Hamiltonian mechanics of Pseudo-Riemannian manifolds

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Abstract

The serious obstacle for a quantum theory of general relativity is that Hamiltonian mechanics fundamentally distinguishes time from other coordinates. Postulating that the Hamiltonian dynamics of spacetime manifold must be entirely described by metric tensor, and energy-momentum tensor, I heuristically derive corresponding equivalent of Hamilton equations and then using the idea of wavefunction of spacetime, quantize the theory.

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1 Introduction

From the paradigm of Quantum Field Theories one would expect to quantise the Newtonian gravitational field ϕ and since that is substituted by $g_{\mu\nu}$ in General Relativity, one expects it is the metric tensor that must be quantised. But in the view adopted in this paper, something is overlooked in the common quest for quantum gravity: before one can do QFT, one needs a non-relativistic quantum mechanics; similarly thus, before one can do quantum gravity, one needs a *neo-Hamiltonian* non-quantic general relativity.

In our current understanding of GR, there are only two entities that are fundamental: the metric tensor $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$. In a retreat to the Hamiltonian

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paradigm, we can only compare these two entities with position q^μ and momentum p^ν, and since $T_{\mu\nu}$ corresponds to p^ν, we conclude that $g_{\mu\nu}$ corresponds to q^μ. Within our current mindset the result is that we must force GR to talk in terms of quantities that quantum mechanics understands, i.e. a new concept of phase space constructed by $g_{\mu\nu}$ and $T_{\mu\nu}$, for which

$$\{g^{\mu\nu}, T_{\rho\sigma}\} = \delta^{\mu\nu}_{\rho\sigma},$$

where { , } is a natural generalisation of the Poisson bracket and $\delta^{\mu\nu}_{\rho\sigma}$ is the generalised Kronecker delta, conveniently given by

$$\delta^{\mu\nu}_{\rho\sigma} = \begin{vmatrix} \delta^\mu_\rho & \delta^\mu_\sigma \\ \delta^\nu_\rho & \delta^\nu_\sigma \end{vmatrix}.$$

Contrary to the common expectation that a quantum theory of gravity must quantise space-time (the metric tensor field), in this theory space-time is unaltered, very much like the case with ordinary (non-relativistic) quantum mechanics in which q^α is not quantised, in this theory space-time is not quantised too.

2 Matter-waves of Manifolds

2.1 Generalisation of de Broglie relation

de Broglie relation

$$p^\mu = \hbar k^\mu$$

where p^μ is the four-momentum and K^μ four-wave-vector, suggests that momentum and wave-vector must be treated similarly (if not equally). We have a generalised notion of momentum in general relativity, but not that of wave vector. We therefore define the *wave tensor* of a matter-wave by

$$T^{\mu\nu} = \hbar K^{\mu\nu}. \tag{1}$$

2.1.1 Wave-blackhole duality

We know that blackholes are very much like ordinary matter. They possess mass, which is the characteristic property of matter. They can also possess electric charge and spin, which are, again, properties of elementary particles. If wave-matter duality is a fundamental law of nature, it might well be the case that blackholes in this case are similar to ordinary matter, too. After the introduction of wave function for a manifold, we are now ready the propose the following principle,

Principle of wave-blackhole duality To each blackhole we associate a wave-tensor defined by its energy-momentum tensor. The wave function of a blackhole with metric $g_{\mu\rho}$ is then given by

$$\psi = e^{iK^{\mu\rho}g_{\mu\rho}}.$$

It is crucial to understand that $T^{\mu\nu} = \hbar K^{\mu\nu}$ is the energy of the blackhole (spacetime) *itself*, not the distributional *source*. I will write about the energy-momentum tensor of spacetime (gravitational field) itself somewhere else, and provide an explicit tensor for it.

3 Analytical Mechanics of Riemannian Manifolds

To describe the dynamics of a matterwave of spacetime, we need to find a way to do Hamiltonian mechanics *of* a manifold. The reader[1] is assumed to be familiar with analytical mechanics; see e.g. Lanczos.

Let $\mathcal{M}$ be a pseudo-Riemannian manifold. We postulate that the metric tensor $\mathfrak{g}$ and energy-momentum tensor $\mathfrak{T}$, completely describe the dynamics of this manifold, making them *canonical variables.* Accordingly $(g_{\mu\nu}, T_{\mu\nu})$ spans the $2n^2$-dimensional *phase space* associated to $\mathcal{M}$.

The fundamental problem that obstructs the passage from classical Hamiltonian mechanics to a Hamiltonian mechanics of manifolds, is that of time. Classical analytical mechanics gives a special role to time (or equivalently a new variable τ, $t(\tau)$ which parametrizes paths in phase space). But there is no methodologically continuous way to get from *metric*, which knows no coordinates, to time. Therefore we cannot expect this new analytical mechanics of manifolds to be a *generalization* in the conventional sense of the word. It would be a parallel similar structure.

We begin with the Hamilton equations

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{2}$$

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \tag{3}$$

The first equation, when its right-hand-side is let equal to zero, gives the conservation of momentum. The analogue of conservation of momentum in GR is given by

$$\nabla^\mu T_{\mu\nu} = 0. \tag{4}$$

This suggests

$$\boxed{\nabla^\mu T_{\mu\nu} = -\partial_{g^{\nu\sigma}} \mathcal{H}^\sigma} \tag{5}$$

for the first Hamilton equation of a manifold, where

$$\partial_{g^{\alpha\beta}} := \frac{\partial}{\partial g^{\alpha\beta}}. \tag{6}$$

Note that for the indices to match, we have to promote $\mathcal{H}$ to an *odd-rank* tensor; I took the minimal possibility (rank-1).
Observe that (5), when its right-hand-side is let equal to zero, yields the conservation of momentum, naturally.

The problem now is that we do not have a definition for a *vector* Hamiltonian $\mathcal{H}^\sigma(g_{\mu\nu}, T_{\mu\nu})$. Let us look at the second Hamilton equation. As the canonical coordinates in GR are $(\mathfrak{g}, \mathfrak{T})$, we expect the GR equivalent of the second equation to be

$$\nabla^\mu g_{\mu\nu} = \partial_{T^{\nu\sigma}} \mathcal{H}^\sigma, \tag{7}$$

[1] And the author! (to avoid the application of the academic maxim *anything not mentioned by the author, is unknown to him*)

where

$$\partial_{T^{\alpha\beta}} := \frac{\partial}{\partial T^{\alpha\beta}}; \tag{8}$$

but the left-hand-side is zero by *metric compatibility condition*. Therefore this second equation turns into an *identity* using which we can *define* a *vector Hamiltonian*. We thus define vector Hamiltonian $\mathcal{H}^\sigma$ by the equation

$$\boxed{\partial_{T^{\nu\sigma}}\mathcal{H}^\sigma = 0} \tag{9}$$

This equation (given appropriate boundary conditions), uniquely determines $\mathcal{H}^\sigma$.

4 Quantization

Using the notion of *wavefunction of spacetime*

$$\Psi = e^{-ik_{\rho\sigma}g^{\rho\sigma}} \tag{10}$$

we can turn canonical variables $T_{\rho\sigma}$, $g^{\rho\sigma}$ hence Hamiltonian itself, into an operator. Note that

$$\partial_{g^{\rho\sigma}}\Psi = -ik_{\rho\sigma}\Psi,$$

therefore

$$\boxed{\hat{T}_{\rho\sigma} := i\hbar\partial_{g^{\rho\sigma}}} \tag{11}$$

Applying the covariant derivative on this, and using (5)

$$i\hbar\nabla^\mu\partial_{g^{\mu\nu}} = \nabla^\mu T_{\mu\nu} = -\partial_{g^{\rho\nu}}\mathcal{H}^\rho \tag{12}$$

$$\boxed{i\hbar\nabla^\mu\partial_{g^{\mu\nu}} + \partial_{g^{\rho\nu}}\mathcal{H}^\rho = 0} \tag{13}$$

Therefore the following set of equations provided a viable candidate for *first quantization* of gravity.

$$\begin{cases} R_{\mu\nu} = 0, \\ \partial_{T^{\mu\nu}}\mathcal{H}^\mu = 0, \\ i\hbar\nabla^\mu\partial_{g^{\mu\nu}}\Psi + \partial_{g^{\rho\nu}}\hat{\mathcal{H}}^\rho\Psi = 0. \end{cases} \tag{14}$$