Hadamard Matrices And Division Algebras Only

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Abstract

The iteration that builds systematic forms for Hadamard matrices doubles the dimension each application equivalent to the Cayley-Dickson algebra doubling process. These systematic forms have a row or column composition rule where the composition of any two rows(columns) is closed for the matrix, such that for sequentially assigned column and row indexes from 0 up, the index of the result is the binary bit-wise exclusive or (xor) of the indexes for the two rows(columns) participating. The Exclusive Or Group of order 2^n and all of its subgroups is isomorphic to the product tables for order 2^n Cayley-Dickson algebra and all of its subalgebras when all resultant –1 signs are dropped. The Boolean xor operator thus forms the bridge between Cayley-Dickson algebras and Hadamard matrices, but only for the division algebra subset. The dimension 8 Hadamard matrix is ubiquitous within Octonion algebraic structure by specifying optimal enumerations of, and relationships between; basis elements, Quaternion subalgebra triplets, proper Octonion orientations, Octonion Algebraic Invariance and Variance.

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It is a known fact the resultant signs of basis element products for division algebras can be expressed by a table where the +1 and –1 sign results form a Hadamard matrix. A systematic construction for relevant order Hadamard matrices uses the following iteration defining H_n from the matrix H_{n-1} starting with the single cell matrix [+1] = H_0:

H_{n-1}	H_{n-1}
H_{n-1}	$-H_{n-1}$

The dimension of these types of Hadamard matrices H_n are 2^n by 2^n matching the dimension of the basis element product table for the nth doubled Cayley-Dickson algebra, where both constructions continue beyond Octonion Algebra, the last division algebra.

Attach integer labels for any systematic Hadamard matrix rows and columns sequentially from zero, left to right and top to bottom. Now consider the row or column composition where a row vector is formed from the product of like column entries for any two same or different rows, or additionally as a column vector formed from the product of like rows for any two same or different columns. The result for every composition of this type will be a row or column of the systematic Hadamard Matrix, this composition definition is closed for the table. In fact, with rows and columns enumerated as mentioned, the composition of any two rows(columns) indexed a and b results in the row(column) a^b, the bit-wise exclusive or (xor) of the binary represented indexes a and b. This composition rule on Hadamard matrices precisely matches Cayley-Dickson algebra basis element products $e_a * e_b = \pm e_{a \wedge b}$ when the basis element product/result indexes are optimally enumerated.

While this composition is interesting and important when the table rows and columns are enumerated as mentioned, we are looking to represent the signs of Cayley-Dickson basis element products within systematic Hadamard matrices, so rows and columns need to be labeled with basis elements. The first row and first column of the systematic form are all +1 entries, making these the appropriate enumerations for the scalar basis element row and column labeled e_0. The remaining diagonal members of the systematic form are not always –1 as they would need to be if the row basis element enumeration is the same as the column enumeration, since all non-scalar basis elements square to –1. There is no loss of generality keeping a sequential basis index enumeration for row labels, respecting the composition rule correspondence with optimal basis element products at least for rows. Our task will then be to find permutations of the non-scalar row basis elements to use as labels for columns such that the Hadamard matrix –1 values properly identify all basis element squares and negated basis element results that are the outcome of anti-commutation if relevant.

H_0 has one element, and trivially covers the algebra of real numbers. For Complex (sub)algebra, H_1 spans its basis product structure in a singular fashion since there is only one non-scalar basis to permute, matching the fact Complex Algebra is singularly defined. We have for H_1

	e_0	e_n
e_0	+1	+1
e_n	+1	–1

Quaternion (sub)algebras will use H_2, where the following row indexes {a, b, c} must be matched bijectively through non-identity permutations to define column indexes {x, y, z}

	e_0	e_x	e_y	e_z
e_0	+1	+1	+1	+1
e_a	+1	–1	+1	–1
e_b	+1	+1	–1	–1
e_c	+1	–1	–1	+1

From the table above we have two choices assigning –1 for basis element squares:

a = x or z
b = y or z
c = x or y

If we choose x = a, this forces y = c which forces z = b. If we choose z = a, this forces y = b, which forces x = c. Thus, the connection to Hadamard matrices tells us there are two different permutation mappings possible, the exact number of orientation choices for Quaternion Algebra. Their basis element product tables are then

	e_0	e_a	e_c	e_b
e_0	+1	+1	+1	+1
e_a	+1	–1	+1	–1
e_b	+1	+1	–1	–1
e_c	+1	–1	–1	+1

	e_0	e_c	e_b	e_a
e_0	+1	+1	+1	+1
e_a	+1	–1	+1	–1
e_b	+1	+1	–1	–1
e_c	+1	–1	–1	+1

The table on the left is represented by the Quaternion six product cyclic right (+) cyclic left (–) basis element product rules (e_c e_b e_a) e.g. $e_c * e_b = +e_a$ and $e_b * e_c = –e_a$, and the representation for the right side table is the Quaternion basis element product rule (e_a e_b e_c), the negation or opposite chiral orientation of the former.

Octonion (sub)algebras will use H_3 and we must find within the next table all permutations on the fixed {a, b, c, d, e, f, g} bijectively placed within {t, u, v, w, x, y, z} that meet the following restrictions. As with Quaternion (sub)algebras above we must accommodate all subalgebras for Octonion Algebra with properly formed Hadamard matrices of appropriate order. Octonion Algebra has seven Quaternion subalgebras. Each of these must have proper 4x4 Hadamard matrix representations isomorphic to what we just went through for H_2 but within the H_3 structure. Once found, all Complex subalgebra H_1 forms will be assured to be legitimate as subforms of H_2. The following is the generic H_3 table:

	e_0	e_t	e_u	e_v	e_w	e_x	e_y	e_z
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_a	+1	–1	+1	–1	+1	–1	+1	–1
e_b	+1	+1	–1	–1	+1	+1	–1	–1
e_c	+1	–1	–1	+1	+1	–1	–1	+1
e_d	+1	+1	+1	+1	–1	–1	–1	–1
e_e	+1	–1	+1	–1	–1	+1	–1	+1
e_f	+1	+1	–1	–1	–1	–1	+1	+1
e_g	+1	–1	–1	+1	–1	+1	+1	–1

There is no loss of generality representing {a, b, c, d, e, f, g} as {1, 2, 3, 4, 5, 6, 7}, a proper Octonion basis set index enumeration. The set of legitimate permutations of {1, 2, 3, 4, 5, 6, 7} bijectively assigned to {t, u, v, w, x, y, z} respecting the seven Quaternion subalgebra Hadamard connections can be easily extracted with a relatively simple computer program. The resultant orientations for each of the Quaternion subalgebras will determine which of the 16 Octonion Algebra orientations that permutation produces. There are only eight permutations of {1, 2, 3, 4, 5, 6, 7} that yield proper H_2 forms for all Quaternion subalgebras, and the following table itemizes the results. The Octonion Algebras they represent are indicated by row entry in the first column:

	t	u	v	w	x	y	z
L0	7	6	5	4	1	3	2
L1	5	3	2	6	1	4	7
L2	1	3	6	7	4	5	2
L3	1	7	2	5	6	3	4
L4	5	7	6	4	3	2	1
L5	3	2	5	7	6	4	1
L6	3	6	1	5	4	2	7
L7	7	2	1	6	3	5	4

Notice all are Left Octonion, and the full complement of Left orientation Octonions are singularly represented. Keeping within Right or Left and not producing both is a common occurrence for Octonion Algebra structural analysis. An important example is PSL(2,7), the automorphism group for the Fano Plane. Its members represent all possible Octonion non-scalar basis element permutations that respect both the partitions for its seven Quaternion subalgebra triplets, and all orientation choices that result in proper Octonion Algebras. PSL(2,7) maps Right Octonion to Right Octonion, and Left Octonion to Left Octonion. This clearly tells us that two separate directed Fano Plane representations are required to fully specify Octonion Algebra. If one would look at a directed Fano Plane representation straight on, then its mirror image, the arrow directions in the classical form following the three triangle sides and the one connecting the side midpoints all go clockwise one way and counter-clockwise the other, yet nothing has changed with the inferred multiplication rules. On the other hand, the three arrows for the vertex bisectors have mirror symmetry. They can be represented two ways, all pointed out of the vertexes (representing all Right Octonion Algebras) or all pointed into the vertexes (representing all Left Octonion Algebras). While there are other arrow possibilities, why make it any more complicated than it has to be?

Having only Left Octonion, and *all* Left is a good thing. The relationship between Left and Right Octonion Algebras is an anti-automorphism where the map between is the involution negating the orientation of every Quaternion subalgebra. Being an anti-automorphism, we can change the only Left H_3 Octonion Algebra table above to only Right H_3 by transposing the table. Since the systematic forms of Hadamard matrices are symmetric, we only need to swap column and row labels, maintaining order. We then have the following:

	t	u	v	w	x	y	z
L/R0	7	6	5	4	1	3	2
L/R1	5	3	2	6	1	4	7
L/R2	1	3	6	7	4	5	2
L/R3	1	7	2	5	6	3	4
L/R4	5	7	6	4	3	2	1
L/R5	3	2	5	7	6	4	1
L/R6	3	6	1	5	4	2	7
L/R7	7	2	1	6	3	5	4

L $\mathbb{O}$	e_0	e_t	e_u	e_v	e_w	e_x	e_y	e_z
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_1	+1	–1	+1	–1	+1	–1	+1	–1
e_2	+1	+1	–1	–1	+1	+1	–1	–1
e_3	+1	–1	–1	+1	+1	–1	–1	+1
e_4	+1	+1	+1	+1	–1	–1	–1	–1
e_5	+1	–1	+1	–1	–1	+1	–1	+1
e_6	+1	+1	–1	–1	–1	–1	+1	+1
e_7	+1	–1	–1	+1	–1	+1	+1	–1

R $\mathbb{O}$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_t	+1	–1	+1	–1	+1	–1	+1	–1
e_u	+1	+1	–1	–1	+1	+1	–1	–1
e_v	+1	–1	–1	+1	+1	–1	–1	+1
e_w	+1	+1	+1	+1	–1	–1	–1	–1
e_x	+1	–1	+1	–1	–1	+1	–1	+1
e_y	+1	+1	–1	–1	–1	–1	+1	+1
e_z	+1	–1	–1	+1	–1	+1	+1	–1

So, what do we have now for Octonion Algebra? Requiring the algebraic basis element product result sign structure to be represented by a Hadamard matrix tells us we have *only* 16 different Octonion orientations. If we require Octonions to be a normed composition algebra where the product of the norms of two algebraic elements is the norm of their product, we also have *only* 16 orientations meeting this test. If we require the algebra of just two Octonion algebraic elements to be associative, i.e. a*(b*a) = (a*b)*a we have *only* 16 different orientations meeting this requirement. The beauty and consistency of Octonion algebraic structure is that all three requirements restrict Octonion Algebra to the same 16 orientations. So, we have lost nothing restricting {a, b c, d, e, f, g} to sequentially {1, 2, 3, 4, 5, 6, 7}. The permutation on any other set would build an isomorphic alias to this restriction, but could be required for an Octonion subalgebra of a higher dimension algebra.

Moving on now to Sedenion algebra, we would expect to build its product signage with H_4 by permuting row indexes 1-15 for use as the indexes for the column basis elements. Interestingly, when we move beyond the division algebra portion of Cayley-Dickson algebras, the Hadamard matrix method stops working. No permutation will produce proper H_2 forms for all 35 Quaternion subalgebras of Sedenion Algebra. If we first restrict consideration to those permutations that yield all 15 non-scalar basis element squares to –1, the best that can be done is 26 proper Quaternion H_2 forms. Here, the improper forms show one of three non-scalar basis elements commutes with the other two instead of anti-commuting. If we remove the preliminary square to –1 restriction, the best that can be done then is 32 of 35 proper H_2 forms. The improper forms here will be combinations of commuting products that should anti-commute, and non-scalar basis element squaring to +1. None of these are proper Sedenion Algebra. Since Sedenion Algebra will be a subalgebra of every greater order Cayley-Dickson algebra, the Hadamard game is over after proper doubling to Octonion Algebra.

While the above connection between Octonion Algebra and Hadamard matrices stands on its own, there is more structure within structure. This is a common occurrence for Octonion Algebra, and the immense structure embodied within its definition becomes more apparent when its Quaternion subalgebra triplets are enumerated by the full complement of three indexes that xor to zero. This is not a close call, notably here from the xor presence in Hadamard row/column compositions and thus division algebras through the earlier discussion, and moreover the correspondence between Exclusive Or Groups and any Cayley-Dickson algebra.

Understanding is best achieved when the noise of unnecessary clutter is removed. The best realization of this begins with standardizing on the xor correspondence for the partitioning of Quaternion subalgebra non-scalar basis triplets, leaving the other 29 ways to do this behind. There is not much, and at the same time too much, to see considering 480 different Octonion Algebra multiplication tables. The meat on the bone is fully appreciated after choosing one of 30 ways to partition the seven Quaternion subalgebra triplets, then fully immersing thought in the resultant 480/30 = 16 Octonion Algebras. The 30 ways are isomorphic representations, nothing more than aliases of each other.

While on the subject, the oft used statement that all 480 Octonion Algebra representations are isomorphic is patently false. No automorphism exists between Left and Right Octonion orientations, the morph is anti-automorphism. *Any* non-symmetric basis element multiplication table and its transpose are distinctly different in structure. The group PSL(2,7) gives us the full complement of reversible automorphisms between any two Left orientations, and between any two Right orientations. Thus, all Left orientations are separately isomorphic, and all Right orientations are separately isomorphic. The fact that all are division algebras is neither here nor there, irrelevant to the question of isomorphism.

Let's look now at the optimal indexing for the seven optimally partitioned Quaternion subalgebra triplets for Octonion Algebra. Since at this point, we do not care about which orientation is used, the orientation non-specific index set representation {a b c} is used instead of the orientation specifying ordered triplet rule (a b c) or (c b a). Itemize as follows:

Triplet[1] = {2 4 6}
Triplet[2] = {1 4 5}
Triplet[3] = {3 4 7}
Triplet[4] = {1 2 3}
Triplet[5] = {2 5 7}
Triplet[6] = {1 6 7}
Triplet[7] = {3 5 6}

The interesting thing about this enumeration is if we look at three selections from the sets Triplet[] indexed by integers matching the indexes for some Quaternion subalgebra triplet, there is a single index common to all three triplets, and it matches the enumeration n of Triplet[n] describing the three chosen values. For instance, looking at Triplet[1], Triplet[2] and Triplet[3], the common index is 4, and Triplet[4] = {1 2 3}. This works for any triple/triplet.

With this in mind, examine the systematic form for H_3 labeling the rows and columns with indexes 0 through 7 sequentially top to bottom and left to right. For non 0 indexed row(column) n we will find the +1 entries show up in the three columns(rows) indexed in Triplet[n]. One could then say the Hadamard matrix form H_3 sources the structures Triplet[n], relevant for both Right and Left Octonion Algebras due to the row/column symmetry. The connection between these triplets and Hadamard matrices will provide the segue to the Hadamard matrix relationship between Quaternion subalgebra triplet orientations and all possible Octonion Algebra orientations, leading then to the important next step consideration of Octonion Algebraic Invariance/Variance where sets of product terms from any complexity sequence of algebraic element products can be sorted such that all set members will either never change sign (algebraic invariants), or based on the particular Octonion Algebra change will either change sign or remain the same sign (algebraic variants), when the orientation of the applied Octonion Algebra is changed up. More on this later on.

Now look again at the **L** $\mathbb{O}$ table

L $\mathbb{O}$	e_0	e_t	e_u	e_v	e_w	e_x	e_y	e_z
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_1	+1	–1	+1	–1	+1	–1	+1	–1
e_2	+1	+1	–1	–1	+1	+1	–1	–1
e_3	+1	–1	–1	+1	+1	–1	–1	+1
e_4	+1	+1	+1	+1	–1	–1	–1	–1
e_5	+1	–1	+1	–1	–1	+1	–1	+1
e_6	+1	+1	–1	–1	–1	–1	+1	+1
e_7	+1	–1	–1	+1	–1	+1	+1	–1

The row labeled e_1 is a representation of all left products by e_1, and the +1 values in columns e_u, e_w, and

e_y imply the ordered triplet product orientations ($e_{u\hat{}1}$ e_1 e_u), ($e_{w\hat{}1}$ e_1 e_w) and ($e_{y\hat{}1}$ e_1 e_y). Since we know we are dealing with a Left Octonion Algebra, e_u, e_w, and e_y do not form a Quaternion subalgebra triplet, and $e_{u\hat{}1}$, $e_{w\hat{}1}$, and $e_{y\hat{}1}$ do. Therefore e_1, e_u, e_w, and e_y form a basic quad whose indexes xor to zero. From this, we have u^w^y = 1. The remaining four column entries with –1 must include e_1 specifying its square, and three more that are the Quaternion subalgebra triplet $e_{u\hat{}1}$, $e_{w\hat{}1}$, and $e_{y\hat{}1}$ whose entries xor to 0. Therefore, we have t^v^x^z = 1. Doing the same on the other rows gives us seven restrictions:

1 = u^w^y = t^v^x^z	1 is one of t, v, x, or y
2 = t^w^x = u^v^y^z	2 is one of u, v, y or x
3 = v^w^z = t^u^x^y	3 is one of t, u, x or y
4 = t^u^v = w^x^y^z	4 is one of w, x, y or z
5 = u^x^z = t^v^w^y	5 is one of t, v, w or y
6 = t^y^z = u^v^w^x	6 is one of u, v, w or x
7 = v^x^y = t^u^w^z	7 is one of t, u, w or z

Assign t = 1. From t^v^x^z = 1 we have v^x^z = 0, a Quaternion subalgebra triplet, and the only one in which indexes v and z will appear. Then from u^v^y^z = 2, we can state that neither u or y can equal 2, and therefore either v or z = 2. Similarly, from w^x^y^z = 4, neither w or y can equal 4, and therefore either x or z = 4. From u^v^w^x = 6, neither u or w can equal 6, and therefore either v or x = 6. If we now pick one of two choices for index v, then x and z will be determined.

If we assign v = 2 with t = 1, then x = 6 and z = 4. Then from t^w^x = 2, 1^w^6 = 2 thus w = 5. From t^u^v = 4 we have u = 7, and from t^y^z = 6 we have y = 3. So [t,u,v,w,x,y,z] = [1,7,2,5,6,3,4].

If we assign v = 6 with t = 1, then x = 4 and z = 2. Then from t^w^x = 2 we have w = 7. From t^u^v = 4 we have u = 3, and from t^y^z = 6 we have y = 5. So [t,u,v,w,x,y,z] = [1,3,6,7,4,5,2].

Both of these [t,u,v,w,x,y,z] assignments are found in the table of eight above. Let's continue with the 1 = t^v^x^y restriction by assigning 1 to each of the other three possibilities in turn to generate the remaining assignments.

Assign v = 1. From t^v^x^z = 1 we have t^x^z = 0. Then from t^u^x^y = 3, we can state that neither u or y can equal 3, and therefore either t or x = 3. Similarly, from w^x^y^z = 4, neither w or y can equal 4, and therefore either x or z = 4. From t^u^w^z = 7, neither u or w can equal 7, and therefore either t or z = 7.

Assign t = 3 for v = 1. Then z = 7 and x = 4. From v^w^z = 3, w = 5. From t^u^v = 4, u = 6. From v^x^y = 7, y = 2. Thus [t,u,v,w,x,y,z] = [3,6,1,5,4,2,7]

Assign t = 7 for v = 1. Then z = 4 and x = 3. From v^w^z = 3, w = 6. From t^u^v = 4, u = 2. From v^x^y = 7, y = 5. Thus [t,u,v,w,x,y,z] = [7,2,1,6,3,5,4]

Assign x = 1. From t^v^x^z = 1 we have t^v^z = 0. Then from u^v^y^z = 2, we can state that neither u or y can equal 2, and therefore either v or z = 2. Similarly, from t^v^w^y = 5, neither w or y can equal 5, and therefore either t or v = 5. From t^u^w^z = 7, neither u or w can equal 7, and therefore either t or z = 7.

Assign t = 5 for x = 1. Then z = 7 and v = 2. From t^w^x = 2, w = 6. From u^x^z = 5, u = 3. From v^x^y = 7, y = 4. Thus [t,u,v,w,x,y,z] = [5,3,2,6,1,4,7]

Assign t = 7 for x = 1. Then z = 2 and v = 5. From v^w^z = 3, w = 4. From u^x^z = 5, u = 6. From v^x^y = 7, y = 3. Thus [t,u,v,w,x,y,z] = [7,6,5,4,1,3,2]

Assign z = 1. From t^v^x^z = 1 we have t^v^x = 0. Then from t^u^x^y = 3, we can state that neither u or y can equal 3, and therefore either t or x = 3. Similarly, from t^v^w^y = 5, neither w or y can equal 5, and therefore either t or v = 5. From u^v^w^x = 6, neither u or w can equal 6, and therefore either v or x = 6.

Assign t = 3 for z = 1. Then v = 5 and x = 6. From v^w^z = 3, w = 7. From u^x^z = 5, u = 2. From t^y^z = 6, y = 4. Thus [t,u,v,w,x,y,z] = [3,2,5,7,6,4,1]

Assign t = 5 for z = 1. Then v = 6 and x = 3. From v^w^z = 3, w = 4. From u^x^z = 5, u = 7. From t^y^z = 6, y = 2. Thus [t,u,v,w,x,y,z] = [5,7,6,4,3,2,1]

As can be seen, all eight assignments found by validating every permutation of [1,2,3,4,5,6,7] for [t,u,v,w,x,y,z] are produced through Boolean algebra expressions. I can't imagine deriving this without the aid of the xor relationships, although it would be in theory possible using any of the other 29 isomorphic Quaternion subalgebra partitioning schemes.

There are additional puzzle piece tidbits of interesting information from all of this concerning Octonion Algebras. Go back and look at the {t, u, v, w, x, y, z} assignment table replacing {t, u, v, w, x, y, z} with {1, 2, 3, 4, 5, 6, 7} to express the basis element index permutations of {1, 2, 3, 4, 5, 6, 7} defining each Octonion orientation Hadamard matrix representation.

	1	2	3	4	5	6	7
L/R0	7	6	5	4	1	3	2
L/R1	5	3	2	6	1	4	7
L/R2	1	3	6	7	4	5	2
L/R3	1	7	2	5	6	3	4
L/R4	5	7	6	4	3	2	1
L/R5	3	2	5	7	6	4	1
L/R6	3	6	1	5	4	2	7
L/R7	7	2	1	6	3	5	4

Every column n does not include the indexes found in Triplet[n], it is 2 up on its basic quad indexes. With a little work one can see that within any single row, forming the xor of the indexes in the three columns found in each Triplet[n] results in the value n.

This table also has an interesting connection to the notion of ordered 9-tuples defined and discussed/used in detail within reference [1]. Briefly for Left and Right Octonion orientations

Left ordered 9-tuple

$(e_c\ e_d\ e_g)$

$\downarrow(e_b\ e_d\ e_f)$ specifies $(e_c\ e_b\ e_a)$, $(e_c\ e_d\ e_g)$, $(e_b\ e_d\ e_f)$, $(e_a\ e_d\ e_e)$, $(e_e\ e_f\ e_c)$, $(e_f\ e_g\ e_a)$, $(e_g\ e_e\ e_b)$

$(e_a\ e_d\ e_e)$

Right ordered 9-tuple

$(e_e\ e_d\ e_a)$

$(e_f\ e_d\ e_b)\downarrow$ specifies $(e_a\ e_b\ e_c)$, $(e_e\ e_d\ e_a)$, $(e_f\ e_d\ e_b)$, $(e_g\ e_d\ e_c)$, $(e_g\ e_f\ e_a)$, $(e_f\ e_e\ e_c)$, $(e_e\ e_g\ e_b)$

$(e_g\ e_d\ e_c)$

From the Quaternion subalgebra triplet orientations one can see the map between is an anti-automorphism, they are representations of Ln and Rn for some n. Centrally located basis element e_d is referred to as the cardinal basis element, and the Quaternion subalgebra triplet on the down arrow side is referred to as the cardinal triplet. Each Octonion Algebra orientation has a bijective map pairing the seven Quaternion subalgebra triplets in cardinal triplet position to the seven non-scalar basis elements in cardinal basis element position, as shown in the following table

	L/R0	**L/R1**	**L/R2**	**L/R3**	**L/R4**	**L/R5**	**L/R6**	**L/R7**
$\{e_6\ e_4\ e_2\}$	e_7	e_5	e_1	e_1	e_5	e_3	e_3	e_7
$\{e_5\ e_4\ e_1\}$	e_6	e_3	e_3	e_7	e_7	e_2	e_6	e_2
$\{e_7\ e_4\ e_3\}$	e_5	e_2	e_6	e_2	e_6	e_5	e_1	e_1
$\{e_1\ e_2\ e_3\}$	e_4	e_6	e_7	e_5	e_4	e_7	e_5	e_6
$\{e_5\ e_7\ e_2\}$	e_1	e_1	e_4	e_6	e_3	e_6	e_4	e_3
$\{e_7\ e_6\ e_1\}$	e_3	e_4	e_5	e_3	e_2	e_4	e_2	e_5
$\{e_6\ e_5\ e_3\}$	e_2	e_7	e_2	e_4	e_1	e_1	e_7	e_4

Within this table if we replace all Triplet[n] row labels with their n values, replace all table entry basis elements with just their indexes, then transpose the table, we reproduce the index permutation assignment table just above.

The map between Octonion Algebras Lm and Ln as well as between Rm and Rn is the involution negating the orientations of all four Quaternion subalgebras that do not include the basis element $e_{m\char`^n}$. Equivalently stated, all basis element product rules including $e_{m\char`^n}$ are unchanged. Not surprising then, if we look at two rows of the index permutation assignment table, L/Rm and L/Rn, the index m^n will appear in the same column. This would clearly depend on how the separate Octonion orientations are enumerated. Once again, the ubiquitous connection between Hadamard matrices and division algebras comes into play.

We previously covered the Hadamard connection first to division algebra basis elements, then the Quaternion subalgebra triplets. Since the orientation of any Octonion Algebra is fully specified by the orientations of its Quaternion subalgebras, we might expect a Hadamard matrix correspondence between the triplets and the proper Octonion Algebra orientations. Review the following table

$\mathbb{O}$	L/R0	L/R1	L/R2	L/R3	L/R4	L/R5	L/R6	L/R7
not {}	+1	+1	+1	+1	+1	+1	+1	+1
$\{e_2\ e_4\ e_6\}$	+1	–1	+1	–1	+1	–1	+1	–1
$\{e_1\ e_4\ e_5\}$	+1	+1	–1	–1	+1	+1	–1	–1
$\{e_3\ e_4\ e_7\}$	+1	–1	–1	+1	+1	–1	–1	+1
$\{e_1\ e_2\ e_3\}$	+1	+1	+1	+1	–1	–1	–1	–1
$\{e_2\ e_5\ e_7\}$	+1	–1	+1	–1	–1	+1	–1	+1
$\{e_1\ e_6\ e_7\}$	+1	+1	–1	–1	–1	–1	+1	+1
$\{e_3\ e_5\ e_6\}$	+1	–1	–1	+1	–1	+1	+1	–1

The last seven Hadamard matrix rows are labeled with Triplet[n] where n is sequentially top to bottom 1 through 7. This allows a meaningful sequential indexing on the Octonion Algebra orientations. Since we have not declared meaning to any yet, the triplet labels are represented with the non-oriented standard {}. The first row is the representation of all products that will not change if the full algebra orientation is changed up, essentially those not governed by any triplet orientation choice. Hence all entries in its row are +1 across all possible Octonion Algebras.

At this point, we have a free choice of what to call either R0 or L0, since choosing one defines the other. Then we can build a Left only or Right only Octonion table by changing the {} row labels to their oriented choices $(e_a\ e_b\ e_c)$ appropriate for our selection for L0 or R0, then removing one of L/R in the column labels. Now we can interpret the meaning of the Hadamard matrix ±1 entries for this connection. The L/R0 column contains all +1 entries, and this is stating the orientations we now have for row labels are those for L/R0 by virtue of this fact. For the remainder of the Hadamard matrix portion, for the algebra selected by the column label, in that algebra the orientation for the Quaternion subalgebra triplet labeling a row is the same as in either L/R0 if the intersection is +1, and is its negation if the intersection is –1.

This Hadamard matrix correspondence between Quaternion subalgebra triplets and Octonion Left or Right Algebras builds all 16 proper Octonion orientations, once again separately for Left and Right algebras. They are given as follows

$\mathbb{O}$	L0	L1	L2	L3	L4	L5	L6	L7
not {}	+1	+1	+1	+1	+1	+1	+1	+1
(e_2 e_4 e_6)	+1	–1	+1	–1	+1	–1	+1	–1
(e_1 e_4 e_5)	+1	+1	–1	–1	+1	+1	–1	–1
(e_3 e_4 e_7)	+1	–1	–1	+1	+1	–1	–1	+1
(e_3 e_2 e_1)	+1	+1	+1	+1	–1	–1	–1	–1
(e_2 e_7 e_5)	+1	–1	+1	–1	–1	+1	–1	+1
(e_1 e_6 e_7)	+1	+1	–1	–1	–1	–1	+1	+1
(e_3 e_5 e_6)	+1	–1	–1	+1	–1	+1	+1	–1

$\mathbb{O}$	R0	R1	R2	R3	R4	R5	R6	R7
not {}	+1	+1	+1	+1	+1	+1	+1	+1
(e_6 e_4 e_2)	+1	–1	+1	–1	+1	–1	+1	–1
(e_5 e_4 e_1)	+1	+1	–1	–1	+1	+1	–1	–1
(e_7 e_4 e_3)	+1	–1	–1	+1	+1	–1	–1	+1
(e_1 e_2 e_3)	+1	+1	+1	+1	–1	–1	–1	–1
(e_5 e_7 e_2)	+1	–1	+1	–1	–1	+1	–1	+1
(e_7 e_6 e_1)	+1	+1	–1	–1	–1	–1	+1	+1
(e_6 e_5 e_3)	+1	–1	–1	+1	–1	+1	+1	–1

These two tables give clear visibility to the difference between Octonion orientations of like chirality. For any Lm(Rm) and Ln(Rn), the involution mapping between them negates the four Quaternion subalgebra triplets that do not include $e_{m\wedge n}$. Within the Left or Right Octonion table, there are no maps between that negate only the three triplets that include a common basis element. This can however be generated by first negating all seven triplet orientations, thus popping between Left and Right tables, and then negating the four triplets that do not include a specified basis element. This gives us the involution map between Lm(Rm) and Rn(Ln) negating the three Quaternion subalgebra triplets that include $e_{m\wedge n}$. For many years now, I have called this the 3:4 Morph Rule: any morph not one of these two involutions or composites thereof will produce an invalid Octonion Algebra.

The final Hadamard Matrix connection to be discussed is the concept of Octonion Algebraic Invariance and Variance. This is discussed in detail within references [1] [3] [4] and its ultimate physics connection: the conservation of energy and momentum, is discussed reference [5]. Rather than repeating, I will just hit the highlights here. If we form the product of two different non-scalar basis elements we have $e_a * e_b = \pm e_c$. The result sign will be determined by the orientation choice used for the Quaternion subalgebra triplet {e_a e_b e_c}. We could say that this product parks us on the triplet-L/R Hadamard table row with label {e_a e_b e_c}. If we then multiply on the left by basis element e_d we will have $e_d * (\pm e_c) = \pm e_e$. The sign of the final result will be the product of the sign from the first product and the sign provided by the orientation choice for {e_c e_d e_e}. This sign product is precisely the Hadamard matrix row composition discussed above, so the second multiplication re-parks us on the row that is the composition of rows {e_a e_b e_c} and {e_c e_d e_e}, which will be the third and final row including e_c. This row tells us precisely within Left or within Right Octonion Algebra what will happen to the sign of the expression $e_d * (e_a * e_b) = \pm e_e$ when we change up the algebra. Since we must include the possibility that a change may be between Left and Right orientations, and this is the map negating all triplet orientations, if we used an odd number of triplet rules in a basis element product string the final result will change sign moving between Left and Right, and will remain the same sign if an even number was used. This process yields 16 different endpoints, the one row labeled not {} and the seven labeled $\{e_x\ e_y\ e_z\}$ doubled by the parity of the triplet rule application count. Since the row composition is closed, any number of basis element products may be included in the creation of the final result. Every different string of products (product histories) ending up on the same final row/parity will either all change sign or all not change sign when the orientation of the applied Octonion Algebra is changed up. If we end up on the not {} row through an even number of triplet rule applications, the result will not change sign for any Octonion orientation change. These product histories are algebraic invariants. All other terminal states are algebraic variants that may or may not change sign when the Octonion orientation is changed, but all in a given state will change or not in unison.

If we individually assign a value of zero to the sum of all product terms in each variant final state, the full Octonion Algebra result for any number of algebraic element products and sums of algebraic element products will be an algebraic invariant since +0 = –0. Each populated variant final state will then yield a homogeneous equation of algebraic constraint describing non-observable phenomenon. The members of the intrinsic algebraic invariant final state will describe all observable phenomenon since it just would not do to allow a sign change for an observation solely based on a choice of proper Octonion orientation.

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