A beautiful relation between two integrals with transcendental functions

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abstract

We give some integral formulas

Introduction

Recall that (Proposed by Srinivasa Raghava, Twitter, 06 ago 2021)

$$\int_0^\infty e^{-x}\sqrt{\sin\left(\pi e^{-x}\right)}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^\infty \frac{e^{-x}}{\sqrt{\sin\left(\pi e^{-x}\right)}}\,dx \tag{1}$$

where $\Gamma(x)$ is the Gamma function and $\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

In this note we give some integrals related to (1).

Integrals

$$\int_0^\pi \sqrt{\sin x}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^\pi \frac{1}{\sqrt{\sin x}}\,dx \tag{2}$$

$$\int_0^1 \frac{1}{\left(1-x^2\right)^{1/4}}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^1 \frac{1}{\left(1-x^2\right)^{3/4}}\,dx \tag{3}$$

$$\int_0^{\pi/2} \sqrt{\sin x}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx \tag{4}$$

$$\int_0^{\pi/2} \sqrt{\cos x}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^{\pi/2} \frac{1}{\sqrt{\cos x}}\,dx \tag{5}$$

$$\int_0^{\pi/2}\left(\sqrt{\sin x}+\sqrt{\cos x}\right)dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^{\pi/2}\left(\frac{1}{\sqrt{\sin x}}+\frac{1}{\sqrt{\cos x}}\right)dx \qquad (6)$$

$$\pi\Gamma(1/4)^4-2\Gamma(1/4)^4\int_0^1\sin^{-1}x^2\,dx=8\pi^3+16\pi^2\int_1^\infty\sin^{-1}\frac{1}{x^2}\,dx \qquad (7)$$

$$\Gamma(1/4)^4\int_0^1\cos^{-1}x^2\,dx=4\pi^3+8\pi^2\int_1^\infty\left(\frac{\pi}{2}-\cos^{-1}\frac{1}{x^2}\right)dx \qquad (8)$$

$$\int_0^\infty\sqrt{\tanh x}\,\operatorname{sech}x\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^\infty\frac{\operatorname{sech}x}{\sqrt{\tanh x}}\,dx \qquad (9)$$

$$\int_0^\infty(\operatorname{sech}x)^{3/2}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^\infty(\operatorname{sech}x)^{1/2}\,dx \qquad (10)$$

$$2\Gamma(1/4)^4+3\Gamma(1/4)^4\int_0^1\ln\left(1+\sqrt{1-x^{4/3}}\right)dx=48\pi^2+24\pi^2\int_0^1\ln\left(1+\sqrt{1-x^4}\right)dx \qquad (11)$$

$$\Gamma(1/4)^4+\Gamma(1/4)^4\int_0^1\frac{x^2}{1+\sqrt{1-x^4}}\,dx=8\pi^2+8\pi^2\int_0^1\frac{dx}{x^{2/3}\left(1+\sqrt{1-x^{4/3}}\right)} \qquad (12)$$

$$\int_0^{\pi/2}\left(\sqrt{\sin x}-\sqrt{\cos x}\right)dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^{\pi/2}\left(\frac{1}{\sqrt{\sin x}}-\frac{1}{\sqrt{\cos x}}\right)dx \qquad (13)$$

$$\int_1^\infty\frac{1}{x\sqrt{x^3-x}}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_1^\infty\frac{1}{\sqrt{x^3-x}}\,dx \qquad (14)$$

$$\int_1^e\frac{\sqrt{\sin(\pi\ln x)}}{x}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_1^e\frac{1}{x\sqrt{\sin(\pi\ln x)}}\,dx \qquad (15)$$

$$\pi\Gamma(1/4)^4-2\Gamma(1/4)^4\int_0^1\cos^{-1}\sqrt{1-x^4}\,dx=8\pi^3+16\pi^2\int_1^\infty\cos^{-1}\sqrt{1-\frac{1}{x^4}}\,dx \qquad (16)$$

$$\Gamma(1/4)^4\int_0^1\sin^{-1}\sqrt{1-x^4}\,dx=4\pi^3+8\pi^2\int_1^\infty\left(\frac{\pi}{2}-\sin^{-1}\sqrt{1-\frac{1}{x^4}}\right)dx \qquad (17)$$

$$\int_0^\infty\frac{e^{-3x}}{\sqrt{1-e^{-4x}}}\,dx=\frac{8\pi^2}{\Gamma(1/4)^4}\int_0^\infty\frac{e^{-x}}{\sqrt{1-e^{-4x}}}\,dx \qquad (18)$$

$$\pi\Gamma\left(1/4\right)^4-4\sqrt{2}\,\Gamma\left(1/4\right)^4\int_0^{1/\sqrt{2}}\sin^{-1}\sqrt{\frac{1-\sqrt{1-4x^4}}{2}}\,dx=8\pi^3+16\sqrt{2}\,\pi^2\int_{\sqrt{2}}^{\infty}\sin^{-1}\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4}{x^4}}}\,dx \tag{19}$$

$$\int_0^1\tanh^{-1}\sqrt{1-x^{4/3}}\,dx=\frac{8\pi^2}{\Gamma\left(1/4\right)^4}\int_0^1\tanh^{-1}\sqrt{1-x^4}\,dx \tag{20}$$

References

[1] G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook*, Part IV, Springer, New York, 2013.

[2] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge university press, 1996.

[3] I.S. Gradshteyn, and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed., Academic Press, Boston, 2000.

[4] B.C. Berndt, *Ramanujan´s Notebook*, Part I, Springer, New York, 1985.

[5] D. Bailey, J. Borwein, N. Calkin, R. Girgensohn, R. Luke, and V. Moll, *Experimental Mathematics in Action*, AK Peters, 2007.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Volumes 1-3, McGraw-Hill, 1953.

[7] S. Raghava, Romanian Mathematical Magazine RMM, Web, 2021.