On the refractive index-curvature relation

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The refractive index-curvature relation is formulated using the second rank tensor of Ricci curvature as a consequence of a scalar refractive index. A scalar refractive index describes linear optics. In a topological space, the linear refractive index is related to the Euler-Poincare characteristic. Because the Euler-Poincare characteristic is a topological invariant then the linear refractive index is also a topological invariant.

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In the geometrical optics, the refractive index-curvature relation which describes ray propagation in a steady (time-independent) state can be derived from the Fermat's principle[1–4]. The refractive index-curvature relation can be written as

$$\frac{1}{R} = \hat{N}\ .\ \vec{\nabla}\ \ln\ n(r) \qquad (1)$$

where $1/R$ is a 1-dimensional space curvature, R is a radius of curvature, $\hat{N}$ is an unit vector along the principal normal or has the same direction with $\vec{\nabla}$ ln $n(r)$ and $n(r)$ is a 1-dimensional space refractive index. Eq.(1) tells us that the rays are therefore bent in the direction of increasing refractive index[1].

The dimension of the curvature in eq.(1) can be extended to any arbitrary number of dimensions[5]. In a $(3+1)$-dimensional space-time, eq.(1) can be written as

$$R_{\mu\nu} = g\ N_\mu\ \partial_\nu \ln n \qquad (2)$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature[5,6], a function of the metric tensor $g_{\mu\nu}$, $g = |(\det\ g_{\mu\nu})|$, is a scalar, a real number. Why do we need to formulate the curvature in eq.(2) as the second rank tensor of Ricci curvature? It is because of the related refractive index in eq.(2) is the zeroth rank tensor, a scalar i.e. a real number.

The zeroth rank tensor (a scalar) of the refractive index describes *an isotropic linear optics*[7]. But, the refractive index can be not simply a scalar[8]. The refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[8]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[7]. Eq.(2) implies that the zeroth rank tensor of the refractive index related to the Ricci curvature describes naturally (an isotropic) linear optics.

We will formulate a curvature in a fibre bundle and we treat the geometical optics as a gauge theory[4]. *Is there a relationship between a fibre bundle and a gauge theory? Why do we need to formulate a curvature in a fibre bundle?* Originally, the fibre bundle and the gauge theory are developed independently. Until it was realized that *the curvature (in the fibre bundle) and the field strength (in Yang-Mills theory) are identical*[9]. *Simply speaking, the curvature in the fibre bundle is the field strength in the gauge theory.*

Because the geometrical optics can be treated as the Abelian $U(1)$ gauge theory[4], so we need to formulate the curvature in the refractive index-curvature relation as *an Abelian curvature form* in a fibre bundle. Probably, this is another reason why we really need to formulate a curvature in a curvature form instead of the Riemann-Christoffel curvature tensor. *A curvature form in a fibre bundle can be an Abelian (or a non-Abelian) which the Riemann-Christoffel curvature tensor can not be an Abelian*[10].

The curvature form, $\Omega_{\rho\sigma}$, can be written as[11,12]

$$\Omega_{\rho\sigma} = \sum R_{\rho\sigma\mu\nu}\ du^\mu \wedge du^\nu \qquad (3)$$

where $R_{\mu\nu\rho\sigma}$ is the fourth rank tensor of Riemann-Christoffel curvature, u^μ, u^ν are local coordinates and $\wedge$ is a notation of the exterior (wedge) product (it satisfies the distributive, anti-commutative and associative laws)[11,12]. *The curvature form, $\Omega_{\rho\sigma}$, is an anti-symmetric matrix of 2-forms*[13,14]. The relation between the Ricci curvature tensor and the Riemann-Christoffel curvature tensor, we call the Ricci-Riemann relation, is $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\sigma\mu\nu}$.

If we reformulate eq.(3) using eq.(2) and the Ricci-Riemann relation, we obtain

$$\Omega_{\rho\sigma} = \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln n\ du^\mu \wedge du^\nu \qquad (4)$$

Eq.(4) shows the relationship between the scalar refractive index and the curvature form in a (3+1)-dimensional space-time. Here, the scalar refractive index is a function of coordinates only (a smooth continuous function of the position[15]) which "lives" in a $(3+1)$-dimensional space-time[4].

Let us introduce the general form of *the curvature matrix*, Ω, which is *a matrix of exterior two-forms* as below[11]

$$\Omega = d\omega - \omega \wedge \omega \qquad (5)$$

where ω is *the connection matrix*. We see that eq.(5) is a non-Abelian, a non-linear equation.

Can the curvature matrix, Ω, in eq.(5) be an Abelian, a linear equation? An Abelian curvature matrix means that the second term in the right hand side of eq.(5), $\omega \wedge \omega$, vanish. It can be done if *the isometry group*, $G = U(1)$, then the Killing vector fields, $\xi_i \in u(1)$ (the Lie algebra of $U(1)$)[4]. So in case of $G = U(1)$[16], we have

$$\Omega = d\omega \tag{6}$$

We see that eq.(6) is an Abelian, a linear equation.

Is there a relationship between the curvature matrix, Ω (5), and the curvature form, $\Omega_{\rho\sigma}$ (3)? Yes (there is)[17]. If $\Omega_{\rho\sigma}$ and $\omega_{\rho\sigma}$ denote *the components* of curvature and connection matrices, Ω and ω, respectively then we can write[11]

$$\Omega = (\Omega_{\rho\sigma}), \quad \omega = (\omega_{\rho\sigma}) \tag{7}$$

So, the curvature matrix (5) can be written using the curvature form[12] as below

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} - {\omega_\rho}^\tau \wedge \omega_{\tau\sigma} \tag{8}$$

In case of the Killing vector fields, $\xi_i \in u(1)$, the curvature form (8) becomes

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} \tag{9}$$

Eq.(9) is *the equation of an Abelian curvature form*. By substituting eq.(9) into eq.(4), we obtain

$$d\omega_{\rho\sigma} = \sum g \ g_{\rho\sigma} \ N_\mu \ \partial_\nu \ln n \ du^\mu \wedge du^\nu \tag{10}$$

We call eq.(10) as *the Abelian curvature form-scalar refractive index relation.*

Let us define *the pfaffian of the curvature matrix* Ω as below[11,18]

$$\text{pf } \Omega \equiv \sum \ \epsilon_{\rho_1\sigma_1 \ldots \rho_{2q}\sigma_{2q}} \ \Omega_{\rho_1\sigma_1} \wedge \ldots \wedge \Omega_{\rho_{2q}\sigma_{2q}} \tag{11}$$

where Ω is *any even-size complex* $2q \times 2q$ *anti-symmetric matrix* (if Ω is *an odd size* complex anti-symmetric matrix, the corresponding pfaffian is defined to be *zero*), $\epsilon_{\rho_1\sigma_1 \ldots \rho_{2q}\sigma_{2q}}$ is the $2q$-th rank Levi-Civita tensor which has value +1 or -1 according as its indices form an even or odd permutation of $1, \ldots, 2q$, and its otherwise zero, and the sum is extended over all indices from 1 to $2q$. Here, $\rho_1 < \sigma_1, \ \ldots \ , \ \rho_{2q} < \sigma_{2q}$ and $\rho_1 < \rho_2 < \ldots < \rho_{2q}$[11,18]. Shortly, the pfaffian of Ω (11) can be rewritten as

$$\text{pf } \Omega = \sum \ \epsilon_{\rho\sigma} \ \Omega_{\rho\sigma} \tag{12}$$

By substituting eqs.(9), (10) into (12) we obtain

$$\text{pf } \Omega = \sum \epsilon_{\rho\sigma} \ \sum g \ g_{\rho\sigma} \ N_\mu \ \partial_\nu \ln n \ du^\mu \wedge du^\nu \tag{13}$$

Using the pfaffian of Ω, *the Gauss-Bonnet-Chern theorem*[19–21] says that[11,20]

$$(-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \text{pf } \Omega = \chi(M^{2q}) \tag{14}$$

where q is *a natural number*, $\chi(M^{2q})$ is *the Euler-Poincare characteristic*[22,23] *of the even dimensional oriented compact Riemannian manifold*, M^{2q}. The Euler-Poincare characteristic is *a topological invariant*[11]. By substituting (13) into (14), the Gauss-Bonnet-Chern theorem (14) becomes

$$\chi(M^{2q}) = (-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \sum \epsilon_{\rho\sigma} \sum g \ g_{\rho\sigma} \ N_\mu \ \partial_\nu \ln n \ du^\mu \wedge du^\nu \tag{15}$$

We see from eq.(15), the scalar refractive index is related to the Euler-Poincare characteristic. *Because the Euler-Poincare characteristic is a topological invariant*[24,25] *then the scalar refractive index should be a topological invariant.*

The pfaffian of the curvature matrices (11) are defined to be zero and non-zero if the curvature matrices are an odd-size and an even size complex antisymmetric matrices respectively. The zero and non-zero pfaffian of the curvature matrices have consequences that the related curvature forms are zero and non-zero respectively. We see from eq.(3) that the zero and non-zero curvature forms in turn have consequences that the Riemann-Christoffel curvature tensors are vanish and not vanish respectively. The vanishing Riemann-Christoffel curvature tensor means that space-time is vacuum. In other words, the Riemann-Christoffel curvature tensor must vanish in vacuum space-time. *So does it mean that the zero and non-zero curvature forms are related to vacuum and non-vacuum space-time (in turn a vanishing and a non-vanishing field strengths or vacuum and non-vacuum gauge potentials)?*

We see from eq.(14) that the zero and non-zero Euler-Poincare characteristics are consequences of the zero and non-zero pfaffian of an odd-size and an even-size of complex antisymmetric curvature matrices respectively. *Does it mean that the zero and non-zero Euler-Poincare characteristics are related to vacuum and non-vacuum space-time (in turn a vanishing and a non-vanishing field strengths or vacuum and non-vacuum gauge potentials)?*

We see from eq.(15), the zero and non-zero Euler-Poincare characteristics have consequences that the scalar refractive indices are zero and non-zero respectively. *Physically, does it mean that the zero and non-zero scalar refractive indices are related to vacuum and non-vacuum space-time? (in turn a vanishing and a non-vanishing field strengths or vacuum and non-vacuum gauge potentials)?*

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[10]The Christoffel symbol does not transform as a tensor, but rather as an object in the jet bundle (Wikipedia, *Christoffel symbols*). If the non-linear term (non-Abelian term) of the Christoffel symbol happens to be zero in one coordinate system, it will in general not be zero in another coordinate system[4].

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[13]WolframMathWorld, *Antisymmetric Matrix*, https://mathworld.wolfram.com/AntisymmetricMatrix.html

[14]An antisymmetric matrix is a square matrix that satisfies the identity $A = -A^T$ where A^T is the matrix transpose. *All* $n \times n$ *antisymmetric matrices of odd size (i.e. if* n *is odd) are singular (determinant of matrix is equal to zero)*. Antisymmetric matrices are commonly called "skew symmetric matrices" by mathematicians[13].

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[21]*Gauss-Bonnet formula expresses the global invariant,* $\chi(M)$*, as the integral of a local invariant*, which is perhaps the most desirable relationship between local and global properties[19].
For *even-dimensional* oriented compact Riemannian manifold, M^{2n}, the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem[20].

[22]Milosav M. Marjanovic, *Euler-Poincare Characteristic - A Case of Topological Convincing*, The Teaching of Mathematics, 2014, Vol. XVII, 1, pp. 21–33.

[23]The Euler-Poincare characteristic starts from *Euler's polyhedron formula (a number)* which appeared first in a note submitted by Euler to the Proceedings of the Petersburg Academy of 1752/53. Henri Poincare who defined *an integer* to be *a topological property* of all other geometric objects. *The Euler-Poincare characteristic is a stable topological property*[22].

[24]Topological Invariant. Encyclopedia of Mathematics. https://encyclopediaofmath.org/wiki/Topological_invariant.

[25]Topological invariant is any property of a topological space that is *invariant* under *homeomorphisms*[24]. Homeomorphisms are, roughly speaking, the mappings that preserve all the topological properties of a given space.