A Proof that $\zeta(n \geq 2)$ is Irrational

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points. In the limit this yields z_n is irrational.

1 Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 8] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \qquad (1)$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [9] for the history of Apery's proof; Havil [5] gives an overview of Apery's ideas and attempts to demystify them. Also of interest is Huylebrouck's [6] paper giving an historical context for the main technique used by Beukers.

Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the work of Rivoal and Zudilin [10, 12]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

Let

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

We show that every rational number in $(0,1)$ can be written as a single decimal using the denominators of a term in z_n as a number basis. But the partial sums can't be expressed with such a single decimal. These two properties yield a proof that all z_n are irrational.

Properties of z_n

We define a decimal set.

Definition 1. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n-1)/j^n\} = \{.1, \ldots, .(j^n-1)\} \textit{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 in base j^n. The decimal set for j^n is

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

The union of decimal sets gives all rational numbers in $(0,1)$.

Lemma 1.

$$\bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in a d_{b^n} and hence in some D_{r^n} with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. □

Next we show $s_k^n \notin \Xi_k^n$.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (2) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 2; cf. Chapter 2, Problem 21 in [2], solution in [7]. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{3}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate [4]. $\square$

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 3. Using Lemmas 2 and 3, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. $\square$

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. $\square$

z_n is irrational

Definition 3. *Let $D_{j^n}^{\epsilon_j}$ be the set of all D_{j^n} decimal sets having an element within ϵ_j of s_j^n.*

Lemma 5. *z_n is rational if and only if there exists a monotonically decreasing sequence ϵ_j such that*

$$\lim_{j\to\infty} \epsilon_j = 0,$$

and

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} \neq \emptyset. \tag{4}$$

Proof. If z_n is rational then, using Lemma 1, $z_n \in D_{j^n}$ for some j. We will designate this D_{j^n} with $D_{j^n}^*$. Define

$$\epsilon_m = z_n - s_j^n$$

and set

$$\epsilon_j = 2\epsilon_m.$$

Then $\epsilon_j \to 0$ monotonically and

$$D^*_{j^n} \subset \bigcap_{j=2}^{\infty} D^{\epsilon_j}_{j^n},$$

so the intersection is not empty. □

Definition 4. *The precision of a decimal base, b, is $1/b$.*

Lemma 6. *Given any ϵ there exists a decimal base b of greater precision than ϵ; that is*

$$\frac{1}{b} < \epsilon.$$

Proof. This is the Archimedean property of the reals [11]. □

We use an obvious property of decimal representations: a number which is not a single digit in number base is between two such numbers and is at most $1/b$ away from either.

Theorem 2. *z_n is irrational.*

Proof. We need to define a sequence ϵ_k. Let

$$\epsilon^*_j = \min\{|x - s^n_j| : x \in \Xi^n_j\}.$$

We know by Corollary 1 that $\epsilon^*_j > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon^*_3$. This excludes the decimal sets of Ξ^n_3 from $D^{\epsilon_3}_3$ at this our first iteration. Assume we can generally do this for the jth iteration. For the $j+1$st iteration, using Lemma 6, there exists a base in Ξ^n_{j+r}, for some r such that $\epsilon^*_{j+r} < \epsilon_j/2$. Set $\epsilon_{j+1} = \epsilon^*_{j+r}$. The procedure gives ϵ values that exclude ever more decimal sets from $D^{\epsilon_j}_{j^n}$. The exclusions are cumulative; they persist. Regroup the series. By Lemma 1, the exclusions are exhaustive, so

$$\bigcap_{j=2}^{\infty} D^{\epsilon_j}_{j^n} = \emptyset$$

Assume that z_n is rational. Using Lemma 1, $z_n \in D_{j^n}$ for some j; designate this with $D^*_{j^n}$. Then by Lemma 5, there exists a sequence $\epsilon_q \to 0$ giving

$$D^*_{j^n} \subset \bigcap_{j=2}^{\infty} D^{\epsilon_j}_{j^n}, \tag{5}$$

but for sufficiently large j, $\epsilon_j < \epsilon_q$, for any fixed ϵ_q and Ξ_j^n will include D^*_{jn}, so $D^{\epsilon_j}_{jn}$ excludes D^*_{jn} contradicting (5). □

2 Conclusion

Although decimal representations of any convergent series in a given base will have partials with an ever greater number of decimal digits, those points must converge to a fixed, single decimal in some base. If this isn't true for a set of bases that includes all rational plausible convergence points as single digits, then the series must converge to an irrational number.

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