigma} = 0$ and would not be defined for s_σ.

However, since $s_\sigma = \sigma + it$ then obtaining common factor 2^{-it} in numerator and 2^{it} in denominator of the fraction, we can express:

$$2^{1-2s_\sigma} = 2^{-2it}\cdot\frac{2^{1-\sigma}-2^{it}}{2^\sigma - 2^{-it}}A(s_\sigma)$$

Now, defining $s_0 \in \mathbb{C}$ such that $s_0 = \frac{1}{2} + it$, we can express previous equation as:

$$2^{2(s_\sigma - s_0)} = \frac{2^\sigma - 2^{-it}}{2^{1-\sigma}-2^{it}}\cdot\frac{1}{A(s_\sigma)} \tag{8}$$

By definition $s_\sigma = \sigma + it$ and $s_0 = \frac{1}{2} + it$ then: $2(s_\sigma - s_0) = 2\sigma - 1$. Thus, developing in trigonometric form $2^{it} = e^{itln2}$ and $2^{-it} = e^{-itln2}$ and since $cos(-x) = cos(x)$ as we know, we obtain:

$$A(s_\sigma)\cdot 2^{(2\sigma-1)} = \frac{2^\sigma - cos(tln2) + isen(tln2)}{2^{1-\sigma} - cos(tln2) - isen(tln2)} \tag{9}$$

and thus:

$$A(s_\sigma) \cdot 2^{(2\sigma-1)} [cos(tln2) + isen(tln2)] = [A(s_\sigma) - 1] \, 2^\sigma + [cos(tln2) - isen(tln2)]$$

As we know: $\{|x| = x : \forall x \geq 0; \forall x \in \mathbb{R}\}$ and $\{|z+w| \leq |z| + |w| : \forall (z,w) \in \mathbb{C}\}$, then by application of modulus and denoting $A(s_\sigma) = A$ by simplicity, we obtain:

$$|A| \cdot 2^{(2\sigma-1)} \leq |A-1| \cdot 2^\sigma + 1$$

Now, denoting $x = 2^{(\sigma - \frac{1}{2})}$ by simplicity and since $(2^{\frac{1}{2}} < 2)$ as we know, we can also express the previous equation as:

$$|A| \cdot 2^{2(\sigma-\frac{1}{2})} \leq |A-1| \cdot 2^{\frac{1}{2}} \cdot 2^{(\sigma-\frac{1}{2})} + 1 \quad \Rightarrow \quad |A|x^2 < 2|A-1|x + 1 \tag{10}$$

However, by Eq.(9) we have: $\{A \in \mathbb{C}\}$, which can be expressed in binomial form as:

$$\begin{aligned} A \;=\; & \frac{[2^\sigma - cos(tln2)] \cdot [2^\sigma - 2^{(2\sigma-1)}cos(tln2)] - 2^{(2\sigma-1)}sen^2(tln2)}{[2^\sigma - 2^{(2\sigma-1)}cos(tln2)]^2 + [2^{(2\sigma-1)}sen(tln2)]^2} + \\ & i \cdot \frac{[2^\sigma - cos(tln2)] \cdot 2^{(2\sigma-1)}sen(tln2) + [2^\sigma - 2^{(2\sigma-1)}cos(tln2)]sen(tln2)}{[2^\sigma - 2^{(2\sigma-1)}cos(tln2)]^2 + [2^{(2\sigma-1)}sen(tln2)]^2} \end{aligned}$$

Thus, we verify that: $\{|A| > 0 : (\sigma \in \mathbb{R}, 0 < \sigma < 1); \forall t \in \mathbb{R}\}$. By simplicity, denoting $\{A = b + id\}$ in previous equation, we know that:

$$\left.\begin{aligned} |A| &= \sqrt{b^2 + d^2} \\ |A-1| &= \sqrt{(b-1)^2 + d^2} = \sqrt{1 - 2b + |A|^2} \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} |A-1| \leq |A| \quad \iff \quad b \geq \frac{1}{2} \\ |A-1| \geq |A| \quad \iff \quad b \leq \frac{1}{2} \end{cases} \tag{11}$$

Where obviously $(1 - 2b + |A|^2 \geq 0)$. Thus, by Eq.(11) the two possible options in Eq.(10) are:

2.1 Case: $|A-1| \leq |A|$.

Then Eq.(10) can be expressed now as:

$$|A|x^2 < 2|A-1|x + 1 \quad \Rightarrow \quad |A|x^2 \leq 2|A|x + 1 \quad \Rightarrow \quad x^2 \leq 2x + \frac{1}{|A|}$$

with $|A| > 0$ and then for any $\{(r, \beta) \in \mathbb{R} : r > 0\}$ we obtain:

$$x^2 + r = 2x + \frac{1}{|A|} \qquad \Rightarrow \qquad x^2 - 2x + \beta = 0 \iff \beta = r - \frac{1}{|A|} \tag{12}$$

2.2 Case: $|A-1| \geq |A|$.

Then, Eq.(10) can be expressed now for any $\{k \in \mathbb{R} : k > 0\}$ as:

$$|A|x^2 < 2|A-1|x + 1 \quad \Rightarrow \quad |A|x^2 + k = 2|A-1|x + 1 \quad \Rightarrow \quad |A|x^2 + k \geq 2|A|x + 1$$

and then again for any $\{(\alpha, \beta) \in \mathbb{R} : \alpha > 0\}$ where $|A| > 0$ we obtain:

$$|A|x^2 + k = 2|A|x + 1 + \alpha \qquad \Rightarrow \qquad x^2 - 2x + \beta = 0 \iff \beta = \frac{k - 1 - \alpha}{|A|} \tag{13}$$

2.3 Final solution.

As we can see we have found the same equation in both options, but obviously β have different parameters in Eq.(12) and Eq.(13). Thus solving for $x = 2^{(\sigma-\frac{1}{2})}$ for any option, we obtain:

$$x^2 - 2x + \beta = 0 \qquad \Rightarrow \qquad 2^{(\sigma-\frac{1}{2})} = 1 \pm \sqrt{1-\beta} \tag{14}$$

However, obviously $(\beta \leq 1)$ since $\{2^{(\sigma-\frac{1}{2})} \in \mathbb{R} : (0 < \sigma < 1)\}$, thus:

$$\beta = 2x - x^2 \leq 1 \quad \Rightarrow \quad x^2 - 2x + 1 \geq 0$$

and solving for $x = 2^{(\sigma-\frac{1}{2})}$:

$$2^{(\sigma-\frac{1}{2})} \geq 1 \quad \Rightarrow \quad \sigma \geq \frac{1}{2} \tag{15}$$

Now, according to Eq.(14) and since $(\beta \leq 1)$, the three options with their two solutions $(\pm)$ are:

2.3.1 $\beta = 0$

1. *Positive solution*

$$2^{(\sigma-\frac{1}{2})} = 1 + \sqrt{1-\beta} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} = 2 \quad \Rightarrow \quad \sigma = \frac{3}{2}$$

   which is outside the strip for nontrivial zeros: $(0 < \sigma < 1)$.

2. *Negative solution*

$$2^{(\sigma-\frac{1}{2})} = 1 - \sqrt{1-\beta} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} = 0$$

   Obviously is not defined.

2.3.2 $\beta < 0$

Then $\{\exists\lambda \in \mathbb{R} : \lambda > 0; \beta = -\lambda\}$. Therefore:

1. *Positive solution*

$$2^{(\sigma-\frac{1}{2})} = 1 + \sqrt{1+\lambda} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} > 2 \quad \Rightarrow \quad \sigma > \frac{3}{2}$$

   which is outside the strip for nontrivial zeros: $(0 < \sigma < 1)$.

2. *Negative solution*

$$2^{(\sigma-\frac{1}{2})} = 1 - \sqrt{1+\lambda} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} = -\alpha \quad \Rightarrow \quad \sigma \in \mathbb{C}$$

   Where $\{\alpha \in \mathbb{R} : \alpha > 0\}$ and therefore is not correct, since $\{\sigma \in \mathbb{R}\}$.

2.3.3 $\beta \in (0, 1]$

1. *Negative solution*

$$2^{(\sigma-\frac{1}{2})} = 1 - \sqrt{1-\beta} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} \leq 1 \quad \Rightarrow \quad \sigma \leq \frac{1}{2} \tag{16}$$

2. *Positive solution*

$$2^{(\sigma-\frac{1}{2})} = 1 + \sqrt{1-\beta} \quad \Rightarrow \quad 2^{(\sigma-\frac{1}{2})} < 2 \quad \Rightarrow \quad \sigma < \frac{3}{2}$$

   which is less restrictive that Eq.(16) and furthermore $(\sigma < 1)$ as we know for nontrivial zeros.

Definitively, according to Eq.(15) and Eq.(16) we have then that if:

$$(\sigma \leq \frac{1}{2}) \wedge (\sigma \geq \frac{1}{2}) \quad \Rightarrow \quad \sigma = \frac{1}{2}$$

by which we can also verify by Eq.(9) that:

$$\sigma = \frac{1}{2} \quad \Rightarrow \quad |A| = |A(s_\sigma)| = 1$$

Therefore we obtain definitely:

$$s_\sigma = \sigma + it \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

Thus, all the nontrivial zeros lie on the critical line $\{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\}$ consisting of the set complex numbers $\frac{1}{2} + it$, thus confirming Riemann's hypothesis.