Gravitational Dynamics from Multifractal Geometry

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Abstract

Emergent gravity builds upon the formal connection between classical Thermodynamics and the field equations of General Relativity. Drawing from the thermodynamic interpretation of multifractals, we speculate here that gravitational dynamics follows from the multifractal geometry of spacetime far above the Fermi scale. Our brief report suggests that a) *informational entropy* corresponds to the Beckenstein-Hawking entropy and b), Newton's constant corresponds to the *information dimension*, a defining attribute of multifractal scaling.

Key words: General Relativity, Emergent gravity, Thermodynamics of spacetime, Multifractal geometry, Black Hole entropy.

In general, interacting Hamiltonian systems are *nonintegrable* and their long-term evolution *chaotic* in the classical sense [1-2, 6, 11-12]. While many-body gravitational systems are non-integrable, chaos is typically suppressed in Quantum Mechanics due to the linearity of the Schrödinger equation. One may reasonably argue that, if *decoherence* sets in somewhere above the Fermi scale, nonlinearly coupled systems of quantum particles or fields are expected to become classical and, as such, evolve towards Hamiltonian chaos in a universal way. This scenario falls outside the framework of Quantum Chaos, where trace formulas and Random Matrix Theory are the main tools of analysis [9-10].

It has been known for quite some time that the structure of the phase-space generated by Hamiltonian chaos is non-differentiable and its characterization typically requires the language of multifractal sets. Let a distribution of phase-space orbits be covered by $N(\varepsilon)$ cells of nearly vanishing size ε. The *information dimension* and *informational entropy* are given by, respectively, [1, 7]

$$D_1 = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \tag{1}$$

$$S_{\text{inf}}(\varepsilon) = -\sum_j P_j \ln P_j = \ln N(\varepsilon) \tag{2}$$

in which

$$N(\varepsilon) \propto \varepsilon^{-D_1} \tag{3}$$

and where P_j denotes the probability of locating the orbit in the j cell. The global information encoded in phase-space is characterized by introducing a *measure* (a generalization of “length”, “area” or “volume” in ordinary geometry) according to

$$\kappa \propto \ln(1/\varepsilon^{\sigma}) = -\sigma \ln \varepsilon; \quad \sigma > 0 \tag{4}$$

such that $\kappa \to \infty$ as $\varepsilon \to 0$. One obtains, by (2), (3) and (4),

$$S_{\text{inf}} \propto D_1 \frac{\kappa}{\sigma} \tag{5}$$

Since both entropy and measure are additive parameters, a reasonable assumption is that (5) applies to a scenario where the phase-space is partitioned into an ensemble of subsets and global averages replace local values, as in

$$\langle S_{\inf} \rangle \propto \langle D_1 \rangle \frac{\langle \kappa \rangle}{\sigma} \tag{6}$$

An intriguing interpretation of (6) is that it echoes the Beckenstein-Hawking entropy formula expressed in natural units ($c = \hbar = 1$),

$$S_{BH} = \frac{1}{4G_N} A \tag{7}$$

where S_{BH} stands for the Black Hole entropy, G_N is Newton's constant and A the area of the event horizon [3]. In this interpretation, $\langle \kappa \rangle / \sigma$ is the analog of the horizon area A and the dimensionless Newton's constant the analog of information dimension, i.e.

$$\boxed{\langle D_1 \rangle \Leftrightarrow \frac{1}{G_N M_p^2}} \tag{8}$$

in which M_P denotes the Planck scale. Here, $\langle D_1 \rangle$ plays the role of a *universal constant*, independent of both σ and $\varepsilon << 1$.

We close with several observations that are relevant for follow up work on the topic:

1) In line with the thermodynamic interpretation of multifractals [1] and the philosophy of emergent gravity [4-5], relations (6)-(8) hint that gravitational dynamics emerges from the *thermodynamics of fractal spacetime.*
2) Allowing the exponent σ to scan through a range of positive and negative values, produces a *degenerate entropy* which fluctuates between 0 and ∞. At least in principle, this possibility offers an unforeseen resolution to the Black Hole information problem [8].
3) The maximal entropy contained in a box of linear size Δ in Quantum Field Theory scales as $S_{QFT} \propto \Delta^3$, while the Beckenstein-Hawking entropy scales as $S_{BH} \propto \Delta^2$ [13]. It is apparent from (6) that this discrepancy goes away if σ varies within a range of continuous values, as "area" and "volume" are naturally blended in the concept of *measure* (4). *This is to say that holographically inspired arguments for the Beckenstein-*

Hawking entropy are no longer needed in the context of multifractal geometry.

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