A Different Way to Prove a Prime Number between 2N and 3N

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Abstract

In this paper we will use a different way to prove that there exists at least a prime number p in between $2n$ and $3n$ where n is a positive integer. The proof extends the Bertrand's postulate - Chebyshev's theorem which states that a prime number exists between n and $2n$. The method to prove this proposition is to analyze the binomial coefficient, a similar method used by Erdős in the proof of Bertrand's postulate.

Introduction

The Bertrand's postulate - Chebyshev's theorem states that for any positive integer n, there is always a prime number p such that $n < p \le 2n$. It was proved in 1850 [1]. In 1932, Paul Erdős [2] used a much simpler method to prove the theorem by carefully analyzing the central binomial coefficient $\binom{2n}{n}$. In 2006, M. El Bachraoui [3] extended the theorem by proving that for any positive integer n, there is a prime number p such that $2n < p \le 3n$. In this paper, the author will use a different method to prove the same extension by analyzing the binomial coefficient $\binom{3n}{n}$. First, we will define and clarify some terms and concepts. Then we will propose the subject of the thesis.

Definition: $\Gamma_{a\ge p>b}\{\binom{3n}{n}\}$ denotes the prime factorization operator of $\binom{3n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{3n}{n}$ in the range of $a \ge p > b$. In this operator, p is a prime number, a and b are real numbers, and $3n \ge a \ge p > b \ge 1$.

It has some properties:

It is always true that $\Gamma_{a\ge p>b}\{\binom{3n}{n}\} \ge 1$ **— (1)**

If there is no prime number in $\Gamma_{a\ge p>b}\{\binom{3n}{n}\}$, then $\Gamma_{a\ge p>b}\{\binom{3n}{n}\} = 1$, or vice versa,
if $\Gamma_{a\ge p>b}\{\binom{3n}{n}\} = 1$, then there is no prime number in $\Gamma_{a\ge p>b}\{\binom{3n}{n}\}$. **— (2)**

For example, $\Gamma_{8\ge p>6}\{\binom{12}{4}\} = 7^0 = 1$. No prime number is in $\binom{12}{4}$ in the range of $8 \ge p > 6$.

If there is at least one prime number in $\Gamma_{a\ge p>b}\{\binom{3n}{n}\}$, then $\Gamma_{a\ge p>b}\{\binom{3n}{n}\} > 1$, or vice versa,
if $\Gamma_{a\ge p>b}\{\binom{3n}{n}\} > 1$, then there is at least one prime number in $\Gamma_{a\ge p>b}\{\binom{3n}{n}\}$. **— (3)**

For example, $\Gamma_{6\ge p>4}\{\binom{12}{4}\} = 5 > 1$. Prime number 5 is in $\binom{12}{4}$ in the range of $6 \ge p > 4$.

Let $v_p(n)$ be the p-$adic\ valuation$ of n, the exponent of the highest power of p that divides n. Similar to Paul Erdős' paper [2], we define *R(p)* by the inequalities $p^{R(p)} \leq 3n < p^{R(p)+1}$, and determine the p-$adic\ valuation$ of $\binom{3n}{n}$.

$v_p\left(\binom{3n}{n}\right) = v_p((3n)!) - v_p((2n)!) - v_p(n!) = \sum_{i=1}^{R(p)}\left(\left\lfloor\frac{3n}{p^i}\right\rfloor - \left\lfloor\frac{2n}{p^i}\right\rfloor - \left\lfloor\frac{n}{p^i}\right\rfloor\right) \leq R(p)$

because for any real numbers a and b, the expression of $\lfloor a + b\rfloor - \lfloor a\rfloor - \lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{3n}{n}$, then $v_p\left(\binom{3n}{n}\right) \leq R(p) \leq \log_p(3n)$, or $p^{v_p\left(\binom{3n}{n}\right)} \leq p^{R(p)} \leq 3n$ **— (4)**

And if $3n \geq p > \lfloor\sqrt{3n}\rfloor$, then $0 \leq v_p\left(\binom{3n}{n}\right) \leq R(p) \leq 1$. **— (5)**

From the prime number decomposition, when $n > \lfloor\sqrt{3n}\rfloor$,

$\binom{3n}{n} = \frac{(3n)!}{n!\cdot(2n)!} = \Gamma_{3n\geq p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{n\geq p>\lfloor\sqrt{3n}\rfloor}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$.

When $n \leq \lfloor\sqrt{3n}\rfloor$, $\binom{3n}{n} \leq \Gamma_{3n\geq p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$.

Thus, $\binom{3n}{n} \leq \Gamma_{3n\geq p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{n\geq p>\lfloor\sqrt{3n}\rfloor}\{\frac{(3n)!}{n!\cdot(2n)!}\} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$.

Since all prime numbers in ($n!$) are not in the range of $3n \geq p > n$,

$\Gamma_{3n\geq p>n}\{\frac{(3n)!}{n!\cdot(2n)!}\} = \Gamma_{3n\geq p>n}\{\frac{(3n)!}{(2n)!}\}$.

Referring to **(5)**, $\Gamma_{n\geq p>\lfloor\sqrt{3n}\rfloor}\{\frac{(3n)!}{n!\cdot(2n)!}\} \leq \prod_{n\geq p} p$.

It has been proved [4] that $\prod_{n\geq p} p < 2^{2n-3}$ when $n \geq 3$.

Thus for $n \geq 3$, $\binom{3n}{n} < \Gamma_{3n\geq p>n}\{\frac{(3n)!}{(2n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor\geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$ **— (6)**

Proposition

For every positive integer n, there exists at least a prime number p such that $2n < p \leq 3n$.

Proof:

By induction on n, for $n = 3$, $\frac{3^{3n-2}}{n(2^{2n-2})} = \frac{3^6}{2^4} = 45\frac{9}{16} < \binom{3n}{n} = \binom{9}{3} = 84$.

If $\binom{3n}{n} > \frac{3^{3n-2}}{n(2^{2n-2})}$ for n stands, then for n +1,

$\binom{3(n+1)}{n+1} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} \cdot \binom{3n}{n} > \frac{3(3n+2)(3n+1)}{(2n+2)(2n+1)} \cdot \frac{3^{3n-2}}{n(2^{2n-2})} > \frac{3^{3(n+1)-2}}{(n+1)(2^{2(n+1)-2})}$

because $\frac{3(3n+2)(3n+1)}{(2n+2)(2n+1)} \cdot \frac{3^{3n-2}}{n(2^{2n-2})} = 3 \cdot \frac{3n+2}{2n+1} \cdot \frac{3n+1}{2n} \cdot \frac{3^{3n-2}}{(n+1)(2^{2n-2})} > 3^3 \cdot \frac{3^{3n-2}}{(n+1)(2^{2n-2})}$

Thus for $n \geq 3$, $\binom{3n}{n} > \frac{3^{3n-2}}{n(2^{2n-2})}$ **— (7)**

Applying **(7)** into **(6)**:

For $n \geq 3$, $\frac{3^{3n-2}}{n(2^{2n-2})} < \Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} \cdot 2^{2n-3} \cdot \Gamma_{\lfloor\sqrt{3n}\rfloor \geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\}$ **— (8)**

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1$ (MOD 6) and $p \equiv 5$ (MOD 6). Thus, $\pi(n) \leq \left\lfloor\frac{n}{3}\right\rfloor + 2 \leq \frac{n}{3} + 2$. **— (9)**

Referring to **(4)** and **(9)**,

$$\Gamma_{\lfloor\sqrt{3n}\rfloor \geq p}\{\frac{(3n)!}{n!\cdot(2n)!}\} = \Gamma_{\lfloor\sqrt{3n}\rfloor \geq p}\{\binom{3n}{n}\} \leq (3n)^{\pi(\sqrt{3n})} \leq (3n)^{\frac{\sqrt{3n}}{3}+2} \quad \textbf{— (10)}$$

Applying **(10)** into **(8)**: $\frac{3^{3n-2}}{n(2^{2n-2})} < \Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} \cdot 2^{2n-3} \cdot (3n)^{\frac{\sqrt{3n}}{3}+2}$

Since for $n \geq 3$, both $2^{2n-3} > 0$ and $(3n)^{\frac{\sqrt{3n}}{3}+2} > 0$

$$\Gamma_{3n \geq p > n}\{\frac{(3n)!}{(2n)!}\} > \frac{3^{3n-2}}{n(2^{2n-2})(2^{2n-3})(3n)^{\frac{\sqrt{3n}}{3}+2}} = \frac{32\cdot\left(\frac{27}{16}\right)^n}{3\cdot(3n)^{\frac{\sqrt{3n}}{3}+3}} = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} \quad \textbf{— (11)}$$

Let $f(x) = \frac{u}{w}$ where x, u, w are real numbers and $x \geq 84$, $u = \frac{32}{3} \cdot \left(\frac{27}{16}\right)^x$, $w = (3x)^{\frac{\sqrt{3x}+9}{3}}$

$$\frac{du}{dx} = \left(\frac{32}{3} \cdot \left(\frac{27}{16}\right)^x\right)' = \frac{32}{3} \cdot \left(\frac{27}{16}\right)^x \cdot ln\left(\frac{27}{16}\right) = u \cdot ln\left(\frac{27}{16}\right)$$

$$\frac{dw}{dx} = \left((3x)^{\frac{\sqrt{3x}+9}{3}}\right)' = \left((3x)^{\frac{\sqrt{3x}+9}{3}}\right)\left(\frac{ln(3x)}{2\sqrt{3x}} + \frac{\sqrt{3x}+9}{3x}\right) = w\left(\frac{ln(3x)+2}{2\sqrt{3x}} + \frac{3}{x}\right)$$

$$f'(x) = \left(\frac{u}{w}\right)' = \frac{w(u)' - u(w)'}{w^2} = \frac{u}{w}\left(ln\left(\frac{27}{16}\right) - \frac{ln(3x)+2}{2\sqrt{3x}} - \frac{3}{x}\right)$$

Let $f_1(x) = ln\left(\frac{27}{16}\right) - \frac{ln(3x)+2}{2\sqrt{3x}} - \frac{3}{x}$

Since when $x > 1$, $f_1{}'(x) = \frac{ln(3x)}{4x\sqrt{3x}} + \frac{3}{x^2} > 0$, $f_1(x)$ is a strictly increasing function.

When $x = 84$, $f_1(x) = ln\left(\frac{27}{16}\right) - \frac{ln(3x)+2}{2\sqrt{3x}} - \frac{3}{x} \approx 0.523 - 0.237 - 0.012 = 0.274 > 0$.

Thus, when $x \geq 84$, $f_1(x) > 0$.

Since when $x \geq 84$, u, w, and $f_1(x)$ are greater than zero, $f'(x) = \frac{u}{w} \cdot f_1(x) > 0$.

Thus $f(x)$ is a strictly increasing function for $x \geq 84$. Then when $x \geq 84$, $f(x+1) > f(x)$.

Let $x = n \geq 84$, then $f(n+1) > f(n) = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}}$

Since for n = 84, $f(n) = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^{84}}{(252)^{\frac{\sqrt{252}+9}{3}}} \approx \frac{1.307E+20}{8.151E+19} > 1$, and since

$f(n+1) > f(n)$, by induction on n, when $n \geq 84$, $f(n) = \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} > 1$. **— (12)**

Applying **(12)** to **(11)**: When $n \geq 84$, $\Gamma_{3n \geq p > n}\{ \frac{(3n)!}{(2n)!} \} > \frac{32}{3} \cdot \frac{\left(\frac{27}{16}\right)^n}{(3n)^{\frac{\sqrt{3n}+9}{3}}} > 1$.

Thus when $n \geq 84$,

$\Gamma_{3n \geq p > n}\{ \frac{(3n)!}{(2n)!} \} = \Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} \cdot \Gamma_{2n \geq p > \frac{3n}{2}}\{ \frac{(3n)!}{(2n)!} \} \cdot \Gamma_{\frac{3n}{2} \geq p > n}\{ \frac{(3n)!}{(2n)!} \} > 1$. **— (13)**

If there is any prime number p such that $2n \geq p > \frac{3n}{2}$, then $(3n)!$ has the factor of p, and $(2n)!$ also has the same factor of p. Thus, they cancel to each other in $\frac{(3n)!}{(2n)!}$ with no prime number in the range of $2n \geq p > \frac{3n}{2}$. Referring to **(2)**, $\Gamma_{2n \geq p > \frac{3n}{2}}\{ \frac{(3n)!}{(2n)!} \} = 1$.

Thus, when $n \geq 84$, $\Gamma_{3n \geq p > n}\{ \frac{(3n)!}{(2n)!} \} = \Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} \cdot \Gamma_{\frac{3n}{2} \geq p > n}\{ \frac{(3n)!}{(2n)!} \} > 1$. **— (14)**

Referring to **(1)**, $\Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} \geq 1$ and $\Gamma_{\frac{3n}{2} \geq p > n}\{ \frac{(3n)!}{(2n)!} \} \geq 1$, from **(14)**, at least one of these two factors is greater than one when $n \geq 84$.

If $n \geq 84$ and $\Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} > 1$, then referring to **(3)**, there exists at least a prime number p such that $2n < p \leq 3n$. **— (15)**

$\Gamma_{\frac{3n}{2} \geq p > n}\{ \frac{(3n)!}{(2n)!} \} = \Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{ \frac{(3n)!}{(2n)!} \}$.

If $\frac{n}{2} \geq 42$ and $\Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{ \frac{(3n)!}{(2n)!} \} = 1$, then from **(14)**, the factor $\Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} > 1$.
Referring to **(3)**, there exists at least a prime number p such that $2n < p \leq 3n$. **— (16)**

If $\frac{n}{2} \geq 42$ and $\Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{ \frac{(3n)!}{(2n)!} \} > 1$, let $m = \frac{n}{2}$, then when $m \geq 42$, there exists at least a prime number p such that $2m < p \leq 3m$. Since $n \geq 84 \geq m \geq 42$, the statement is also valid for n.

Thus, when $n \geq 84$, if $\Gamma_{3 \cdot (\frac{n}{2}) \geq p > 2 \cdot (\frac{n}{2})}\{ \frac{(3n)!}{(2n)!} \} > 1$, then $\Gamma_{3n \geq p > 2n}\{ \frac{(3n)!}{(2n)!} \} > 1$, and there exists at least a prime number p such that $2n < p \leq 3n$. **— (17)**

From **(16)** and **(17)**, no matter $\Gamma_{\frac{3n}{2} \geq p > n}\{ \frac{(3n)!}{(2n)!} \}$ is equal to 1 or greater than 1, there exists at least a prime number p such that $2n < p \leq 3n$ when $n \geq 84$. **— (18)**

Table 1 shows that when $1 \le n \le 84$, there is a prime number p such that $2n < p \le 3n$. **— (19)**

Thus, the proposition is proven by combining **(15)**, **(18)**, and **(19)**: For every positive integer n, there exists at least a prime number p such that $2n < p \le 3n$.

Table 1: For $1 \le n \le 84$, there is a prime number p such that $2n < p \le 3n$.

$2n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28
p	3	5	7	11	13	17	17	19	23	29	29	31	31	37
$3n$	3	6	9	12	15	18	21	24	27	30	33	36	39	42
$2n$	30	32	34	36	38	40	42	44	46	48	50	52	54	56
p	37	41	41	43	43	47	47	53	53	59	59	61	61	67
$3n$	45	48	51	54	57	60	63	66	69	72	75	78	81	84
$2n$	58	60	62	64	66	68	70	72	74	76	78	80	82	84
p	67	71	71	73	73	79	79	83	83	89	89	97	97	101
$3n$	87	90	93	96	99	102	105	108	111	114	117	120	123	126
$2n$	86	88	90	92	94	96	98	100	102	104	106	108	110	112
p	101	103	103	107	107	109	109	113	113	127	127	131	131	137
$3n$	129	132	135	138	141	144	147	150	153	156	159	162	165	168
$2n$	114	116	118	120	122	124	126	128	130	132	134	136	138	140
p	137	139	139	149	149	151	151	157	157	163	163	167	167	173
$3n$	171	174	177	180	183	186	189	192	195	198	201	204	207	210
$2n$	142	144	146	148	150	152	154	156	158	160	162	164	166	168
p	173	179	179	181	181	191	191	193	193	197	197	199	199	211
$3n$	213	216	219	222	225	228	231	234	237	240	243	246	249	252

References

[1] M. Aigner, G. Ziegler, *Proofs from THE BOOK*, Springer, 2014, 16-21

[2] P. Erdős, *Beweis eines Satzes von Tschebyschef*, Acta Sci. Math. (Szeged) **5** (1930-1932), 194-198

[3] M. El Bachraoui, *Prime in the Interval* [2*n*, 3*n*], International Journal of Contemporary Mathematical Sciences, Vol.1 (2006), no. 13, 617-621.

[4] Wikipedia, https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate, Lemma 4.