The $4n$-region map is four-colorable

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Abstract

In this paper, we try to solve the four-color problem for a special case; i.e., a map with $4n$-region.

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1 Introduction

The four-color theorem states that no more than four colors are required to color the regions of any map such that no two adjacent regions have the same color. Fallacious proofs of this problem were given independently by Kempe [Kem79] and Tait [Tai80]. In 1890, Heawood [Hea90] exposed a flaw in Kempe's proof. In 1912, Birkhoff [Bir13] showed the reducibility of Birkhoff diamond. The four color theorem was proven in 1977 by Appel and Haken [AH77, AHK77].

In this paper, the $4n$-region map is decomposed into smaller pieces of four-region maps. We color each smaller maps and then we unify them. In §2 we bring several terminologies of map theory. In §3 we color the $4n$-region map with our algorithm.

2 Preliminaries

Definition 2.1. A country is a region in a map.

Definition 2.2. A C-reducible configuration is a configuration which can be proved to be reducible only after it has been modified in some way.

Definition 2.3. A D-reducible configuration is a configuration for which every coloring of the surrounding ring is a good coloring, or can be converted into one by one or more Kempe-chain changes of color.

Definition 2.4. A map is a collection of countries or regions separated by boundary lines.

Definition 2.5. A minimal criminal is a map with a certain number of regions which cannot be colored with four (or any other given number of) colors, while any map with fewer regions can be so colored.

Definition 2.6. The neighboring countries are two countries with a boundary line in common.

Definition 2.7. A region is a general term for a country, county or state in a map.

Definition 2.8. The ring-size is the number of regions surrounding a configuration. If there are k surrounding regions, the configuration has ring-size k.

Definition 2.9. The Shimamoto horseshoe is the configuration with ring-size 14 whose D-reducibility would have implied the four-color theorem.

Axiom 2.10. The 1-region map is not four-colorable.

Axiom 2.11. The 2-region map is not four-colorable.

Axiom 2.12. The 3-region map is not four-colorable.

Axiom 2.13. The 4-region map is four-colorable.

The biggest mistake of the method of Kempe-chains is that this method can deliver us into a minimal criminal.

3 Main Results

This below is Kempe's method for coloring any map [Kem79].

Kempe's Algorithm

1. Locate a region with five or fewer neighbors.

2. Cover this region with a blank piece of paper (a patch) of the same shape but slightly larger.

3. Extend all the boundaries which meet this patch and join them together at a single point within the patch which amounts to shrinking the region to a point. It has the effect of reducing the number of regions by 1.

4. Repeat the above procedure with the new map, continuing until there is just one regions left.

5. Color the single remaining regions with any of the four colors.

6. Reverse the above process, stripping off the patches in reverse order, until the original map is restored. At each stage, color the restored regions with any available color until the entire map is colored with four colors.

Appel and Haken tested the configurations of ring-size 14 by a computer [AH77]. And Koch tested the C-reducibility of the configurations of ring-size 11. Then, Appel extended the result to configurations of ring-sizes 12, 13 and 14 [AHK77].

Here, we introduce an algorithm for coloring a $4n$-region map.

Algorithm

1. Given a rectangular map with four identical regions. Fix the colors: red, blue, green and yellow. We denote the map by M_1.

2. Copy the map M_1 and denote the map by M_2. Join the map M_2 on the right side of the map M_1.

3. Repeat n times the step 2. Copy the map M_j and denote the map by M_{j+1}. Join the map M_{j+1} on the right side of the map M_j. Hence, we join all maps $M_1, \ldots, M_n$ in a serial rectangular map.

Thus, by using the above algorithm we obtain that the $4n$-region map is four-colorable.

References

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