# Amazing formulas related to Pi

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abstract

*In this note we give some formulas related to Pi:*

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415926535 \ldots$$


keywords: number Pi , integrals , series .

## Formulas

Entry 1.

$$2\sqrt{3}\,\ln\left(\frac{1+\sqrt{3}}{2}\right) - \frac{\pi}{6} = \int_0^1 \sin^{-1}\left(\frac{4(1-x^2)}{3+3x^2+\sqrt{1+34x^2+x^4}}\right) dx \tag{1}$$

Entry 2.

$$\pi = 3\sqrt{3}\int_0^1 \frac{1+x^6}{(1+x^2+x^4)^2}\,dx \tag{2}$$

$$\frac{\pi}{3\sqrt{3}} = \int_{1/2}^1 \frac{1+x^6}{(1+x^2+x^4)^2}\,dx + \sum_{n=0}^{\infty}(n+1)\,2^{-6n-1}\left(\frac{1}{6n+1} - \frac{1/2}{6n+3} + \frac{1/16}{6n+5} + \frac{1/64}{6n+7} - \frac{1/128}{6n+9} + \frac{1/1024}{6n+11}\right) \tag{3}$$

$$\begin{aligned}\frac{\pi}{3\sqrt{3}} = &\sum_{n=0}^{\infty}(n+1)\,2^{-6n-1}\left(\frac{1}{6n+1} - \frac{1/2}{6n+3} + \frac{1/16}{6n+5} + \frac{1/64}{6n+7} - \frac{1/128}{6n+9} + \frac{1/1024}{6n+11}\right) + \\ &\sum_{n=0}^{\infty}(n+1)\,2^{-n-2}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(-1)^k\left(\frac{1-2^{-2k-2m-1}}{2k+2m+1} + \frac{1-2^{-2k-2m-7}}{2k+2m+7}\right)\end{aligned} \tag{4}$$

Entry 3.

$$\pi\sqrt{2\left(\sqrt{2}-1\right)} = 2\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{1+n^2}\right) + 4\int_0^1\left(\sum_{n=0}^{\infty}\frac{x(x+n)}{1+\left(1+(x+n)^2\right)^2}\right)dx \tag{5}$$

Entry 4.

$$\frac{\pi^2}{72} = \int_{1/3}^{1/2}\frac{1}{\sqrt{1-x^2}}\cos^{-1}\left(\frac{1-x}{2x}\right)dx \tag{6}$$

$$\frac{\pi^2}{6} = \int_0^1 \frac{1}{\sqrt{x(1-x)}}\tan^{-1}\left(\frac{17-12x}{4\sqrt{21}}\right)dx = \int_0^1 \frac{1}{\sqrt{x(1-x)}}\tan^{-1}\left(\frac{5+12x}{4\sqrt{21}}\right)dx \tag{7}$$

Entry 5.

$$\pi\left(\frac{1}{8}+\frac{\sqrt{2\left(\sqrt{2}-1\right)}}{2}\right)=$$

$$\sqrt{2\left(\sqrt{2}-1\right)}\tan^{-1}\left(\frac{7}{2\sqrt{17+13\sqrt{2}}}\right)-\frac{1}{2}\tan^{-1}\left(\frac{1649}{5743}\right)+\sqrt{\frac{1+\sqrt{2}}{2}}\ln\left(\frac{5+8\sqrt{2}-4\sqrt{5\sqrt{2}-1}}{13}\right)+ \tag{8}$$

$$2\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{1+n^2}\right)+4\sum_{n=2}^{\infty}\sum_{k=0}^{\infty}c(k)\left(\frac{n^{-2k-1}-(n+1)^{-2k-1}}{2k+1}-n\frac{n^{-2k-2}-(n+1)^{-2k-2}}{2k+2}\right)$$

$$c(k)=\left(\frac{1+i}{2}\right)\left((-1+i)^k-i(-1-i)^k\right),\ k=0,1,2,\ \ldots;\ i=\sqrt{-1} \tag{9}$$

Entry 6.

$$\frac{3\sqrt{3}}{2\pi\sqrt[3]{4}}\left(\Gamma\left(\frac{2}{3}\right)\right)^3=\int_0^{\infty}\left(1-\left(\sqrt[3]{\sqrt{\frac{1}{4}+\frac{1}{27x^6}}+\frac{1}{2}}-\sqrt[3]{\sqrt{\frac{1}{4}+\frac{1}{27x^6}}-\frac{1}{2}}\right)^{3/2}\right)dx \tag{10}$$

$$\frac{27}{8\pi}\left(\Gamma\left(\frac{2}{3}\right)\right)^3=\int_0^{\infty}\left(1-\left((\cosh x)^{2/3}-(\sinh x)^{2/3}\right)^{3/2}\right)\frac{\cosh(2x)}{(\sinh(2x))^{4/3}}dx \tag{11}$$

Remark: Γ(x) is the Gamma function.

Entry 7.

$$\frac{\pi}{\sqrt{15}}\sqrt[3]{\frac{1}{2\sqrt{5}}+\frac{3}{4}\sqrt[3]{\frac{1}{2\sqrt{5}}+\ldots}}=$$

$$\sum_{n=0}^{\infty}(-1)^n 2^{-2n-2}\left(\frac{(2n)!}{(2/3)_{2n+1}}F\left(\frac{2}{3},2n+1,2n+\frac{5}{3},-1\right)+\frac{(2n)!}{(1/3)_{2n+1}}F\left(\frac{1}{3},2n+1,2n+\frac{4}{3},-1\right)\right) \tag{12}$$

Remark: $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 8.

$$2\sqrt{3}\ln\left(\frac{1+\sqrt{3}}{2}\right)-\frac{\pi}{6}+\frac{4}{3}-\frac{7}{6\sqrt{3}}=$$

$$\int_{1/2\sqrt{3}}^{1}\left(\sqrt[3]{\sqrt{\frac{1}{27x^6}+\frac{74}{27x^4}-\frac{1}{9x^2}}+\frac{5}{3x^2}-\frac{1}{27}}-\sqrt[3]{\sqrt{\frac{1}{27x^6}+\frac{74}{27x^4}-\frac{1}{9x^2}}-\frac{5}{3x^2}+\frac{1}{27}}\right)dx \tag{13}$$

Entry 9.

$$\pi=\frac{33}{2\sqrt{21}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{121}{336}\right)^n\sum_{k=0}^{n}\binom{2n+1}{2k}\binom{2k}{k}\left(\frac{3}{11}\right)^{2k} \tag{14}$$

$$\pi=\frac{645}{44\sqrt{21}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{215^2}{88^2\cdot 21}\right)^n\sum_{k=0}^{2n+1}\binom{2n+1}{k}\binom{2k}{k}\left(\frac{9}{215}\right)^k \tag{15}$$

Entry 10.

$$\frac{\pi}{\sqrt{3}} - 2 + \frac{3}{2}\ln 2 + \ln 3 = \int_0^1 \ln\left(\sqrt{x^2 + \sqrt{48x^2 + x^4}} + \sqrt{24 + x^2 + \sqrt{48x^2 + x^4}}\right) dx \tag{16}$$

Entry 11. For $a > \cosh^{-1}\left(2\sqrt[3]{\frac{1}{3\sqrt{3}}}\right)$, $u = \frac{1}{\sinh^2 a \cosh a}$, we have

$$\frac{a}{\sinh^2 a \cosh a} + \frac{1}{\sinh a} + \tan^{-1}(\sinh a) - \frac{\pi}{2} = \int_0^u \cosh^{-1}\left(\sqrt[3]{\frac{1}{2x} + \sqrt{\frac{1}{4x^2} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2x} - \sqrt{\frac{1}{4x^2} - \frac{1}{27}}}\right) dx \tag{17}$$

Entry 12. For $a > 0$, we have

$$2\tan^{-1}(e^a) - \frac{\pi}{2} = \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1 - e^{-(2k+1)a}}{2k+1}\right) \tag{18}$$

$$2\tan^{-1}(e^a) - \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(1 - e^{-a})^{n+1}}{n+1} F\left(-n,\, n+1,\, n+2,\, \frac{1 - e^{-a}}{2}\right) \tag{19}$$

$$2\tan^{-1}(e^a) - \frac{\pi}{2} = (1 - e^{-a}) \sum_{n=0}^{\infty} \frac{\left(1 - e^{-2a}\right)^n 2^{-n}}{n+1} F\left(-n,\, 1,\, n+2,\, -\frac{1 - e^{-a}}{1 + e^{-a}}\right) \tag{20}$$

$$2\tan^{-1}(e^a) - \frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1 - e^{-2a}}{2}\right)^{n+1} F\left(2n+2,\, 1,\, n+2,\, \frac{1 - e^{-a}}{2}\right) \tag{21}$$

Remark: $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 13.

$$\frac{\pi}{6} = \ln\left(\frac{1 + \sqrt{3}}{2}\right) + 2\sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n-2}}{n+2} F\left(1 + \frac{n}{2},\, 1 + \frac{n}{2},\, 2 + \frac{n}{2},\, \frac{1}{4}\right) \tag{22}$$

$$\frac{\pi}{6} = \ln\left(\frac{1 + \sqrt{3}}{2}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n-1}}{n+2} \sum_{k=0}^{[n/2]} \frac{1}{k!} \left(1 - k + \frac{n}{2}\right)_k \tag{23}$$

Remark: $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 14.

$$\frac{\pi}{8} = \int_0^1 \frac{\sin(\ln x)}{\left(1 + x^2 + 2x\cos(\ln x)\right)\ln x} dx \tag{24}$$

$$\frac{\pi}{8} = \int_0^\infty \frac{e^{-x}\sin x}{\left(1 + 2e^{-x}\cos x + e^{-2x}\right)x} dx \tag{25}$$

Entry 15.

$$\frac{1}{\sqrt{2}} - \frac{\sqrt{2}\ln 2}{\pi} = \int_0^1 \left(\prod_{n=1}^{\infty} \frac{(4n-2)^2 - (2-x)^2}{(4n-2)^2 - (1-x)^2}\right) dx \tag{26}$$

Entry 16.

$$\sqrt{\pi} = \int_0^1 \sqrt{\frac{1}{x}\tanh^{-1}\left(\frac{1-x}{1+x}\right)}\, dx \tag{27}$$

$$\sqrt{\pi} = 2\int_0^1 \frac{1}{1+x}\sqrt{\frac{\tanh^{-1} x}{1 - x^2}}\, dx \tag{28}$$

$$\sqrt{\pi} = 2\int_0^\infty \frac{\sqrt{\sinh^{-1} x}}{1 + x^2 + x\sqrt{1 + x^2}}\, dx \tag{29}$$

$$\sqrt{\pi} = 2\int_1^\infty \frac{\sqrt{\cosh^{-1} x}}{x^2 - 1 + x\sqrt{x^2 - 1}}\, dx \tag{30}$$

Entry 17. For $a = \frac{1}{6}\sqrt{21 - 4\sqrt{11}}$ , we have

$$\left(\frac{\pi}{3}\right)^4 = \left(\sum_{n=1}^\infty \frac{(-1)^{n-1}\sin((2n-1)a\pi)}{(2n-1)^2}\right)^2 - \left(\sum_{n=1}^\infty \frac{(-1)^n \cos(2na\pi)}{(2n)^2}\right)^2 \tag{31}$$

Entry 18.

$$\frac{1}{12\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 = \int_0^\infty \left(1 - \left(\frac{2x}{x + \sqrt{4 + x^2}}\right)^{1/4}\right) dx \tag{32}$$

$$\frac{5}{24\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 - 1 = \int_1^\infty \left(1 - \left(\frac{2\sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}\right)^{1/4}\right) dx \tag{33}$$

Remark: $\Gamma(x)$ is the Gamma function.

Entry 19.

$$\pi \tanh 1 = \int_{-\tanh 1}^{\tanh 1} \cos^{-1}\left(\tanh^{-1} x\right) dx \tag{34}$$

$$\pi \tanh 1 = \int_{-1}^{1} \frac{\cos^{-1} x}{\cosh^2 x}\, dx \tag{35}$$

$$\pi\left(\frac{1}{2} - \frac{1}{e^2 + 1}\right) = \int_{1/(1+e^2)}^{1/(1+e^{-2})} \cos^{-1}\left(\ln\sqrt{\frac{x}{1 - x}}\right) dx \tag{36}$$

$$\pi = \int_0^1 \left(\frac{e^{-x}}{\cosh x} + \frac{e^{\sqrt{1-x^2}}}{\cosh\sqrt{1 - x^2}}\right)\frac{1}{\sqrt{1 - x^2}}\, dx \tag{37}$$

$$\pi = \int_0^{\pi/2} \frac{e^{\cos x}}{\cosh(\cos x)}\, dx + \int_0^{\pi/2} \frac{e^{-\sin x}}{\cosh(\sin x)}\, dx \tag{38}$$

$$\int_{1/(1+e^2)}^{1/(1+e^{-2})} \sin^{-1}\left(\ln\sqrt{\frac{x}{1 - x}}\right) dx = 0 \tag{39}$$

$$-\int_{1/(1+e^2)}^{1/2} \sin^{-1}\left(\ln\sqrt{\frac{x}{1 - x}}\right) dx = \int_{1/2}^{1/(1+e^{-2})} \sin^{-1}\left(\ln\sqrt{\frac{x}{1 - x}}\right) dx \tag{40}$$

$$\int_{1/2}^{1/(1+e^{-2})} \sin^{-1}\left(\ln\sqrt{\frac{x}{1 - x}}\right) dx = \frac{\pi}{2\left(1 + e^{-2}\right)} - \int_0^{\pi/2} \frac{1}{1 + e^{-2\sin x}}\, dx \tag{41}$$

$$\int_0^{\pi/2} \tanh(\sin x)\, dx = \frac{\pi}{2}\tanh 1 - \int_0^{\tanh 1} \sin^{-1}\left(\tanh^{-1} x\right) dx \tag{42}$$

$$\int_0^{\pi/2} \tanh(\sin x)\, dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}\, 2^{4n-1}\left(2^{2n} - 1\right) B_n}{n}\left(\frac{n!}{(2n)!}\right)^2 \tag{43}$$

where $B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\}$ are the Bernoulli numbers.

$$\int_{1/(1+e^2)}^{1/2} \sin^{-1}\sqrt{\ln\left(e\sqrt{\frac{1-x}{x}}\right)\ln\left(e\sqrt{\frac{x}{1-x}}\right)}\,dx = \int_{1/2}^{1/(1+e^{-2})} \sin^{-1}\sqrt{\ln\left(e\sqrt{\frac{1-x}{x}}\right)\ln\left(e\sqrt{\frac{x}{1-x}}\right)}\,dx \tag{44}$$

$$\pi = \int_{1/(1+e^2)}^{1/(1+e^{-2})} \frac{1}{(1-x)\sqrt{1-\left(\ln\sqrt{\frac{x}{1-x}}\right)^2}}\,dx \tag{45}$$

$$\pi = \int_{1/(1+e^2)}^{1/(1+e^{-2})} \frac{1}{x\sqrt{1-\left(\ln\sqrt{\frac{1-x}{x}}\right)^2}}\,dx \tag{46}$$

$$\pi\left(\frac{1}{2}-\frac{1}{1+e^2}\right) = \int_1^{e/\cosh 1} \cos^{-1}\left(\ln\sqrt{\frac{x}{2-x}}\right)dx + \int_{e^{-1}/\cosh 1}^{1} \sin^{-1}\left(\ln\sqrt{\frac{2-x}{x}}\right)dx \tag{47}$$

Entry 20.

$$\pi\left(\sqrt{1+\sqrt{2}}-\sqrt{2}\right) = \sum_{n=0}^{\infty}\frac{2^{-n}}{2n+3}\sum_{k=0}^{[n/2]}\frac{(-1)^k(-1/2)_{n-2k}}{(n-2k)!} + \sum_{n=0}^{\infty}\frac{2^{-n-1}}{2n+3}\sum_{k=0}^{n}\frac{(-1/2)_{n-k}}{(n-k)!}\sum_{m=0}^{[k/2]}\binom{k-m}{m}(-2)^{-m} \tag{48}$$

Entry 21.

$$1+\frac{\ln 2}{3}-\frac{\pi\sqrt{3}}{9} = \int_0^1 \sqrt[3]{\frac{x}{1+\sqrt{1-x^2}}}\,dx \tag{49}$$

Entry 22.

$$\pi = \int_{e^{-1}}^{e} \frac{2x}{(1+x^2)\sqrt{1-\ln^2 x}}\,dx = \int_{e^{-1}}^{e} \frac{2x}{(1+x^2)\sqrt{\ln\left(\frac{e}{x}\right)\ln(e x)}}\,dx \tag{50}$$

Entry 23. For $u = \frac{2\sqrt[3]{2}}{9}$, we have

$$\frac{8\pi}{27} = \int_0^u \sqrt[4]{x}\left(\sec\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}\,x^{3/2}}{2}\right)\right)-\sec\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}\,x^{3/2}}{2}\right)\right)\right)dx \tag{51}$$

$$\frac{5}{6\sqrt{3}}+\frac{4\ln 2}{9\sqrt{3}}-\frac{4\pi}{27} = -\int_0^{1/4}\sqrt[4]{x}\,\sec\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}\,x^{3/2}}{2}\right)\right)dx \tag{52}$$

Entry 24.

$$\frac{\sqrt{\pi}}{8}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 = -\int_0^1 \frac{1}{\sqrt{1-x^2}}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{2}\right)}{\sqrt{3-\sqrt{1-x^2}}}+\frac{\ln\left(\frac{1+\sqrt{1-x^2}}{2}\right)}{\sqrt{3+\sqrt{1-x^2}}}\right)dx \tag{53}$$

Remark: Γ(x) is the Gamma function.

Entry 25. For $\phi = \frac{1+\sqrt{5}}{2}$, we have

$$\frac{\pi}{\sqrt{3}} = 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{\phi}{3\,\alpha}\right)^n\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{(-1)^k}{2\,n-2\,k+1}\left(\frac{3}{\phi^2\sqrt{5}}\right)^k \tag{54}$$

where

$$\alpha = \sqrt{\frac{1}{\sqrt{5}} + \phi\sqrt{\frac{1}{\sqrt{5}} + \phi\sqrt{\frac{1}{\sqrt{5}} + \ldots}}} = \phi + \cfrac{1}{\phi\sqrt{5} + \cfrac{1}{\phi + \cfrac{1}{\phi\sqrt{5} + \ldots}}} \tag{55}$$

Entry 26. For $\phi = \frac{1+\sqrt{5}}{2}$ , we have

$$\frac{\pi}{\sqrt{3}} = 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{\phi\sqrt{5}\,\beta}{3}\right)^n\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{(-1)^k}{2\,n-2\,k+1}\left(\frac{3}{\phi^2\sqrt{5}}\right)^k \tag{56}$$

where

$$\beta = -\cfrac{1/\sqrt{5}}{\phi + \cfrac{1/\sqrt{5}}{\phi + \cfrac{1/\sqrt{5}}{\phi + \ldots}}} \tag{57}$$

Entry 27.

$$-\frac{\sqrt{\pi}}{32}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 = \int_0^1 \frac{\ln x}{\sqrt{1+x^4+\sqrt{1+2\,x^4-3\,x^8}} - \sqrt{1+x^4-\sqrt{1+2\,x^4-3\,x^8}}}\,dx \tag{58}$$

Remark: $\Gamma(x)$ is the Gamma function.

Entry 28.

$$\pi = 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{2\,n}\sum_{k=0}^{\left[\frac{2\,n+1}{3}\right]}\binom{2\,n-2\,k}{n-k}\binom{2\,n-2\,k+1}{k}\frac{(-1)^k\,(16/27)^k}{2\,n-2\,k+1} \tag{59}$$

$$\pi = 3\sum_{n=0}^{\infty}\binom{2\,n}{n}\frac{(3/4)^{2\,n}}{2\,n+1}\,{}_3F_2\left(-\frac{2\,n+1}{3}, \frac{1-2\,n}{3}, -\frac{2\,n}{3};\ \frac{1}{2}-n, \frac{1}{2}-n;\ 1\right) \tag{60}$$

Remark: ${}_3F_2\,(a, b, c; d, e; x)$ is the hypergeometric function.

Entry 29.

$$\pi = \int_0^1 \frac{1}{\sqrt{1-x^2}}\left(\frac{\ln\left(\frac{(2-x)\left(1-\sqrt{1-x^2}\right)}{x^2}\right)}{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)} + \frac{\ln\left(\frac{(2-x)\left(1+\sqrt{1-x^2}\right)}{x^2}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx \tag{61}$$

Entry 30. For $a = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$ , $b = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$ , we have

$$\frac{341\,\pi^3}{2^{15}\,b} = \sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\left(1-\frac{a}{2}\right)^k\binom{n+k+1}{n-k}\sum_{m=0}^{n-k}\binom{n-k}{m}\frac{(-1)^m}{(k+m+1)^3} \tag{62}$$

$$\frac{341\,\pi^3}{2^{15}\,b} = 8\sum_{n=0}^{\infty}\frac{1}{(1+a)^{n+2}}\sum_{k=0}^{n}(-1)^k\left(\frac{b^2}{1+a}\right)^k\binom{n+k+1}{n-k}\sum_{m=0}^{n-k}\binom{n-k}{m}\frac{(-2)^m}{(m+1)^3} \tag{63}$$

$$\frac{341\,\pi^3}{2^{15}\,b} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\,F\left(\frac{n+1}{2}, -\frac{n-1}{2}, \frac{3}{2}, \frac{b^2}{4}\right) \tag{64}$$

$$\frac{341\,\pi^3}{2^{15}\,b} = \frac{4}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{2}{3}\right)^k \sum_{m=0}^{k}\binom{k}{m}\frac{a^{k-m}}{(k+m+1)^3} \tag{65}$$

Remark: $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 31.

$$\pi = \int_0^{\infty}\frac{x}{\left(1+x^2\right)^{7/8}+\left(1+x^2\right)^{9/8}}\,dx \tag{66}$$

$$\frac{\pi}{2} = \int_0^{\infty}\frac{x}{\left(1+x^2\right)^{3/4}+\left(1+x^2\right)^{5/4}}\,dx \tag{67}$$

Entry 32.

$$\pi\sqrt{2\left(\sqrt{6}+2\right)} = 4\sqrt{2\left(\sqrt{6}+2\right)}\,\tan^{-1}\left(\frac{\sqrt{2}-\sqrt{\sqrt{6}-2}}{\sqrt{2}+\sqrt{\sqrt{6}-2}}\right) + \frac{8}{\sqrt{3}}\sum_{n=0}^{\infty}\left(1-\sqrt{\frac{2}{3}}\right)^n \frac{2^{2n}}{2n+1}\binom{2n}{n}^{-1} \tag{68}$$

$$\pi = 4\sum_{n=0}^{\infty}\left(\frac{1}{2}-\frac{1}{2}\sqrt{\sqrt{\frac{3}{2}}-1}\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k} + 4\sqrt{\sqrt{\frac{2}{3}}\left(1-\sqrt{\frac{2}{3}}\right)}\sum_{n=0}^{\infty}\left(1-\sqrt{\frac{2}{3}}\right)^n\frac{2^{2n}}{2n+1}\binom{2n}{n}^{-1} \tag{69}$$

Entry 33. For $\lambda = \sinh 2$ , we have

$$\pi = \frac{4\lambda\sqrt{2}}{2+\lambda^2}\sum_{n=0}^{\infty}2^{-3n}\times\sum_{k=0}^{n}2^{3k}\binom{2n-2k}{n-k}^2\left(\frac{\lambda^2}{2+\lambda^2}\right)^k\sum_{m=0}^{k}\frac{(-1)^m\,2^{5m}}{(2m+1)^2}\binom{k}{m}\binom{2n-2k+4m+2}{n-k+2m+1}^{-1}\binom{n-k+2m+1}{n-k}^{-1} \tag{70}$$

Entry 34. For $u_{n+1} = \sqrt[3]{8\sqrt{3}+12\,u_n}$ , $u_0 = 0$ , we have

$$\frac{4}{9}\pi = \tan^{-1}\left(\sqrt{3}+u_1\right) + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{u_{n+1}-u_n}{1+\left(\sqrt{3}+u_n\right)\left(\sqrt{3}+u_{n+1}\right)}\right) \tag{71}$$

$$\frac{4}{9}\pi = \tan^{-1}\left(\sqrt{3}+\sqrt[3]{8\sqrt{3}}\right) + \tan^{-1}\left(\frac{\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}-\sqrt[3]{8\sqrt{3}}}{1+\left(\sqrt{3}+\sqrt[3]{8\sqrt{3}}\right)\left(\sqrt{3}+\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}\right)}\right) + \tan^{-1}\left(\frac{\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}}-\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}}{1+\left(\sqrt{3}+\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}\right)\left(\sqrt{3}+\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}+12\sqrt[3]{8\sqrt{3}}}}\right)}\right) + \ldots \tag{72}$$

Entry 35. For $u_{n+1} = \sqrt[3]{2368+336\,u_n}$ , $u_0 = 0$ , we have

$$\frac{4}{9}\pi = \tan^{-1}\left(\sqrt{11+u_1}\right) + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{11+u_{n+1}} - \sqrt{11+u_n}}{1+\sqrt{11+u_n}\,\sqrt{11+u_{n+1}}}\right) \tag{73}$$

$$\frac{4}{9}\pi = \tan^{-1}\left(\sqrt{11+\sqrt[3]{2368}}\right) + \tan^{-1}\left(\frac{\sqrt{11+\sqrt[3]{2368+336\sqrt[3]{2368}}} - \sqrt{11+\sqrt[3]{2368}}}{1+\sqrt{11+\sqrt[3]{2368}}\,\sqrt{11+\sqrt[3]{2368+336\sqrt[3]{2368}}}}\right) + \ldots \tag{74}$$

Entry 36.

$$\frac{\pi}{14} = \sin^{-1}\left(\frac{1}{2+4\sin\left(3\sin^{-1}\left(\frac{1}{2+4\sin\left(3\sin^{-1}\left(\frac{1}{2+\ldots}\right)\right)}\right)\right)}\right) \tag{75}$$

$$\frac{\pi}{9} = \sin^{-1}\left(\frac{\sqrt{3}}{4} - \frac{1}{4}\tan\left(\sin^{-1}\left(\frac{\sqrt{3}}{4} - \frac{1}{4}\tan\left(\sin^{-1}\left(\frac{\sqrt{3}}{4} - \ldots\right)\right)\right)\right)\right) \tag{76}$$

Entry 37.

$$\int_0^{\infty}\int_0^{\infty}\frac{x+y}{e^{2x+2y}-1}\,dx\,dy = \frac{90\,\zeta(3)}{\pi^4}\int_0^{\infty}\int_0^{\infty}\frac{x^2+y^2}{e^{2x+2y}-1}\,dx\,dy \tag{77}$$

$$\frac{90\,\zeta(3)}{\pi^4}\int_0^{\infty}\int_0^{\infty}\frac{x^2+y^2}{e^{2x+2y}-1}\,dx\,dy = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n-1}\,B_n}{(2n)!}\sum_{k=0}^{2n}\binom{2n}{k}\frac{1}{(2n-k+1)(k+1)} + \sum_{n=1}^{\infty}\frac{(2n+2)\,e^{-2n} - (2n+1)\,e^{-4n}}{4n^3} \tag{78}$$

Remark: $\zeta(x)$ is the Riemann zeta function, $B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \ldots\right\}$ are the bernoulli numbers.

Entry 38. For $w = \cos w$, $w = 0.739085\ldots$, we have

$$\sqrt{\pi} = \int_0^{\infty}\left(\sin\left(\frac{w^2x^2}{2} + \frac{1}{2x^2}\right) + \cos\left(\frac{w^2x^2}{2} + \frac{1}{2x^2}\right)\right)dx \tag{79}$$

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{wy^2}{2} + \frac{w}{8x^2}\right)\cos\left(\frac{wx^2}{2} + \frac{w}{8y^2}\right)dx\,dy \tag{80}$$

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{x^2y^2}\sin\left(\frac{wy^2}{8} + \frac{w}{2x^2}\right)\cos\left(\frac{wx^2}{8} + \frac{w}{2y^2}\right)dx\,dy \tag{81}$$

Entry 39.

$$\frac{1}{2} + \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}} = \int_0^1\left(\sqrt[3]{\sqrt{\frac{x^6}{27} + \frac{1}{x^2}} + \frac{1}{x}} - \sqrt[3]{\sqrt{\frac{x^6}{27} + \frac{1}{x^2}} - \frac{1}{x}}\right)dx \tag{82}$$

Rematk: $\Gamma(x)$ is the Gamma function.

Entry 40.

$$\pi = k\cos\left(\frac{k}{9}\cos\left(\frac{k}{9}\cos\left(\frac{k}{9}\ldots\right)\right)\right) \tag{83}$$

where

$$k = \frac{18}{7}\prod_{n=1}^{\infty}\left(1 - \left(\frac{7}{18n}\right)^2\right)^{-1} = 324\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(18n-9)^2-4} = \frac{18}{7} + 252\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(18n)^2-49} \tag{84}$$

Entry 41.

$$\pi = 8\tan^{-1}\left(\frac{\sqrt{2\left(17+\sqrt{17}\right)} - \sqrt{17} - 1}{4}\right) + 2\tan^{-1}\left(\frac{1}{4}\right) \tag{85}$$

$$\pi = 16\tan^{-1}\left(\frac{\sqrt{2\left(17+\sqrt{17}\right)} - \sqrt{17} - 1}{4}\right) - 4\tan^{-1}\left(\frac{3}{5}\right) \tag{86}$$

$$\pi = \frac{32}{3}\tan^{-1}\left(\frac{\sqrt{2\left(17+\sqrt{17}\right)} - \sqrt{17} - 1}{4}\right) - \frac{4}{3}\tan^{-1}\left(\frac{7}{23}\right) \tag{87}$$

Entry 42.

$$\pi\sqrt{\sqrt{2}+1} = \int_0^\infty \frac{1}{\sqrt{\sqrt{2e^x - 1} - 1}}\,dx \tag{88}$$

$$\pi = \int_0^\infty \tan^{-1}\left(\frac{16+8x^2}{63+8x^2+x^4}\right)dx \tag{89}$$

$$\frac{\pi^2}{8} - \pi\tan^{-1}\left(\frac{\sqrt{\sqrt{2}-1}}{1+\sqrt{\sqrt{2}+1}}\right) = \int_0^{\pi/4}\sin^{-1}\sqrt{\frac{\tan x}{2-\tan x}}\,dx \tag{90}$$

$$\pi\tan^{-1}\left(\frac{\sqrt{\sqrt{2}-1}}{1+\sqrt{\sqrt{2}+1}}\right) = \int_0^1 \frac{1}{\sqrt{1-x^2}}\tan^{-1}\left(\frac{2x^2}{1+x^2}\right)dx \tag{91}$$

$$\pi\tan^{-1}\left(\frac{\sqrt{\sqrt{2}-1}}{1+\sqrt{\sqrt{2}+1}}\right) = \int_0^\infty \frac{\tan^{-1}(\tanh x\tanh(2x))}{\cosh x}\,dx \tag{92}$$

$$\sqrt{2}\,\pi\tan^{-1}\left(\frac{\sqrt{\sqrt{2}-1}}{1+\sqrt{\sqrt{2}+1}}\right) = \int_0^1 \frac{\tan^{-1}x}{(2-x)\sqrt{x(1-x)}}\,dx \tag{93}$$

$$\pi = \int_0^\infty \ln\left(\frac{144+\left(5+4x^2\right)^2}{16+\left(3+4x^2\right)^2}\right)dx \tag{94}$$

$$2\pi = \int_0^{2\ln(13/5)}\sqrt{\frac{5}{e^x-1} - \frac{3e^x}{e^x-1} + \frac{2\sqrt{-36+41e^x-4e^{2x}}}{e^x-1}}\,dx \tag{95}$$

Entry 43.

$$\pi\sqrt{2\sqrt{2}+2} = \int_0^\infty \frac{1}{x^{3/2}}\ln\sqrt{(1+x)^2+x^2}\,dx \tag{96}$$

$$\pi\sqrt{2\sqrt{2}-2} = \int_0^\infty \frac{1}{x^{3/2}}\tan^{-1}\left(\frac{x}{1+x}\right)dx \tag{97}$$

$$\pi\sqrt{2\sqrt{2}+2} = \int_0^\infty \frac{\ln\left(1+2x^2+2x^4\right)}{x^2}\,dx \tag{98}$$

$$\pi\sqrt{2\sqrt{2}-2} = \int_0^\infty \frac{2}{x^2}\tan^{-1}\left(\frac{x^2}{1+x^2}\right)dx \tag{99}$$

Entry 44.

$$\pi\ln\left(1+\sqrt{2}+\sqrt{2+2\sqrt{2}}\right) = \int_0^\infty \frac{\ln\left((\sinh x)^4+(\cosh x)^4\right)}{\cosh x}\,dx \tag{100}$$

$$\pi\ln\left(\frac{1+\sqrt{2}+\sqrt{2+2\sqrt{2}}}{4}\right) = \int_0^1 \frac{\ln\left(1+x^4\right)}{\sqrt{1-x^2}}\,dx \tag{101}$$

$$\pi\tan^{-1}\left(1+\sqrt{2}-\sqrt{2+2\sqrt{2}}\right) = \int_0^1 \frac{\tan^{-1}x^2}{\sqrt{1-x^2}}\,dx \tag{102}$$

## Endnote

Entry 45. For $m \in \mathbb{N} = \{1, 2, 3, 4, \ldots\}$, we have

$$\pi = 2^{m+1}\sum_{n=0}^{\infty}(s_m)^{n+1}\sum_{k=0}^{[n/2]}\binom{2k}{k}\binom{n}{n-2k}\frac{2^{-2k}}{2k+1} \tag{103}$$

where

$$s_1 = \frac{\sqrt{2}}{2+\sqrt{2}},\ s_2 = \frac{\sqrt{2-\sqrt{2}}}{2+\sqrt{2-\sqrt{2}}},\ s_3 = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2+\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \ldots \tag{104}$$

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