SERIES

Edgar Valdebenito

10 Feb 2022

Abstract. We give some series for Pi

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ... \tag{1}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \tag{2}$$

where $\tan^{-1}(x)$ is the arctangent function.

In this note we give some formulas for Pi.

Some series for Pi

Entry 1. For $u = \left(\sqrt{13 + 4\sqrt{2}} - 2\sqrt{2} - 1\right) / 2$, we have

$$\pi = 16 \sum_{n=0}^{\infty} u^{n+1} \sum_{k=[n/2]}^{n} \frac{(-1)^k}{2k+1} \binom{2k+1}{2n-2k+1} \tag{3}$$

Entry 2.

$$\pi = 8 \sum_{n=0}^{\infty} (-1)^n u^{2n+1} \sum_{k=0}^{[n/2]} \frac{1}{2n-4k+1} \binom{2n-2k}{2k} \tag{4}$$

where

$$u = -\frac{1}{2}\sqrt{\frac{a-2}{3}} + \frac{1}{2}\sqrt{-\frac{4}{3} - \frac{a}{3} + 4\sqrt{\frac{3}{a-2}}} \tag{5}$$

$$a = \left(91 + 6\sqrt{267}\right)^{1/3} - 11\left(91 + 6\sqrt{267}\right)^{-1/3} \tag{6}$$

Entry 3. For $u = \left(\sqrt{1 + \sqrt{5} - \sqrt{2 + 2\sqrt{5}}}\right) / 2\sqrt{2}$, we have

$$\pi = 8 \sum_{n=0}^{\infty} u^{2n+1} \sum_{k=0}^{[n/2]} \frac{2^{4k}}{2n-4k+1} \binom{2n-2k}{2k} \binom{2n-4k}{n-2k} \tag{7}$$

Entry 4.

$$\pi = 8\sqrt{u}\sum_{n=0}^{\infty}(-1)^n u^n \binom{2n}{n}\sum_{k=0}^{[n/2]}\frac{2^{-4k}}{2n-4k+1}\binom{n}{2k}\binom{2n-4k}{n-2k}^{-1} \tag{8}$$

where

$$u = \frac{1}{4}\sqrt{2+a} - \frac{1}{4}\sqrt{4-a+\frac{4}{\sqrt{2+a}}} \tag{9}$$

$$a = \frac{1}{3}\left(135-6\sqrt{249}\right)^{1/3} + \frac{1}{3^{2/3}}\left(45+2\sqrt{249}\right)^{1/3} \tag{10}$$

Entry 5.

$$\pi = 8\sum_{n=0}^{\infty}u^{2n+1}\sum_{k=1+\left[\frac{n-1}{2}\right]}^{n}\frac{(-1)^k}{2k+1}\binom{2k+1}{2n-2k} \tag{11}$$

where

$$u = \frac{1}{2}\sqrt{\frac{2+a}{3}} - \frac{1}{2}\sqrt{\frac{4-a}{3}+4\sqrt{\frac{3}{2+a}}} \tag{12}$$

$$a = \left(89-6\sqrt{159}\right)^{1/3} + \left(89+6\sqrt{159}\right)^{1/3} \tag{13}$$

Entry 6.

$$\pi = 8\left(1+u^2\right)\sum_{n=0}^{\infty}(-1)^n u^{2n+1}\sum_{k=0}^{n}\sum_{m=0}^{k}\frac{1}{2m+1}\binom{k+m}{k-m}\binom{n-m}{n-k} \tag{14}$$

where

$$u = \frac{2\left(\sqrt{2}-1\right)}{3} + \frac{1}{3}\left(\frac{2}{-121+89\sqrt{2}+\sqrt{4\left(8\sqrt{2}-9\right)^3+\left(89\sqrt{2}-121\right)^2}}\right)^{-1/3} + \frac{\left(8\sqrt{2}-9\right)}{3}\left(\frac{2}{-121+89\sqrt{2}+\sqrt{4\left(8\sqrt{2}-9\right)^3+\left(89\sqrt{2}-121\right)^2}}\right)^{1/3} \tag{15}$$

Entry 7. For $u = \sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\ldots}}}$, we have

$$\pi = 2\sqrt{3}\left(1-\sum_{n=0}^{\infty}(-1)^n u^{-n-2}\sum_{k=[n/3]}^{[n/2]}\frac{(-1)^k}{6k-2n+3}\binom{k}{n-2k}\right) \tag{16}$$

remark: $u = \left(\frac{1+i\sqrt{3}}{2}\right)^{1/3} + \left(\frac{1+i\sqrt{3}}{2}\right)^{-1/3}$.

Entry 8.

$$\pi = 2\sqrt{3}\left(1-\sum_{n=0}^{\infty}u^{-n-2}\sum_{k=[n/3]}^{[n/2]}\frac{(-1)^k}{6k-2n+3}\binom{k}{n-2k}\right) \tag{17}$$

where

$$u = \sqrt{3 - \frac{1}{\sqrt{3 - \frac{1}{\sqrt{3 - \ldots}}}}} = \frac{1 - i\sqrt{3}}{2}\left(\frac{1 + i\sqrt{3}}{2}\right)^{1/3} + \left(\frac{1 + i\sqrt{3}}{2}\right)^{2/3} \tag{18}$$

Entry 9.

$$\pi = 8\sum_{n=0}^{\infty} u^{n+1} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{6k - 2n + 1} \binom{4k - n}{n - 2k} f(n - 2k, k) \tag{19}$$

where

$$f(k, n) = \begin{cases} 1 & 0 \le k \le n \\ 0 & k > n \end{cases} \tag{20}$$

$$u = \sqrt{2} - 1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} - 1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} - 1 + \ldots\right)^3\right)^3 \tag{21}$$

remark: $u^3 - \left(\sqrt{2} + 1\right)u + 1 = 0$.

Entry 10.

$$\pi = 8u - \frac{8}{3}u^3 - 8u^4 + \frac{8}{5}u^5 + 8u^6 + 8\sum_{n=6}^{\infty} u^{n+1} \sum_{k=[n/3]}^{[n/2]} \frac{(-1)^k}{6k - 2n + 1} \binom{4k - n}{n - 2k} \tag{22}$$

where

$$u = \sqrt{2} - 1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} - 1 + \left(\sqrt{2} - 1\right)\left(\sqrt{2} - 1 + \ldots\right)^3\right)^3 \tag{23}$$

remark: $u^3 - \left(\sqrt{2} + 1\right)u + 1 = 0$.

Entry 11.

$$\pi = 6\sum_{n=0}^{\infty} (-1)^n u^{2n+1} \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{(-1)^k}{2n - 2k + 1} \binom{2n - 2k + 1}{k} \tag{24}$$

where

$$u = \frac{1}{3^{2/3}}\left(\frac{3\sqrt{3} + \sqrt{39}}{2}\right)^{1/3} - \left(\frac{2}{3\left(3\sqrt{3} + \sqrt{39}\right)}\right)^{1/3} \tag{25}$$

Entry 12.

$$\pi = 2\sum_{n=0}^{\infty} u^{-2n-1}\left(\binom{2n}{n}\frac{2^{-2n}}{2n + 1} + \sum_{k=0}^{n}\binom{2k}{k}\binom{n + k}{n - k}\frac{2}{2k + 1}\right) \tag{26}$$

where

$$u = \sqrt{\frac{7}{3} + \left(\frac{262}{27} + \sqrt{\frac{172}{27}}\right)^{1/3} + \left(\frac{262}{27} - \sqrt{\frac{172}{27}}\right)^{1/3}} \tag{27}$$

Entry 13.

$$\pi = 16u + 16\sum_{n=1}^{\infty} (-1)^n u^{n+1} \sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]} \frac{(-1)^k}{2k + 1} \binom{2k + 1}{n - 2k} \tag{28}$$

where

$$u = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\left(\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\right)^{1/2}} \tag{29}$$

Entry 14.

$$\pi = \sum_{n=0}^{\infty} 2^{-3n} \binom{2n}{n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \binom{2k}{k}^{-1} \left(2^{k+1} + 2^{-k-1}\right) \tag{30}$$

$$\pi = \frac{16}{5} \times \sum_{n=0}^{\infty} 5^{-2n} \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} \binom{2n}{2n-2k} \tag{31}$$

$$\pi = 2 \times \sum_{n=0}^{\infty} 6^{-n} \sum_{k=0}^{[n/2]} \frac{(-1)^k 2^{2k}}{2k+1} \binom{2n-2k}{n-k} \binom{n-k}{k} \binom{2k}{k}^{-1} \tag{32}$$

$$\pi = 3 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \sum_{k=0}^{[n/2]} \frac{1}{2n-4k+1} \binom{2n-2k}{n-k} \binom{n-k}{k} \binom{2n-4k}{n-2k}^{-1} \tag{33}$$

$$\pi = 3 \times \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n} \sum_{k=0}^{n} \frac{(-1)^k 2^{2k}}{2k+1} \binom{n}{k} \binom{2k}{k}^{-1} \tag{34}$$

$$\pi = \frac{18}{5} \sum_{n=0}^{\infty} (-1)^n 50^{-n} \binom{2n}{n} \sum_{k=0}^{n} \frac{2^k \times 3^{2k}}{2k+1} \binom{n}{k} \binom{2k}{k}^{-1} \tag{35}$$

$$\pi = \frac{2}{3} \times \sum_{n=0}^{\infty} 6^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} \binom{2k}{k}^{-1} 2^{2k} \left(2^{2k+2} + 1\right) \tag{36}$$

$$\pi = \frac{2}{3} \times \sum_{n=0}^{\infty} 6^{-2n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{2n-2k}{n-k} 3^{2k} \times 2^{-k} \left(2^{2k+2} + 1\right) \tag{37}$$

$$\pi = \frac{9}{4} \times \sum_{n=0}^{\infty} 2^{-4n} \binom{2n}{n} \sum_{k=0}^{n} \frac{1}{2k+1} \binom{n}{k} \left(\frac{3}{2}\right)^{2k} \tag{38}$$

$$\pi = \sum_{n=0}^{\infty} 2^{-3n-1} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{2n-2k}{n-k} \left(2^{2k+2} + 1\right) \tag{39}$$

Entry 15.

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \sum_{k=0}^{n} \frac{(-1)^k 5^{-k}}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} \left(2^{-2n+2k-1} + 2^{-k} \times 3^{-2n+2k-1}\right) \tag{40}$$

Entry 16.

$$\pi = 2\sqrt{2} \sum_{n=0}^{\infty} \left(\frac{2-\sqrt{2}}{4}\right)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{n-2k} \left(\sqrt{2} + 1\right)^{2k} \tag{41}$$

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2\sqrt{3} - 3}{6}\right)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{n-2k} \left(2 + \sqrt{3}\right)^{2k} \tag{42}$$

Entry 17.

$$\pi = 9 \times \sum_{n=0}^{\infty} 2^{-4n-1} \sum_{k=0}^{n} \binom{2n-2k}{n-k} 2^k \left(\frac{(-1)^k}{3k+1} - \frac{2^k}{6k+3}\right) \tag{43}$$

Entry 18.

$$\pi = 4\left(\frac{\sqrt{5}-1}{2}\right) + 4\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{2k+1}{n-2k} \tag{44}$$

$$\pi = 6\left(\frac{2\sqrt{3}}{3+\sqrt{9+12\sqrt{3}}}\right) + 6\sum_{n=1}^{\infty}\left(\frac{2\sqrt{3}}{3+\sqrt{9+12\sqrt{3}}}\right)^{n+1}\sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{2k+1}{n-2k} \tag{45}$$

$$\pi = 8\left(\frac{\sqrt{4\sqrt{2}-3}-1}{2}\right) + 8\sum_{n=1}^{\infty}\left(\frac{\sqrt{4\sqrt{2}-3}-1}{2}\right)^{n+1}\sum_{k=\left[\frac{n-1}{4}\right]}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{2k+1}{n-2k} \tag{46}$$

Entry 19.

$$\pi = \frac{5}{2}\sum_{n=0}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^{2n}\sum_{k=0}^{n}\frac{(-1)^k 2^{-2k}}{(1-2k)(2n-2k+1)}\binom{2k}{k}\binom{2n-2k}{n-k}^{-1} \tag{47}$$

Entry 20.

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k}\left(3\left(\frac{1}{2}\right)^{n+1} + 3^{-2k-1}\left(\frac{3}{5}\right)^{n+1}\right) \tag{48}$$

Entry 21.

$$\pi = 8\sum_{n=0}^{\infty}(-1)^n u^{2n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2n-2k+1}\binom{2n-2k+1}{2k} \tag{49}$$

where

$$u = \frac{1}{2}\sqrt{\frac{2+a}{3}} - \frac{1}{2}\sqrt{\frac{4-a}{3} + 4\sqrt{\frac{3}{2+a}}} \tag{50}$$

$$a = \left(89 - 6\sqrt{159}\right)^{1/3} + \left(89 + 6\sqrt{159}\right)^{1/3} \tag{51}$$

Entry 22.

$$\pi = \frac{8}{3}\sum_{n=0}^{\infty}\left(\frac{2}{9}\right)^n\sum_{k=0}^{n}\frac{(-1)^k 2^{-k}}{2k+1}\binom{2n}{2n-2k} \tag{52}$$

$$\pi = \frac{96}{29}\times\sum_{n=0}^{\infty}29^{-2n}\sum_{k=0}^{n}\frac{(-1)^k 12^{2k}}{2k+1}\binom{2n}{2n-2k} \tag{53}$$

$$\pi = \frac{560}{169}\times\sum_{n=0}^{\infty}13^{-4n}\sum_{k=0}^{n}\frac{(-1)^k 70^{2k}}{2k+1}\binom{2n}{2n-2k} \tag{54}$$

Entry 23.

$$\pi = \sum_{n=0}^{\infty}2^{-n}\left(2+(-2)^{-n}\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{n}{n-2k}\left(\sqrt{2}-1\right)^{n-2k} \tag{55}$$

$$\pi = 2\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^{n+1}\left(2+(-1)^n\right)\sum_{k=0}^{[n/2]}\frac{(-1)^k 2^{-2k}}{2k+1}\binom{n}{n-2k} \tag{56}$$

$$\pi = 3\sum_{n=0}^{\infty}\left(\frac{2-\sqrt{3}}{4}\right)^n\sum_{k=0}^{[n/2]}\frac{(-1)^k 2^{4k}}{2k+1} \tag{57}$$

$$\pi = 3 \sum_{n=0}^{\infty} \left(\frac{2-\sqrt{3}}{4} \right)^n \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{n-k} \tag{58}$$

Entry 24.

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \, 12^{-n} \sum_{k=0}^{n} \frac{2^{2k}}{2k+1} \binom{2n-2k}{n-k} \tag{59}$$

$$\pi = 3 \times \sum_{n=0}^{\infty} 2^{-4n} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{2n-2k}{n-k} \left(\frac{16}{3} \right)^k \tag{60}$$

$$\pi = 2 \times \sum_{n=0}^{\infty} 6^{-n} \sum_{k=0}^{n} \frac{(-1)^k \, 2^k}{2k+1} \binom{2n-2k}{n-k} \tag{61}$$

Endnote

Entry 25. For $0 < p < q$, $0 < \theta < \frac{\pi}{4}$, we have

$$\frac{\pi}{2} + \theta = \tan^{-1}\left(\frac{p}{q}\right) + 2\tan^{-1}\left(\exp\left(\frac{\sqrt{p^2+q^2}}{q} \sum_{n=0}^{\infty} (-1)^n \left(\frac{(\tan\theta)^{n+1} - (p/q)^{n+1}}{n+1} \right) \sum_{k=0}^{[n/2]} (-1)^k \left(\frac{p}{q}\right)^{n-2k} \binom{2k}{k} 2^{-2k} \right)\right) \tag{62}$$

where $\exp(x) = e^x$.

Example 1: $p = 1$, $q = 2$, $\theta = \pi/6$:

$$\pi = \frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right) + 3\tan^{-1}\left(\exp\left(\sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\left(2\sqrt{3}\right)^{-n-1} - 4^{-n-1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} \right)\right) \tag{63}$$

$$\pi = -\frac{12}{5}\tan^{-1}\left(\frac{1}{3}\right) + \frac{24}{5}\tan^{-1}\left(\exp\left(\sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\left(2\sqrt{3}\right)^{-n-1} - 4^{-n-1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} \right)\right) \tag{64}$$

Example 2: $p = 1$, $q = 3$, $\theta = \pi/6$:

$$\pi = \frac{3}{2}\tan^{-1}\left(\frac{1}{3}\right) + 3\tan^{-1}\left(\exp\left(\sqrt{10} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\left(3\sqrt{3}\right)^{-n-1} - 9^{-n-1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} (3/2)^{2k} \right)\right) \tag{65}$$

$$\pi = -\frac{12}{5}\tan^{-1}\left(\frac{1}{2}\right) + \frac{24}{5}\tan^{-1}\left(\exp\left(\sqrt{10} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\left(3\sqrt{3}\right)^{-n-1} - 9^{-n-1} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} (3/2)^{2k} \right)\right) \tag{66}$$

Example 3: $p = \sqrt{2} - 1$, $q = 1$, $\theta = \pi/6$:

$$\pi = \frac{48}{13}\tan^{-1}\left(\exp\left(\sqrt{4+2\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\left(\sqrt{6}+\sqrt{3}\right)^{-n-1} - \left(\sqrt{2}+1\right)^{-2n-2} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} \left(2\sqrt{2}-2\right)^{-2k} \right)\right) \tag{67}$$

Example 4: $p = 1$, $q = 4$, $\tan\theta = 1/2$:

$$\pi = \frac{4}{3}\tan^{-1}\left(\frac{1}{3}\right) + \frac{4}{3}\tan^{-1}\left(\frac{1}{4}\right) + \frac{8}{3}\tan^{-1}\left(\exp\left(\sqrt{17} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(2^{-3n-3} - 2^{-4n-4} \right) \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} 2^{2k} \right)\right) \tag{68}$$

Example 5:

$$\pi = \frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right) + 3\tan^{-1}\left(\frac{\sqrt{5}-1}{2} \exp\left(\sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(2\sqrt{3}\right)^{-n-1} \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} \right)\right) \tag{69}$$

$$\pi = -\frac{12}{5}\tan^{-1}\left(\frac{1}{3}\right) + \frac{24}{5}\tan^{-1}\left(\frac{\sqrt{5}-1}{2} \exp\left(\sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(2\sqrt{3}\right)^{-n-1} \sum_{k=0}^{[n/2]} (-1)^k \binom{2k}{k} \right)\right) \tag{70}$$

References

[1] Apelblat, A.: Tables of Integrals and Series. Verlag Harri Deutsch, 1996.

[2] Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.

[3] Bailey, D.H., and Broadhurst, D.J.: "Parallel Integer Relation Detection: Techniques and Applications," Mathematics of Computation, vol. 70, no. 236 (Oct 2000), pg. 1719-1736.

[4] Blatner, D.: The Joy of π . Walker Publ., 1999.

[5] Borwein, J.M., and Bailey, D.H.: Mathematics by Experiment: Plausible Reasoning in the 21st Century, AK Peters, Natick, MA, 2008.

[6] Borwein, J.M., Bailey, D.H. and Girgensohn, R.: Experimentation in Mathematics: Computational Paths to Discovery, AK Peters, Natick, MA, 2004.

[7] Castellanos, D.: The ubiquitous Pi. Part I. Math. Magazine, 61, 1988, 67-98.

[8] Castellanos, D.: The ubiquitous Pi. Part II. Math. Magazine, 61, 1988, 148-163.

[9] Petkovsek, M., Wilf, H., and Zeilberger, D.: $A = B$. A. K Peters, Wellesley, MA, 1996.