PYTHAGOREAN TRIPLES AND SQUARE PELL NUMBERS

Julian Beauchamp

ABSTRACT. In this short paper I demonstrate a simple connection between primitive Pythagorean triples of the form $\{X, Y, Z = Y + 1\}$ and the squares of the Pell Numbers. I conjecture that when X is equal to one of the numerators of continued fraction convergents to $\sqrt{2}$, then and only then can Y or Z be a square, and only then a square Pell Number.

Introduction

For any primitive Pythagorean triple of the form $(X, Y, Z = Y + 1)$, it appears that Y or Z will always, and apparently only be a square Pell number if X is a member of the number sequence (A001333).

Let $a_{(n)}$ be the numerators of continued fraction convergents to $\sqrt{2}$ (A001333), where $a_{(0)} = 0$, such that:

$$a_{(n)} = 1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119...$$

The closed formula for this sequence is:

$$\frac{(1-\sqrt{2})^n + (1+\sqrt{2})^n}{2}.$$

Also let $b^2_{(n)}$ be the square Pell numbers (A079291), where $b^2_{(0)} = 0$, such that:

$$b^2_{(n)} = 0, 1, 4, 25, 144, 841, 4900, 28561, 166464, 970225, 5654884, ...$$

When $X = a_{(n)}$ $(n > 1)$, then if n is even, Y is always a square Pell number, and if n is odd, then Z is always a square Pell number. For example:

$$(3, \mathbf{4}, 5), (7, 24, \mathbf{25}), (17, \mathbf{144}, 145), (41, 840, \mathbf{841}), (99, \mathbf{4900}, 4901)...$$

Thus, when $X = a_{(n)}$ $(n > 1)$, then
$(X, Y, Z) = (a_{(n)}, b^2_{(n)}, b^2_{(n)} + 1)$ when n is even, or:
$(X, Y, Z) = (a_{(n)}, b^2_{(n)} - 1, b^2_{(n)})$, when n is odd.

As a closed formula, $(X, Y, Z) =$

$$\left(\frac{(1-\sqrt{2})^n + (1+\sqrt{2})^n}{2}, \frac{[(1-\sqrt{2})^n + (1+\sqrt{2})^n]^2 - 4}{8}, \frac{[(1-\sqrt{2})^n + (1+\sqrt{2})^n]^2 + 4}{8}\right).$$

I conjecture that Y and Z can only be squares when $X = a_{(n)}$ $(n > 1)$.

THE RECTORY, VILLAGE ROAD, WAVERTON, CHESTER CH3 7QN, UK
Email address: julianbeauchamp47@gmail.com

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