Critical line of nontrivial zeros of Riemann zeta function $\zeta(s)$

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Abstract

In this paper, we find a curious and simple possible solution to the critical line of nontrivial zeros in the strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$ of Riemann zeta function $\zeta(s)$. We show that exists $s_\sigma \in \mathbb{C}$ such that if $\{s_\sigma = \sigma + it : (\sigma \in \mathbb{R}, 0 < \sigma < 1); (\forall t \in \mathbb{R})\}$ with i as the imaginary unit, then exactly satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0 \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

Therefore, all the nontrivial zeros lie on the critical line $\{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\}$ consisting of the set complex numbers $\frac{1}{2} + it$, thus confirming Riemann's hypothesis.

1 Introduction.

There is a large and extensive bibliography on the Riemann zeta function and its zeros, so we will not go into further details of it. Basically, Riemann zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by the absolutely convergent infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Leonhard Euler already considered this series for real values of s. He also proved that it equals the Euler product:

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

where the infinite product extends over all prime numbers p. However, we can also define the Riemann zeta function Eq.(1) as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad \Rightarrow \quad \zeta(s) = \frac{1}{2^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \right)$$

Which can also be expressed as:

$$\zeta(s) = \frac{1}{2^s} \left[\zeta(s) + B(s) \right] \iff B(s) = \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^s} \tag{2}$$

Thus, by Eq.(2) we can definitely express the Riemann zeta function as:

$$\zeta(s) = (2^s - 1)^{-1} B(s) \tag{3}$$

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As is well known, the Riemann zeta function $\zeta(s)$ and the Dirichlet eta function $\eta(s)$ satisfy the relation:

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s) \tag{4}$$

Thus, by Eq.(3) we can now express the Dirichlet eta function as:

$$\eta(s) = \left(\frac{1 - 2^{1-s}}{2^s - 1}\right) B(s) \tag{5}$$

2 Proof.

By Eq.(2), Eq.(4) and Eq.(5) we can obtain:

$$2^{1-s} = 1 - \frac{\eta(s)}{\zeta(s)} = 2^s \cdot \frac{\zeta(s) - \eta(s)}{\zeta(s) + B(s)} \quad \Rightarrow \quad 2^{1-2s} = \frac{\zeta(s) + \left(\frac{2^{1-s}-1}{2^s-1}\right) B(s)}{\zeta(s) + B(s)} \tag{6}$$

However, exists $s_\sigma \in \mathbb{C}$ such that $\{s_\sigma = \sigma + it : (\sigma, t) \in \mathbb{R}\}$ with i as the imaginary unit, such that exactly satisfy:

$$\lim_{s \to s_\sigma} \zeta(s) = \zeta(s_\sigma) = 0$$

Therefore, calculating $(\lim_{s \to s_\sigma})$ in Eq.(6), we obtain:

$$\lim_{s \to s_\sigma} \left(2^{1-2s}\right) = \lim_{s \to s_\sigma} \frac{\zeta(s) + \left(\frac{2^{1-s}-1}{2^s-1}\right) B(s)}{\zeta(s) + B(s)} \quad \Rightarrow \quad 2^{1-2s_\sigma} = \frac{\left(\frac{2^{1-s_\sigma}-1}{2^{s_\sigma}-1}\right) B(s_\sigma)}{B(s_\sigma)}$$

However, since $[B(s_\sigma) \to 0 \iff \zeta(s_\sigma) \to 0]$ by Eq.(3), we obtain an indeterminacy of the type $\frac{0}{0}$. Then by successive applications of the L'hôpital rule until any nth derivative $B^{(n)}(s_\sigma) \neq 0$, that is: $(\forall j < n : B^{(j)}(s_\sigma) = 0)$, we obtain:

$$2^{1-2s_\sigma} = \frac{\left(\frac{2^{1-s_\sigma}-1}{2^{s_\sigma}-1}\right) B^{(n)}(s_\sigma)}{B^{(n)}(s_\sigma)} \qquad \Rightarrow \qquad 2^{1-2s_\sigma} = \frac{2^{1-s_\sigma} - 1}{2^{s_\sigma} - 1}$$

As $s_\sigma = \sigma + it$ then obtaining common factor 2^{-it} in numerator and 2^{it} in denominator of the fraction, we can express:

$$2^{1-2s_\sigma} = 2^{-2it} \cdot \frac{2^{1-\sigma} - 2^{it}}{2^\sigma - 2^{-it}}$$

Now, defining $s_0 \in \mathbb{C}$ such that $s_0 = \frac{1}{2} + it$, we can express previous equation as:

$$2^{2(s_\sigma - s_0)} = \frac{2^\sigma - 2^{-it}}{2^{1-\sigma} - 2^{it}} \tag{7}$$

Since by definition $s_\sigma = \sigma + it$ and $s_0 = \frac{1}{2} + it$ then $2(s_\sigma - s_0) = 2\sigma - 1$. Thus, developing in trigonometric form $2^{it} = e^{itln2}$ and $2^{-it} = e^{-itln2}$, we obtain:

$$2^{(2\sigma-1)} = \frac{2^\sigma - cos(tln2) + isen(tln2)}{2^{1-\sigma} - cos(tln2) - isen(tln2)} \tag{8}$$

since obviously as we know $cos(-x) = cos(x)$. Thus, by simplifying we have:

$$2^{(2\sigma-1)} = \frac{cos(tln2) - isen(tln2)}{cos(tln2) + isen(tln2)}$$

which by application of modulus, that is:

$$\left|2^{(2\sigma-1)}\right| = \left|\frac{cos(tln2) - isen(tln2)}{cos(tln2) + isen(tln2)}\right| \quad \Rightarrow \quad \left|2^{(2\sigma-1)}\right| = \frac{|cos(tln2) - isen(tln2)|}{|cos(tln2) + isen(tln2)|}$$

since for any $\{z \in \mathbb{C} : |z| = |\overline{z}|\}$ we definitely obtain:

$$\left|2^{(2\sigma-1)}\right| = 1 \quad \Rightarrow \quad 2\sigma - 1 = 0 \quad \Rightarrow \quad \sigma = \frac{1}{2}$$

Therefore, since by definition $s_\sigma = \sigma + it$, we obtain that for :

$$\zeta(s_\sigma) = 0 \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

Exactly, by Eq.(7) and Eq.(8) for $\sigma = \frac{1}{2}$ we can verify:

$$\left|2^{2(s_\sigma - s_0)}\right| = \frac{\left|\left(2^{\frac{1}{2}} - cos(tln2)\right) + isen(tln2)\right|}{\left|\left(2^{\frac{1}{2}} - cos(tln2)\right) - isen(tln2)\right|} = 1$$

Thus, since by definition $s_0 = \frac{1}{2} + it$, we have:

$$2(s_\sigma - s_0) = 0 \quad \Rightarrow \quad s_\sigma = s_0 \quad \Rightarrow \quad s_\sigma = \frac{1}{2} + it$$

Thus, all the nontrivial zeros lie on the critical line $\{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\}$ consisting of the set complex numbers $\frac{1}{2} + it$, thus confirming Riemann's hypothesis.