# AN ALTERNATIVE MODEL FOR SPACE

**ADRIAAN VAN DER WALT**

*In Science and Philosophy, all new ideas and the reasoning and evidence in their support are entitled to testing and scrutiny of debate and informed and qualified review and the courteous judgement of those qualified by knowledge and skill to give it.*

In this spirit I dedicate this compendium, as I did originally with DIVISION THREE, to PARMENIDES OF ELEA with the wish that this time the debate will end in his favour.

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# Foreword and Introduction

This compendium of my writings is somewhat like a travelogue emulating the travels of the princes of Serendib.

My journey started when I was fortunate enough to study at the University of Potchefstroom from 1960. This was the time when Academia was on the cusp of changing from places that defined themselves as “places of learning” into places that defined themselves as “places where knowledge is generated and sold”. The vice chancellor at that time was an educationist of international standing and the University resisted the trend by introducing a compulsory course which exposed me to basic Philosophy, the Philosophy of Science and the History of the Natural- and Mathematical Sciences. My post graduate study was in Physics. This turned me into a scientist because my natural inclination is that of an Epicurean which made me to automatically embrace the Scientific Method as it is described in DIVISION ONE. I then drifted into Mathematics and Applied Mathematics (Mechanics in those days). When I was 41 years old I returned to Academia to teach Mathematics at the University of Pretoria in South Africa.

My real quest started while reading up on the Foundations of Mathematics at that time and I realised that while Cantor[1] used subscripted variables to prove that it is not possible to make a list of all real numbers between zero and one, I can make that same list in the same notation by using the actual digits instead of subscripted variables and so to generate a list of all possible infinite permutations of strings of digits. This list was countable. An apparent contradiction like this made me turn to the scientific method to look for the cause of the discrepancy. However, I also realised that Cantor’s proof has been accepted by all Mathematicians for more than a century, which implied overwhelmingly that it was I who must be in the wrong. I then classified myself as just another “crackpot” and distanced myself from the practice of Mathematics as well as from the mathematical community while I earned my keep by doing more than my share of teaching.

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1 **Georg Ferdinand Ludwig Philipp Cantor** February 19, 1845 – January 6, 1918

In December of 1999 I constructed Example A of DIVISION THREE, PART ONE Section 1. This convinced me that I was not a “crackpot”, and that Cantor’s proof is simplistic. I considered his proof to be simplistic because Cantor assumed that an infinite vector that differs on the main diagonal from any given row vector in an infinite matrix, differs from all row vectors in the matrix.  This is a direct generalisation of a property of finite dimensional matrices which is not valid for matrices with more rows than columns. The infinite matrix of Example A is the limit of finite matrices where the number of rows is many more than the number of columns. But my colleagues did not share my opinion, and most of them would not even listen to me.

At first I focussed on finding the fallacy in Cantor’s argument itself. At last I concluded that the most likely error was this: Cantor proved that a list of all real numbers cannot be made. From this he concluded that there must be more than countable many real numbers. But an equally valid conclusion from his proof is that real numbers cannot be listed at all, i.e. a list of one real number cannot be made. An investigation into this possibility led to the analysis in section 2 of DIVISION TWO which led in turn to the introduction of the classification of real numbers either as value specified or as component specified, and everything that followed from that.

The origin of the expression “elephant in the room” is a Russian story of a man in a museum who was so intrigued by the small exhibits that he never noticed the presence of an elephant in the hall. Therefore the real serendipitous moment in my thinking happened when a colleague, in exasperation, one day told me that I did not understand what Mathematics is. After consideration I realised that he was right, and at that moment the Elephant came into view.

I was intent on looking for the discrepancy in the details of Cantor’s argument because at that time I considered Mathematics to be a science and as such subjected to the scientific method. Thus I was looking for the assumption in the argument that would clear up the discrepancy when changed. But Mathematics originated in Philosophical Antiquity, as discussed in DIVISION ONE, and therefore its assumptions are considered to be fixed and true. Apart from Geometry and numbers, Mathematics is not subjected to the constraints of how we experience reality, and therefore Mathematics is not a science. Hence it is not subjected to the Scientific Method. It therefore never needed nor experienced “reality checks”[2] like those that Alchemy got from Mendeleev and Astrology got from Kepler.

It is simplest to consider Mathematics to be a logical structure built on fundamental philosophical (or Metaphysical) assumptions about Space and Numbers. Thus different but equally valid assumptions in this regard could have been made while Mathematics as such remains “true”.

The elephant, when at last I saw it clearly, was that two different assumptions about the nature of space and hence two different sub-models based on these assumptions, could and should be formed for Mathematics. In the end it turned out that Cantor’s proof belongs to one model, called here the Euclidean Model (Based on assumptions about Space called here the Euclidean Cosmology), and that my list belongs to a different model, called here the Leibnitzean Model (Based on assumptions about Space called here the Leibnitzean Cosmology). The conclusions about the two lists of numbers are therefore not comparable and each is correct in its own model.

This insight also revealed that Infinitesimals and Calculus, as perceived today, are the uncomfortable embedding of Leibnitz’ work, which is fundamentally continuous, in the Euclidean sub-model, which is fundamentally discrete.

Then it dawned on me that the implications of my insight are even wider and that it extends to Metaphysics – in particular Ontology - where it has a revolutionary impact. While traditional Metaphysics assumes that whatever is continuous is composed of discrete entities called points, my work does not assume the existence of points but assumes conversely that whatever is discrete can be analysed using continuous entities (volumes, areas or lines).

This compendium constitutes some form of “reality check” for Mathematics: Apart from the baby elephant called here the “Cauchy numbers” that were introduced as an extension to the number system for use in the Leibnitzean Cosmology, there is almost nothing of

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2 Although Mathematics is not a science, the Mathematical Sciences, like Mathematical Physics, are. A reality check for the Mathematical Sciences - according to the scientific method - means that contradictions should not only be checked against physical assumptions, but should also be checked against Mathematical assumptions. It turned out that the particle/wave duality of Mathematical Physics does not pose a fundamental problem when using the Leibnitzean Cosmology and the Metaphysics of Parmenides of Elea.

consequence that is presented in these documents that is anything else but a rearrangement of known Mathematics to suit the constraints of the two cosmologies, with a few new definitions added to assign names to entities that were not named before.

It is unfortunate that I now have to ask the reader's indulgence with how these documents are presented. I was born in 1942 which now puts me in my seventies. I am acutely aware that whatever is in my head will vanish when my mind is lost in the not-so-distant future either through death or through senility, and that anything that is not printed on proper paper will therefore be gone within a few years. This lends an urgency to my writing and this, together with the loss of clarity that comes with age, may make these writings onerous to read. Furthermore, many of the thoughts shared here are still not that far removed from the intuitive level and will in future need a lot of clarification by minds better than mine. Therefore please look on reading these writings as a serendipitous journey for the reader too.

It was also never possible to find a peer with whom I could argue my ideas in a proper academic manner. This means that I can claim all the credit for myself, but I also have to assume sole responsibility for all the mistakes. Therefore all the mistakes that were not discovered by myself are still in the script and in the logic. Please look on this as a challenge to be overcome. However, the positive result of not being able to interact with peers was that I was never distracted and I could fit the progress of more than one lifetime into one.

Chronologically, the article presented in DIVISION THREE was posted on VIXRA in January 2015 when all my various thoughts finally gelled into one coherent structure. I posted it in the hope of eliciting criticism – which did not happen – and also to have the ideas posted somewhere where they can be accessed by others. It was only after 2015 that I clearly saw the elephant in the room and my focus moved from discrediting Cantor's proof to formulating the new alternative model for Space (Surprisingly, it turned out that the first part of DIVISION THREE, which was a critique of Cantor's work, became an argument for the existence of two sub-models in Mathematics). I then wrote a second article presented in DIVISION TWO in order to express my thinking after this change of focus and citing the relevant arguments and examples.

Later, after wasting the time of two very able colleagues, I realised that it is not fair of me to expect from anybody to extract from my work in one hour what took me almost 40 years to see. I also realised that a paradigm drift in my thinking was the main impediment to successful communication – I was talking in an altered paradigm. I therefore wrote a third document presented in DIVISION ONE in an effort to describe this paradigm and make it easier for others to follow my thoughts.

I therefore recommend reading DIVISION ONE first and then DIVISION TWO referring to DIVISION THREE where some of the ideas are presented more fully though immature, and then read DIVISION THREE. A complete description of my original thinking is presented in DIVISION THREE which I inserted here in full and unaltered – apart from changing some outlay and spelling - as it was when first published on VIXRA[3] in 2015.

As noted above, my ideas matured more after 2015. Since then I was trying to get them into a form in which they can be preserved and shared. Because of my declining mental abilities it was not possible to combine all my ideas into a single well-ordered document, so that I finally decided to simply put everything that I have written in a single compendium with the intention to print it on paper. The price to be paid for this is *ad nauseam* repetition.

In the process of getting my thoughts onto paper the friend of my old age in my new country, Trevor Roberts, was of inestimable[4] help. Seeing that he is a lawyer, it is appropriate that I thank him for aiding and abetting me. I also thank him for the quote from a radio discussion of his used on the title page at the dedication to Parmenides of Elea.

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3 It can be found on the link http://viXra.org/abs/1501.0153
4 An expression that is trite but, in this case, literally true.

# In A Nutshell

### An Alternative Model for Space and the Consequential Extension of the Number System

When intelligence dawned on the human race, the harshness of existence and the lack of understanding of how things happen must have been overwhelming. Somewhere in the chaos of this perceived randomness the early humans would have seen the vague shadows of recurring patterns and then transformed them into superstitions. Like a baby learning to speak, these superstitions were guesses about causes and consequences and, as experience increased, these assumptions were probably refined. The scientific method, which emerged millennia later, could have had its roots in this original inquisitiveness and the consequential assumptions about cause and effect.

Mathematics emerged from such times, and it is only natural that alternative assumptions about the basic phenomena that are described by Mathematics should be made – especially if contradictions emerge that have to be explained away. Such a contradiction motivated this study. It started when it emerged that, using numerals of the digits, a list of all infinite permutations of digits can be made, thus showing them to be countable. But, using subscripted variables, Cantor proved that the cardinality of the set of all infinite permutations of digits is more than countable. Investigation of this apparent contradiction finally led to the conclusion that Mathematics divides into two irreconcilable subsystems based on two different ways that Space can be described: one in which Cantor's proof is valid (Called the Euclidean Cosmology), the other in which the list of permutations of numerals is valid (Called the Leibnitzean Cosmology).

The following is a description of the intellectual path that led to this conclusion:

About two and a half millennia ago the foundations for the modern Mathematical Sciences were laid down in ancient Greece. Two of the assumptions that were made then were described by Euclid[5] as (a) there exists a piece of space with no extent called a point and (b) Space is synthesised from such entities. These assumptions had two strange consequences: firstly that there exist more than countable many such points and secondly that it is possible to override the limitations of inductive logic and perform infinitely many operations to completion (here "infinity" is a synthetic[6] natural number that is larger than all other natural numbers). These assumptions underpin the Mathematical Sciences at present.

Looking more closely at this, one sees that having no extent means that points have zero volume, area or length. This made Space complete because any set of nested intervals of which the lengths tend to zero now had a point as limit. The model of Space that results from the fundamental assumption that it consists of points is called the Euclidean Cosmology. The two consequences mentioned above are results of this assumption because the non-zero length of a line has to be the sum of the zero lengths of its constituent points. This necessitated the introduction of the concept of "more than countable" and also caused the implicit introduction of the Axiom of Choice into Mathematics.

About two millennia later Leibnitz[7] set out to determine the tangent to the graph of the function y = f(x) in the XOY – plane when x=a. In this quest he ended up with sequences of lines of which the lengths $\{\delta x_n\}$ and $\{\delta y_n\}$ tend to zero and of which the ratio of their limits was the gradient of the required tangent. This posed a dilemma for him. Firstly, the paradigm of the Euclidean Cosmology required that the limits of these lines must be points and hence be of zero length. Secondly, he intended to use these limits as variables in his theory and, by their very nature, variables cannot be identically zero and be of any use. He resolved the dilemma by introducing a new entity which he called an "infinitesimal" (Which is pidgin Latin for "the little thing at infinity") to be the limit of the sequences at the place where the tangent touches the

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5 Euclid Ca. 400 to 300 BC
6 *Die ganze Zahl shuf der liebe Gott, alles uebrige ist Menschenwerk*. - Leopold Kornecker
7 Gottfried Wilhelm (von) Leibniz 1/7/1646 – 14/11/1716

graph. In so doing he avoided dividing a zero by a zero. To him infinitesimals were intervals that are short enough to be the limits of sequences of intervals of which the lengths converge to zero, but not so short that manipulation of their lengths is meaningless.

But this created a structural discrepancy in the Euclidean Cosmology: While zero and infinitesimal numbers can comfortably co-exist in a number system, a point and an infinitesimal are two different spatial entities and a sequence of nested intervals of which the lengths tend to zero cannot have two different spatial entities as limits. These conflicting requirements are resolved here by formulating an alternative model for Space based on infinitesimals rather than on points.

This alternative model for Space (the Leibnitzean Cosmology) is based on using “being continuous” as the fundamental property of space. It is described by two alternative assumptions: Firstly that all spatial entities have extent and, secondly, that Space is such that any piece of space can always be divided into two pieces of space of which the total extent equals the extent of the original piece.

In this alternative model for Space, infinitesimals can be re-introduced by defining them as sets of nested intervals of which the lengths converge to zero. This is an extension of the ideas of Bernhard Riemann[8]. Therefore, in this model Space can be analysed using nested spatial entities. The two consequences of the Euclidean assumptions do not occur in this model and “infinite” regains its literal meaning of “not bounded”.

Thus, in contrast to points that are discrete by nature, infinitesimals are continuous by nature. This alternative model for Space, and the resulting alternative paradigm, is a unification of the work of Leibnitz, Riemann and L’Hospital[9], with insights from Topology and Functional Analysis - both of which were developed after the deaths of Leibnitz and L’Hospital.

The contribution from L’Hospital is that he pointed the way to define the infinitesimal numbers. He realised that the value assigned to an indefinite form generated at x=a by the quotient h(a)/g(a) of two functions h and g with the values h(a)=0 and g(a)=0, does not result

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8 Georg Friedrich Bernhard Riemann 17 September 1826 – 20 July 1866
9 Guillaume François Antoine, Marquis de l'Hôpital 1661 – 2 February 1704

from the values of h and g at x=a but depends on Cauchy sequences of numbers {h($x_n$)} and {g($x_n$)} generated by a Cauchy sequence of points {$x_n$} that converges to the point a.  The value that is then assigned to the indefinite form follows from a comparison of the rates of convergence of {h($x_n$)} and {g($x_n$)}.

This insight into the nature of indefinite forms was used to revisit the standard theory for the development of the real numbers in order to define a new (abstract) type of number that represents rate. This was achieved by defining the Cauchy sequences that form the equivalence classes that are the real numbers to be a new type of number called Cauchy Numbers. Cauchy numbers that are equivalent to zero are then identified as the Infinitesimal numbers (Thus the lengths of the intervals that form an infinitesimal are the components of an infinitesimal number). Rules for the manipulation of these numbers in a way that suits the needs of the theory of Leibnitz then follows from the rule of L'Hospital.

This relationship between the work of Leibnitz and L'Hospital was never exploited, and Leibnitz's work is at present still embedded in the classic Euclidean paradigm.

In the alternative paradigm based on the Leibnitzean Cosmology, whatever is discrete is formed from continuous entities and it is never required to sum zeros to a non-zero number. Therefore inductive logic is not violated and, as mentioned, the word "infinite" has its original meaning of "not bounded". Furthermore the real numbers are countable in this cosmology.

In Physics a particle can be defined as an infinitesimal, and with this definition a particle is not discrete but is continuous and this fits in with the ideas of Parmenides of Elea that were rejected in ancient Greece. This may therefore potentially eliminate the particle/wave duality because such a particle can now exist throughout all space.

From all this it can be seen that the Leibnitzean Cosmology is a much simpler model of Space than the Euclidean Cosmology. One should expect that the application of Mathematics in the Mathematical Sciences will divide between these two models depending on which one is best suited to the problem. This should be much like the way in which one of the three models for Mechanics in Physics is selected on the basis of which one is the most appropriate to use under given circumstances.

# Table of Contents

# DIVISION ONE: Paradigm Description

## 1. Introduction

### 1.1 An Alternative Paradigm

Why did an alternative paradigm develop?

To answer this question one has to look at the history of Mathematical thought for the last two-and-a-half millennia. The main players in this narrative are Euclid[10], Leibnitz[11] and L'Hospital[12]

A paradigm is a standard pattern on which thinking is based. The current paradigm of Mathematics is based on the fundamental assumptions about space as described by Euclid. This paradigm expresses the fundamental assumption that whatever is continuous is formed from discrete points. This assumption is here called the Euclidean Cosmology.

One should note that once someone is used to a specific paradigm, any thinking that is not in accordance with that paradigm somehow feels "wrong". The most obvious example of this is when a person who is steeped in Newtonian Mechanics is exposed to Quantum Mechanics for the first time. It takes some time to realise that, although this model for Mechanics is "strange", it is equally valid as a model because its results are more accurate than those of Newtonian Mechanics when it is applied on a sub-atomic scale.

Euclid is the first player in the history that is looked at here. He wrote a text on Geometry that remained the standard basic work on the topic for almost two and a half millennia. In this text he defined a point as "That

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10 Euclid Ca. 400 to 300 BC
11 Gottfried Wilhelm (von) Leibniz 1/7/1646 – 14/11/1716
12 Guillaume François Antoine, Marquis de l'Hôpital 1661 – 2 February 1704

which has no extent". Since any non-spatial entity, e.g. smell, has no extent, this should be interpreted as "A piece of space that has no extent". The present paradigm for thinking about space is - as it was at the time of Leibnitz - built on the assumption that these entities are the fundamental building blocks of space: Space itself is formed by combining points; surfaces by combining single layers of points; and lines by combining strings of points.

In this paradigm a set of nested lines of which the lengths tend to zero has a point as limit. Therefore the limit of the set of lines is a line consisting of a single point and hence it is a line of zero length.

The second player was Leibnitz. His quest was to determine both the tangent to the graph of the function y = f(x) at the point x=a in the XOY - plane, and the area under the graph of the function between x=a and x=b. In this quest he ended up with sequences of lines of which the lengths $\{\delta x_n\}$ and $\{\delta y_n\}$ tend to zero. This posed a dilemma for him. Firstly, the paradigm of the Euclidean approach required that these limits must be lines of zero length. Secondly, he used these limits as variables in his theory called Calculus and, by their very nature, variables cannot all be identically zero. He solved this dilemma by introducing a new entity which he called an "infinitesimal" (Which is pidgin Latin for "the little thing at infinity"). To him infinitesimals are intervals that are short enough to be the limits of sequences of intervals of which the lengths converge to zero, but not so short that manipulation of their lengths is meaningless.

The third player is L'Hospital. His contribution is short and decisive: In certain cases quantities that are zero or infinite can be manipulated to yield valid results that are not themselves either zero or infinite. These cases are called the indefinite forms $0/0$ and $0 \cdot \infty$. Note that this notation is purely symbolic because it does not refer to actual values of any functions. If an indefinite form should result from function values at the point a, then these symbols refer to the limits of the values of these functions at sequences of points that converge to the point a.

Today, having the advantage of our knowledge of transfinite numbers, we can recognise that an infinitesimal number is a second type of cisfinite number (the first being the number zero). It is a number that is equivalent to zero in the sense that it can be the length of the a line that is the limit of a sequence of nested intervals of which the lengths converge to zero, but it differs from the number zero in that it is not identically zero. According to the first part of DIVISION TWO the cardinality of the set of all points is $\aleph_1$ while the cardinality of the set of infinitesimals is $\aleph_0$.

The number zero can coexist with infinitesimal numbers in the same number system. However, any set of nested intervals of which the lengths converge to zero can have only one limit. Therefore Leibnitz' infinitesimals and points cannot coexist in the same model for space. This is the reason for the development of a second paradigm for use in Mathematics where the role of points is taken over by infinitesimals and space is no longer synthetized by combining points, but is analysed by using infinitesimals. This change implies that any discrete object is formed from continuous space and hence it rules out the existence of points as spatial entities. This defines the Leibnitzean Cosmology.

But the relationship between the work of Leibnitz and L'Hospital was never exploited, and Leibnitz's infinitesimals remained embedded in the classic paradigm which still precludes such a link. The alternative paradigm described here unifies the work of Leibnitz and L'Hospital. The example given in paragraph 4.4 below should help to clarify this connection.

**Remark**

It took more than thirty years to develop the Leibnitzean Cosmology and its accompanying paradigm. The reason for writing a description of this alternative paradigm is to help the reader to develop this paradigm in the span of a few days by reading the results stated in the other divisions. Therefore this is an attempt to describe an initial paradigm for thinking in the Leibnitzean Cosmology. The hope is that this will save the reader some time and effort and make the Leibnitzean Cosmology feel less "strange".

Note that, in the context of this exposition of the alternative paradigm, the short discussion on logic and the structure of science given in paragraph 3 below validates the use of the word "Model" when referring to science or to human insight into perceived reality.

## 2. Space

This alternative paradigm is distinguished from the current paradigm in the way that space is modelled. Whereas the current paradigm is based on discrete points, the alternative paradigm is based on continuous lines, surfaces and volumes.

Between 600 B.C. and 500 B.C. there were already discussions in ancient Greece about how to best describe space. One of the proposals at that time was that lines (vectors) should be the tools for describing space. This idea was not exploited. However, if this would have been done, an alternative model for space, and thus an alternative paradigm for thinking about space, would have been formed with lines as the fundamental descriptive tools for space and not points. In such a paradigm the concept of “point” as a piece of space would be redundant and the assumption that space is formed from points would not be present. This approach has been revived here because it turned out to be the natural setting to examine the consequences of the ideas of Leibnitz about space.

**Space is extent.**

Some of the properties that we assign to space from our experience are:

- A piece of space can be isolated, and every piece of space has a measurable (non-zero) volume.
- Any piece of space can always be divided into two pieces of space of which the sum of the volumes equal the volume of the original piece.
- The interface between adjacent pieces of space is a surface, which is a property of the two pieces of space and has an area.
- The interface between two intersecting surfaces is a line which is a property of the two surfaces and has a length.
- The interface between two intersecting lines is a point which is a property of the two lines and has a place in space.

- Space also has the properties of direction and distance.

This is but a short partial list of the properties of space as we experience it. Note that this list attempts to describe the properties of space outside of the present paradigm. Therefore in this list a point is not a spatial entity but simply the place in space where a line ends – i.e. the point of the line.

## 3. A List of Some Aspects of the Alternative Paradigm

- A realisation that there are two irreconcilable ways to model space in Mathematics. The first way is a discrete model based on the introduction of the concept “point”. The second is a continuous model based on the introduction of the concept “infinitesimal”. This creates two irreconcilable substructures in Mathematics. (This conclusion is motivated in the first section of DIVISION TWO)
- The first model is a discrete model which is based on the assumption that spaces, surfaces and lines are synthesized from points, and that points are spatial entities that are the discrete limits of nested spaces, surfaces and lines of which the maximum diameters of their volumes, areas or lengths tend to zero. This will be called the **Euclidean Cosmology**.
- The second model is a continuous model that is based on the assumption that space can be analysed using nested spaces, surfaces and lines of which the maximum diameters of their volumes, areas or lengths tend to zero and are focussed at places in space. These places can be indicated by discontinuities like the ends of a lines. These nested sets of spaces, surfaces or lines of which the maximum diameter of their volumes, areas or

lengths tend to zero are called Infinitesimals[13]. This will be called the **Leibnitzean Cosmology**.

- This use of the concept of two substructures emulates Physics where there are three substructures for Mechanics: one based on the hypothesis that mass is independent of motion, the second that the speed of light is independent of the motion of the observer and the third that energy can only change by discrete quantities.

- The realisation that irrational numbers cannot be represented by numerals in the same way as rational numbers. This is based on the reasoning at the end of DIVISION TWO and partially motivates the extension of the number-concept to define and include the Cauchy numbers.

- The number system is extended by defining a Cauchy Number as any Cauchy sequence[i] (i.e. Cauchy numbers are the elements of the equivalence classes of Cauchy sequences that are the real numbers) written in vector form.

- Addition, subtraction, multiplication and division for Cauchy numbers are defined.

- An infinitesimal number is defined as a Cauchy number that belongs to the equivalence class of Cauchy sequences that forms the real number zero. (Or, to put it differently, it is a Cauchy number that is equivalent to zero). Thus the lengths of the nested intervals forming an infinitesimal is an infinitesimal number.

- An infinite Cauchy number is a sequence that diverges to infinity. (i.e. it is the inverse - under division - of an infinitesimal number)

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13 This definition of an infinitesimal changes an infinitesimal from being a spatial entity into a sequence. Thus a point, if defined, can be the limit of an infinitesimal. Hence the division of Mathematics into two cosmologies as described above is no longer necessary on the grounds that a sequence cannot have more than one limit. Therefore it is only the fundamental assumption in the Euclidean Cosmology that space is formed from points that prevents these two cosmologies from being unified.

- It is re-asserted that the word “number” refers to an abstract entity and that the word “numeral” refers to a symbol that depicts a number in some way.
- This extension of the concept of number to include the Cauchy numbers was done in order to extend the scope of numbers according to the following scheme of objectives: A natural number is introduced to describe an abstract property of a set of discrete objects; a rational number to describe the concept of the relativity of two numbers; a real number to describe the abstract property of extent for volumes, surfaces and lines. These classes of numbers are now augmented by defining Cauchy numbers to describe the abstract concept of rate.
- The principal Theorem of the Integral Calculus is re-interpreted by defining cascades of infinitesimals that form directed sets on which an integral can be defined as a net. This is done in DIVISION THREE.
- It is pointed out that, if the Leibnitzean Cosmology is used in the physical sciences to model a particle, an alternative assumption about the nature of particles is obtained when a particle is modelled as an infinitesimal and not as a point. In this case the particle/wave duality becomes meaningless because a particle, when defined as an infinitesimal, is not localised or discrete but is continuous and can extend through all space. The particle would merely be perceived as being at the focus of an infinitesimal. It is pointed out that this is in accord with the Metaphysics of the Greek philosopher Parmenides of Elea.

## 4. Background

In order to make sure that the relevant background to the material that is presented here is stated in a common format, the following is a short synopsis of what should be familiar to any practicing scientist and mathematician.

### 4.1 Philosophy

The Mathematical sciences originated in ancient Athens and was founded by philosophers who insisted on rigorous logic. There are three logical processes involved here:

- **Deduction**: The sure (forward) way of inference, e.g. 1) When it rains it is wet 2) It is raining 3) Hence it is wet.
- **Induction**: The unsure (backwards) way of inference, e.g. 1) when it rains it is wet 2) It is wet 3) Hence it may be raining.
- **Abduction**: an inspired guess. E.g. when it was hot and humid in old Rome and the air got muggy, a fever spread that was assumed to be caused by the “bad-air” (Mal aria).

Note that any deductive argument must of necessity start with abduction – a first statement (also called a *principle,* a *hypothesis* or an *axiom*) that is accepted as true – for example “When it rains it is wet”.

The time before the enlightenment (roughly before 1750) is called philosophical antiquity. In those days a statement was taken to be either a truth or a lie. Therefore, when the Greeks defined an axiom as a “self-evident truth” they implied firstly that it is an abductive statement because nothing else is evidence for it (this means that it does not follow deductively or inductively from any other statement), and secondly that it is true (it is not a lie).

After the enlightenment philosophical thinking developed to the point where the initial abductive statement is considered to be true (in order to start the sequence of statements in a deductive argument) but that it is only true as far as the argument is concerned and that the result of the argument is only true modulo the truth of this initial assumption.

Note that all our knowledge are the results of inductive arguments: Everything we know is the result of what we consider to be the cause of the information gathered by our senses and transmitted to our brains (all knowledge are therefore conclusions that were drawn from evidence gathered by our senses).

Legal trials are good examples of processes based on evidence using inductive reasoning. In the practice of law it is accepted that circumstantial evidence cannot lead to necessarily true conclusions. In court, evidence is presented to “prove” the guilt of the accused, but unless the accused confesses his guilt or was caught in the act, his guilt can never be known. Hence the legal requirement is that, when based on circumstantial evidence, the accused can only be found guilty “beyond reasonable doubt”.

## 4.2 The Scientific Method

The scientific method splits the inductive process of accumulating knowledge into a sequence of deductive processes[14]. A new deductive process in such a sequence is started whenever evidence appears which implies that the abductive first assumption of the current deductive argument cannot deductively explain this new evidence. The current abductive assumption is then replaced by a new abductive assumption which is such that the deductive argument still yields all the results of the old deductive argument, but in addition explains the new evidence. In this process an ever better understanding of the cause of the observed evidence is formed without ever finding the “true” cause – or even ever progressing to the stage of “beyond reasonable doubt”. In this scheme the abductive assumption is usually called a hypothesis and in a specific case the argument is sometimes called a model.

One should note that this creates a tree of hypothesis since the results of preceding arguments can be used in later arguments. Thus, whenever a discrepancy occurs, it is necessary to identify the relevant hypothesis in the line of assumptions that will cause the argument to explain the new evidence without contradicting any of the old evidence. The set of all trees of assumptions (and the evidence supporting them) is the body of Science.

Only disciplines which are based on evidence can use the scientific method. This immediately rules out religious systems and some aspects of Mathematics. All these disciplines originated during philosophical antiquity and hence are regarded as incontrovertible truths by their “believers”. In these systems evidence contrary to the original abductive assumptions are usually ignored, replaced by fallacies[15] (untrue

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14 An illustration of this method is divining the meaning of unknown words without the use of a dictionary: At the first encounter the meaning of a word is guessed from its context and every time the word is encountered thereafter the concept of what it means is refined by using the new context. This is also the only possible way that a baby can learn to speak.
15 Cantor’s fallacy as described in DIVISION THREE belongs to this class. See the EPILOGUE TO DIVISION ONE.

statements universally accepted as true) or are dogmatically classified as lies.

Therefore these systems cannot be adapted, but parallel systems based on new abductive assumptions have to be founded – as is evidenced by the plethora of different religions and sects.

# 5. The Euclidean and the Leibnitzean Cosmologies

## 5.0 The Basic Construction

Start with any line and cut it at a convenient place. This introduces a discontinuity that defines a place in space called the origin. This place in space is indicated by the end of either of the two resulting lines. Name this place in space as the origin O and the line as the X-axis.

With the choice of a suitable scale it is then possible to construct a line of which the length is any given rational number. Hence, for a given rational number a, a line OA of length a can be constructed on the X-axis. With the usual conventions about axes applied, the interval formed by the line OA on the X-axis can be indicated by (0;a) i.e. the line starting at x=0 and ending at x=a. Note that, because this argument is solely about lines and their lengths, the concept of "point" is not involved, hence the concept of open and closed intervals is absent.

Let $\delta = (\delta_1; \delta_2; \delta_3; \delta_4;\dots)$ be a Cauchy sequence of rational numbers that converges to zero but is written in vector form; hence it is a Cauchy number equivalent to zero and as such it is an infinitesimal number.

Using the method above, the lines $OB_n$ and $OC_n$ of lengths $a-\delta_n$ and $a+\delta_n$ can be constructed for each component of δ. The line $B_nC_n$ forms the interval $(a-\delta_n; a+\delta_n)$. As $\delta_n$ takes ever smaller values when n gets larger and larger, a nested set of intervals, here called an infinitesimal, is formed and this infinitesimal is focussed at the end A of the line OA.

## 5.1 The Euclidean Cosmology

The foundations of the Mathematical Sciences were laid during the era of philosophical antiquity in Greece – mainly in Athens. There the "truths" of Mathematics were discovered by intellectual philosophical debate and formulated as axioms.

The process was driven by philosophers in Athens who would, in terms of the present classification of sciences, be called "Geometers". For all practical purposes the philosophers in Athens revered geometry.

Thus the "truths" of Mathematics are based on the insights of geometers whose aim was to study unchanging discrete geometrical figures and who drew their geometric figures in sand.

Not all philosophers of the era agreed with their conclusions – notably the philosopher Parmenides who lived in Elea in Italy. His Metaphysics also included specific views on motion and on “coming into being”. Today these thoughts would be considered as relevant to Physics, so that he would probably be called a Physicist today. Very little of his own writings survived and our knowledge of his ideas come almost entirely from his detractors and from the well-known paradoxes posed by his eromenos Zeno.

The remark in section 1 about Euclid’s definition of a point can now be extended: Euclid defined a point as “that which has no extent” which was seen to mean “a piece of space with no extent”. But because this definition belongs to philosophical antiquity, it should by right have been formulated as “There exists a piece of space with no extent”, and this statement was to be taken as true in the absolute sense.

After the enlightenment, this statement could not be taken as true in the absolute sense anymore, so that it became an abductive statement to be taken as true in order to start a deductive sequence of arguments. In this case the tree of arguments following from this assumption will be called the Euclidean Cosmology which, in current times, can be taken as true modulo this assumption.

(In order to justify the use of the word “Cosmology” here, one notes that the Greek word “cosmos” means “order” or “world” and the word “logos” means “to talk about”).

In the Euclidean cosmology the set of nested intervals constructed above in paragraph 5.0 is focussed at a place in space which is at the end of the line OA. But the numbers $\delta_n$ converge to zero and hence the length of the intervals converge to zero. In the Euclidean cosmology the point at A belongs to all these intervals and hence it belongs to the limit when n tends to infinity. Thus there is a piece of space of length zero, the point at A, to serve as limit for the sequence of nested intervals. In this sense space is complete in the Euclidean cosmology.

The Euclidean cosmology then posit the following assumption: “Space is formed from points”. This implies that any continuous entity is formed from discrete entities. However, note that the linguistic use of the word “discrete” implies that any discrete entity is identifiable.

This assumption is the source of the concept “more than countable”:

Consider the unit interval and let A be the set of points that form this line of unit length. Then the **sum** of the lengths of all the points in A, (not the limit!) must be one. But the length of every point is zero, so that the sum of the lengths of a finite number of points is zero, which implies that the total length of a countable number must also be zero. Therefore, for all the zeros to add up to the non-zero number “one”, the cardinality of A must be more than countable.

Note that the Euclidean cosmology is excellent for describing discrete states, e.g. the places where motion starts and ends, but is awkward in describing the rate of movement between these places, where it assumes that every point of a body is positioned at some point of space at every instant of time.

## 5.2 The Leibnitzean Cosmology

If the concept of point (as a spatial entity) is not introduced, then the set of nested intervals above has no limit and it becomes a never-ending sequence of intervals that is focussed at the end A of the line OA. Thus, space is not complete anymore.

Each interval in this never-ending set of intervals is a continuous line of non-zero length. Note that, In the Leibnitzean cosmology, whenever a line is cut it divides into two sublines both of non-zero length. Hence the Leibnitzean model is based on continuous intervals that are all of non-zero length. This is also true for volumes and areas, and hence discrete objects can always be analysed into continuous entities in this cosmology.

One should also note that in this cosmology the absence of points precludes the existence of open and closed volumes, areas and intervals in **geometric** space. However, note that other types of spaces, for example probability spaces, are **not** inherently geometric spaces, but are models formed using concepts from geometric spaces.

Any finite straight line has a length because it begins and ends somewhere in space. However, in this cosmology, the line itself is not necessarily discrete because it cannot be identified unless additional information is available. To motivate this statement, one should look at section 2.1 of DIVISION TWO where the line of length $\sqrt{2}$ is identified from a geometrical construction, but the line of which the length represents the value of the real number formed from the Fibonacci sequence cannot be known and hence this line is unidentified even though it exists.

Central to the Leibnitzean cosmology are the differences in the implications from deductive and inductive processes:

When the length of a line is known, like $\sqrt{2}$ above, then it is easy to find a sequence of lines of which the (rational) lengths converge to this length – as was done in section 2.1 of DIVISION TWO. This is a deductive process since the limit of this Cauchy sequence of rational numbers is known *a priori* (is said to be **identified**).

When the length of a line is unknown, as in the case of the line based on the Fibonacci sequence in that paragraph, then a sequence of lines of which the lengths converge to the desired length can still be set up, but the length of this line can never be known *a posteriori* – not "beyond a reasonable doubt". This is an inductive process.

In the Leibnitzean cosmology a real number is still defined as the limit of a Cauchy sequence of rational numbers (or equivalently as an equivalence class of Cauchy numbers), and from the above follows that every real number is associated with the length of a line (hence with a place in space). In line with the constraints of deductive and inductive reasoning, the value of the real number – which is the length of the line - is only known if the Cauchy sequence that defines it is set up to converge to that value. Hence the value of a real number, as the limit of a given Cauchy sequence, always exists but can never be known (identified) unless additional information is available.

Thus there are two classes of real numbers: **Value described** real numbers where the limit of the Cauchy sequence is given *a priori*, and **Component described** real numbers where only the Cauchy sequence is given *a priori*. Value described real numbers are identified and can be tested for equality. Component described real numbers are not identified and cannot be tested for equality unless additional information is available. Note that a component described real number is the equivalence class to which the Cauchy sequence that describes it belongs. Two such real numbers can only be shown to be equal by showing that the two Cauchy sequences that define them belong to the same equivalence class of Cauchy sequences - as is discussed in section 5.4 below.

The real line is still defined, but every real number now corresponds to the end of some line from the origin – i.e. there is a one-to-one correspondence between the real numbers and lines from the origin.

## 5.3 The Cauchy Numbers

As noted above, the Euclidean cosmology is based on the concept of discreteness: in the Euclidean cosmology all numbers are associated with points on the real line and as such are discrete. This is ideal for modelling discrete quantities such as marbles in a jar, money in an account and ratios like odds at the racetrack. This model is however not easy to use when modelling continuous abstract concepts like rates of change.

But rates of convergence/divergence can be defined for convergent/divergent sequences, and hence sequences are ideal vehicles for modelling rates. Extending the concept of number to include the sequences that form the equivalence classes that are the real numbers (and then later extending the definition of Cauchy numbers to include divergent sequences), enable numbers in the Leibnitzean cosmology to describe rates.

These numbers are the Cauchy numbers. They can be described more fully as:

- Each non-infinite Cauchy number is a convergent sequence of rational numbers.
- The real numbers are equivalence classes of Cauchy numbers (as usual).
- The Cauchy numbers that converge to zero are called infinitesimal numbers,
- The Cauchy numbers that converge to non-zero real numbers are called the rated numbers.
- The Cauchy numbers that diverge to infinity are called the infinite numbers.

This structure can be illuminated by looking at a compactification of the real numbers as presented in section 3.1.2.1 of PART TWO of DIVISION THREE.

## 5.4 Arithmetic for Cauchy Numbers

Arithmetic for Cauchy numbers is defined component-wise as shown in section 3.1.1 of PART TWO of DIVISION THREE. For example:

If a = ($a_1$; $a_2$; $a_3$…) and b = ($b_1$; $b_2$; $b_3$;…) then a/b = ($a_1/b_1$ ; $a_2/b_2$ ; $a_3/b_3$ ;…) provided that not more than a finite number of the rational numbers $b_n$ are zero.

The same holds for the other three arithmetic operations.

A Cauchy sequence is a never ending sequence of rational numbers, and as such it is not a symbol and thus it cannot serve as the numeral for a number. Therefore equality of two Cauchy numbers cannot be determined directly. However, two Cauchy numbers are equivalent if it can be shown that they belong to the same equivalence class. This constitutes the concept of "equality" for Cauchy numbers in a way similar to rational numbers, where "equality" also means that two rational numbers belong to the same equivalence class.

Arithmetic for Cauchy numbers enable the calculation of rates using algebraic operations in lieu of limit operations.  In essence this is an extension to, and generalisation of, the rules of L'Hospital (or L'Hopital).

In Euclidean cosmology numbers are considered to be equal if they have the same value – i.e. if they are represented by the same point on the real line or if their difference is equal to zero. As argued above, numbers in Leibnitzean cosmology, being Cauchy sequences, are considered to be equal if they belong to the same equivalence class (or if their difference is an infinitesimal number – see DIVISION TWO 1.6.3).

For example:

In the Euclidean cosmology, L'Hospital states that if h(a) =0 and g(a)=0 then

$$\lim_{x \to a} \frac{h(x)}{g(x)} = \frac{h'(a)}{g'(a)}$$

As indicated in paragraph 1.6.1 of DIVISION TWO, this equality in the Leibnitzean cosmology means that the Cauchy sequence formed by the left hand side as x converges to a, belongs to the same equivalence class as the right hand side when written as a Cauchy number.

Let

$$dx = (\delta_1; \delta_2; \delta_3; \ldots.) \text{ with } \delta_n \to 0$$

be an infinitesimal number and let

$$a = (a_1; a_2; a_3; \ldots.)$$

be the number a written as a Cauchy number.

Then:

$$\lim_{x \to a} \frac{h(x)}{g(x)} = \lim_{n \to \infty} \frac{h(a + \delta_n)}{g(a + \delta_n)}$$

$$= \lim_{n \to \infty} \frac{h(a_n + \delta_n)}{g(a_n + \delta_n)}$$

Provided that h and g are differentiable. This limit is a real number and therefore is an equivalence class of Cauchy numbers. The Cauchy sequence

$$\{ \frac{h(a_n + \delta_n)}{g(a_n + \delta_n)} \; ; \; n = 1, 2, 3, \ldots. \}$$

has this real number as limit and it therefore belongs to this equivalence class.

But this Cauchy sequence is the Cauchy number

$$\frac{h(a + dx)}{g(a + dx)}$$

So that the Left Hand Side (as a real number) is the equivalence class containing the Cauchy number

$$\frac{h(a+dx)}{g(a+dx)}$$

For the right hand side one notices that h(a) = 0 and g(a) = 0 so that dh and dg can be written as functions of a and dx as:

dh(a,dx) = h(a+dx)-h(a) = h(a+dx)

dg(a,dx) = g(a+dx)-g(a) = g(a+dx)

Substituting:

$$\frac{h(a+dx)}{g(a+dx)} = \frac{dh(a,dx)}{dg(a,dx)} = \frac{dh(a,dx)}{dx} \cdot \frac{dx}{dg(a,dx)} = \frac{h'(a)}{g'(a)}$$

Where all the equal signs mean “is in the same Equivalence class” and hence mean “is the same Cauchy number”

Thus the above rule of L’Hospital, when written using Cauchy numbers, is

$$\frac{h(a+dx)}{g(a+dx)} = \frac{h'(a)}{g'(a)}$$

Meaning that these two Cauchy numbers belong to the same equivalence class.

## 6. Matter in the Leibnitzean Cosmology

In Greece of old, the geometers considered moving bodies to be geometrical shapes of which every point of the body is located at some point of space at every moment of time. Today this description of motion still forms part of Euclidean Cosmology and of Physics.

Parmenides[16] also studied motion, but from a physical perspective rather than from geometry. Through the ages his ideas about this were considered as laughable because he said among other things that (1) motion is an illusion because everything is one and that (2) nothing can come into being where it has not existed before. Looking at these ideas of Parmenides one can understand why they were not accepted, because anyone who ever stubbed his toe knows that motion is not an illusion and by looking through a window at passing people, one can see them to come into being where they have not existed a second before!

However, looking through a window one can also see that the earth is flat.

Therefore one should note that the word “illusion”, as used here, can have more than one interpretation: For instance, seeing a wave moving over water for the first time can be interpreted as an illusion because water is seen to move over water, but after the wave has passed all the water is still where it was. Nothing actually came into being where it did not exist before and it is only our interpretation of what we see that noted change. Therefore our interpretation of what we see is biased by what we expect through comparison with other experiences. Thus an illusion may happen when the brain is not necessarily misled, but when the brain does not have the right information to correctly interpret what is seen because what happens is outside the range of human perception.

In the Euclidean cosmology a particle is modelled as a point and as such it is discrete, has mass and it is localised; hence it is common to think of an electron as a little solid sphere. Points (as spatial entities) do not exist In the Leibnitzean cosmology and modelling a particle as an infinitesimal is the only viable choice. Hence, in the Leibnitzean cosmology, a particle is continuous and is not localised but consists as a

16 Parmenides of Elea. Born c. 515 B.C.

set of nested volumes focussed at the place in space where the particle is observed. Therefore everything is one because the nested volumes reach out through all space, and movement is simply the changing of where in space the infinitesimal is focussed. This makes movement an illusion because nothing actually moves, and neither does the particle come into being where it did not exist before!

This gives a different perspective on the particle/wave duality. This duality has been with us since Huygens' Principle and the photoelectric effect. But these two cases did not bother us, because light has no mass and therefore a photon is not a "real" particle. However, the converse seems wrong when something that is supposed to be solid "dematerialises". This conflict is eliminated when a particle is modelled as an infinitesimal. Therefore the particle/wave duality should not exist in the Leibnitzean Cosmology.

This reveals a possible alternative way of handling this discrepancy in Physics - namely starting from an alternative abductive assumption about the fundamental nature of space and in so doing also of matter.

# 7. Epilogue to division one

What is referred to as “Cantor’s fallacy” in DIVISION THREE belongs to the class of fallacies where, in systems stemming from philosophical antiquity, evidence contrary to the original abductive assumptions are replaced by untrue assumptions that are then universally accepted as true. On first sight the (well hidden) logical fallacy is the assumption that the limit of a Cauchy sequence can be inferred from the terms of the sequence - contrary to the correct use of the appropriate inductive argument.

This fallacy is hidden in the common practice of using a Cauchy sequence as a numeral for an irrational number, irrespective of whether the number is value specified or component specified. It is further obscured by the different meanings of the word “Infinite” when used in Mathematics.

But, looking closer at the underlying cause of this fallacy, it appears to be even more fundamental.

As pointed out above, the assumption of Euclidean Cosmology that continuous space is composed of points has as a consequence that more than countable many zeros must be summed to a non-zero sum. The process of addition requires that the points of (say) the unit interval must be identified one by one and for each such point a zero must be added to the sum. When the last of the points forming the interval has been processed, the total of all the zeros is exactly one and the interval has been completely deconstructed. This implies that it is possible to perform an infinite (more than countable) number of actions (identifications followed by additions). This is an implicit introduction of the axiom of choice into the Euclidean cosmology. This is discussed in more detail at the end of DIVISION TWO. It also implies that the result of these more than countable number of actions is discrete and can be identified *a posteriori*.

Therefore the meaning “never ending” for the word “infinite” does not exist in the Euclidean Cosmology. In the Euclidean Cosmology the word “infinite” means only “a natural number larger than all other natural numbers”, and the phrase “infinite decimal fraction” refers to a string of digits of infinite length in some abstracted reality where the values of all the digits are known. This could explain why ‘Abstract Mathematics’ is such a suitable name to use for Euclidean Cosmology. However, it also raises concern about the use of ordinary digits and numerals in Abstract Mathematics. This concern is reflected in the non-acceptability of the proof of Theorem 4.1 of Section 4 of Part One of DIVISION THREE in the Euclidean Cosmology.

In the Leibnitzean cosmology it is never required to add zeros to get a non-zero sum, and the rules of logical induction prohibits the *a posteriori* identification of the limit of a component specified real number. Therefore a real number, in the Leibnitzean cosmology, is a place on the axis where a “never ending” Cauchy sequence is focussed. In the Leibnitzean Cosmology the expression “Infinite string of digits” means “never ending string of digits” and the expression “a natural number larger than all other natural numbers” has no meaning.

**Thus, in the end, “Cantor’s Fallacy” is the assumption that Mathematics is a single model and therefore the application of the concepts of Euclidean Cosmology to Leibnitzean Cosmology and vice versa is acceptable in certain situations.**

It is therefore highly appropriate to use Cantor’s name for this fallacy, even though his diagonal proof turned out to be simplistic only when viewed from the perspective of Leibnitzean Cosmology. In the Euclidean Cosmology his argument is not simplistic but is absolutely correct, but the list of all decimal fractions in Theorem 4.1 in DIVISION THREE cited above is not acceptable. In the Leibnitzean cosmology Cantor’s diagonal proof is not acceptable and Theorem 4.1 is correct.

Therefore Cantor’s correct use of the concepts of Euclidean Cosmology opened the door to where the Leibnitzean Cosmology was found even though the fallacy was mine – looking at the Euclidean cosmology from the perspective of the Leibnitzean Cosmology.

# DIVISION TWO: Space and Numbers

## 1. Models of Space

Science's fundamental assumption about space is that it is infinitely divisible. This assumption is accepted by all scientists and mathematicians. It means that any piece of space, be it volume, area or line, no matter how small, can be divided over and over without ever changing its nature.

### 1.0 Introduction: Basic Construction

Start with any line and cut it at a convenient place. This creates a discontinuity that defines a place in space. This place in space can be indicated by the ends of either of the two half-lines formed. Name this place in space as O and call it the origin.

With the choice of a suitable scale it is then possible to construct an interval of which the length is any given rational number. Hence, using the line as axis and keeping the usual rules for axes in mind, the line OA of length a can be constructed for any given positive rational number a. As per conventions, the interval formed by the line OA on the X-axis can be indicated by (0;a) i.e. the line (or vector) starting at x=0 and ending at x=a. Note that this argument is solely about lines and their lengths so that the concept of "point" is not involved and that therefore the concept of open and closed intervals is absent as well. Thus x=0 means "at the origin" and x=a means "at the end A of line OA of length a"

Let δ be a positive rational number that is less than a. Construct the lines OB and OC of lengths a-δ and a+δ. The line BC forms the interval (a-δ; a+δ) of length 2δ on the X-axis with its centre at x=a. Intervals like these are then used in the following construction:

Consider the unit interval from 0 to 1 on the X-axis. Thus d(0;1) = 1 where d is the distance between the two ends of the interval. This unit interval can now be partitioned repeatedly, i.e. cut up into shorter lines, by forming sub-intervals in the following way:

Starting with the whole interval as partition zero, form successive partitions by dividing each interval of the previous partition into three equal parts. In this way the $n^{th}$ partition will consist of $3^n$ intervals, each of length $3^{-n}$. If $x = a_i^n$ is at the centre of the $i^{th}$ interval of the $n^{th}$ partition, then

$$a_i^n = \frac{2i-1}{2} 3^{-n} \quad \text{For i = 1, 2, 3, …, } 3^n \text{ and n = 0, 1, 2, …. [1.0.1]}$$

But the length of the whole interval is the sum of the lengths of the parts so that:

$$1 = \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} , a_i^n + \tfrac{1}{2}3^{-n}) \text{ for n=0, 1, 2, ….} \qquad [1.0.2]$$

Therefore

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} , a_i^n + \tfrac{1}{2}3^{-n}) \qquad [1.0.3]$$

The equation [1.0.3] is based solely on the property of space that any given line can always be sub-divided.

There are two fundamentally different approaches to describing the infinite divisibility of space. The first way is a consequence of what was decided in Athens at about 500 B.C. It is a process of synthesising space and is studied directly below. It is referred to here as the approach of Euclid. The other is a process of analysing space, as is done above, and the above construction follows from the approach of Leibnitz at about 1700 A.D.

## 1.1 Euclid

The basic assumptions of the Mathematical sciences as practiced today were made in ancient Greece. At that time the concept of “point” as a spatial entity was introduced by philosophers who were fundamentally geometers. A notable exception was Parmenides of Elea who also argued about the nature of motion (change and coming into being) and thus, in today’s classification, was fundamentally a Physicist.

The Athenians looked at volumes, areas and lines that extend in three, two and one directions and then extended this triplet to include a fourth entity. This entity was defined as “That which has no extent”.

In DIVISION ONE sections 1 and 4.1 it is argued that this is not a definition, but should be treated as an axiom: “There exists a piece of space with no extent”, and that in this axiom the word “exists” has its literal meaning, namely that a point is a spatial entity, i.e. it is a piece of space.

The introduction of this concept not only gave a tool to refer to a specific place in space, but made space complete in the sense that any nested sequence of spheres, discs or lines of which the volumes, areas or lengths converge to zero in such a way that their maximum diameters converge to zero, will have a spatial object (a point) as limit.

For intervals as used in section 1.0, this means that, if a is any point, the sequence of nested intervals

$$\{(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) : n = 0,1,2,3\ldots\}$$

has an interval (a), consisting of the single point a, as limit. This is because a is a point that belongs to all of these intervals. Thus

$$\lim_{n\to\infty}(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) = (a)$$

Also

$$d(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) = 3^{-n}$$

So that

$$\lim_{n\to\infty} d(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}) = d(a) \qquad [1.1.1]$$

Here d(a) is the length of a single point and thus d(a) = 0.

But a second assumption about space was made by the philosophers in Greece. This assumption was that space is formed (synthesised) from points. This implies that any interval consists of a string of points.

This is a second fundamental assumption about the nature of space (The first was that space is infinitely divisible). On a philosophical level this assumption means that anything that is continuous is formed from discrete entities. This assumption will be called the **Euclidean Cosmology**.

This second assumption is the reason why the concept of “more than countable” had to be introduced into Mathematics:

Let B = { $\alpha_\beta$ } be the set of points forming the unit interval and let the cardinality of B be Γ. Each $\alpha_\beta$ is a point and thus d($\alpha_\beta$) = 0. But, according to the second assumption above, the total length of the interval must be the **sum** (not the limit) of the lengths of all the points forming the interval:

$$1 = \sum_{\beta}^{\Gamma} d(a_\beta) \qquad [1.1.2]$$

(Where the notation does not conform to any standards, but is hopefully self-explanatory.)

In this sum every d($\alpha_\beta$) is zero. But the sum of a finite number of zeros is zero, and the sum of a countable number of zeros, being the limit of its partial sums, must then also be zero. This implies that Γ cannot be either finite or countable. Thus Γ has to be more than countable i.e. it requires more than countable many zeros to add up to the non-zero number one.

Equation [1.1.1] is also true for any one of the real numbers $\alpha_\beta$ so that the same argument as above results in

$$\lim_{n\to\infty} \mathrm{d}(a_\beta - \tfrac{1}{2}3^{-n}, a_\beta + \tfrac{1}{2}3^{-n}) = d(a_\beta)$$

So that equation [1.1.2] becomes

$$1 = \sum_{\beta}^{\Gamma} \lim_{n\to\infty} \mathrm{d}(a_\beta - \tfrac{1}{2}3^{-n}, a_\beta + \tfrac{1}{2}3^{-n}) \qquad [1.1.3]$$

Note that this assumption is also an implicit introduction of the axiom of choice into the Euclidean Cosmology: In order to perform the sum of zeros, consecutive points from the interval has to be chosen one by one and for each a zero has to be added to the sum. Moreover, when the last point is processed, the interval would have been deconstructed and the sum would be exactly one. Thus it is not only possible to perform an infinite number of operations, but in the end the result is discrete and identified.

## 1.2 Leibnitz: Integrals

Leibnitz and Newton both developed tools to study motion. Newton used fluxions and Leibnitz used infinitesimals to describe the rates at which bodies moved and accelerated. Although Newton's notation is still sometimes used in Mechanics, the notation introduced by Leibnitz survived in general use.

The word "infinitesimal" is pidgin Latin for "a little thing at infinity" and, even today when physicists use the word, the intended meaning is "a number that is zero, but not completely so".

Leibnitz introduced the notation

$$\int_a^b f(x)dx$$

for his process to determine the area under the graph of y=f(x) between the values x=a and x=b. This is called an integral and dx is an infinitesimal number (and hence f(x)dx is an infinitesimal number too) and the integral sign is an elongated "S" to denote an infinite (but also not completely so) sum of infinitesimals. Therefore an integral can be imagined to be an (almost) infinite sum of (almost) zeros, and hence it is alike to the indefinite form $\infty \cdot 0$ as studied by L'Hospital. This aspect is discussed in more detail in the following paragraphs.

The evaluation of integrals like these is done using Riemann sums. To do this the interval (a,b) is partitioned into sub-intervals and the areas of rectangles, each with a part of the partition as base and with its height equal to the function value at some point of that part of the partition, are summed to approximate the area under the curve. The required area under the curve is then the limit of these sums as the number of parts in the partition tends to infinity in such a way that the length of each part tends to zero.

In order to come to an understanding of the way that Leibnitz was thinking, one notes that Leibnitz introduced the concept of "infinitesimal" to study integrals and rates of movement. Thus it is advantageous to look at a simple integral In order to get a grip on his thinking about this concept. The following is the simplest possible integral. It is to determine the area under the line f(x)=1 between the values x=0 and x=1: hence it is the area of a square of side length one.

Thus:

$$\int_0^1 1 \cdot dx = 1$$

In this case the height of all the rectangles in the Riemann sum is one, and the Riemann sum is not an approximation anymore but is exactly one for all possible partitions. Thus, using the set of partitions as described in section 1.0 the integral becomes:

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} 1 \cdot d(a_i^n - \tfrac{1}{2}3^{-n} \ , \ a_i^n + \tfrac{1}{2}3^{-n}) \qquad [1.2.1]$$

Which turns out to be a restatement of equation [1.0.3].

This is an extremely simple set of partitions, but as such they are suitable to study the process of refining partitions:

Drawing a few lines of unit length and filling in the partitions for n=0, n=1, n=2… the following conclusions become obvious (but can rigorously be shown to be true):

- Once x=$a_i^n$ is in the middle of a part of a partition for some value of n and some value of i, it will be in the middle of a part of a partition for all subsequent values of n and the consequential value of i.
- This property can be used to generate sequences of nested intervals using parts from consecutive partitions as n increases, e.g.

  {($\frac{1}{2} - \frac{1}{2}3^{-n}$ , $\frac{1}{2} + \frac{1}{2}3^{-n}$ ) ; n=0, 1, 2, ….} ;

  {($\frac{1}{6} - \frac{1}{2}3^{-n}$ , $\frac{1}{6} + \frac{1}{2}3^{-n}$ ) ; n=1, 2, 3, ….} ;

  {($\frac{5}{6} - \frac{1}{2}3^{-n}$ , $\frac{5}{6} + \frac{1}{2}3^{-n}$ ) ; n=1, 2, 3, ….} ;

etc

In DIVISION THREE such sets of nested intervals are called **infinitesimals,** which are defined in DIVISION THREE: PART TWO: Section 3.1.3 as:

**Definition**

A sequence of nested volumes, areas or lines of which the maximum diameters converge to zero is called an **infinitesimal volume**, - **area** or - **line** (or simply an "**infinitesimal**" as is the common practice)

## 1.3 The need for two models for Space

There are two fundamentally different abductive assumptions about the nature of Space to be considered here. The first is what was decided in Athens at about 500 B.C. and is referred to here as the approach of Euclid. The other follows the approach of Leibnitz at about 1700 A.D. The approach of Leibnitz turns out to be in line with the thoughts of the Greek philosopher Parmenides of Elea who lived during the sixth and the fifth century B.C.

These two models are distinguished in that the Euclidean approach leads to Space being synthesised using points while the Leibnitzean approach leads to Space being analysed using infinitesimals. The question arises of whether these two models for space are compatible.

Two heuristic arguments to evaluate the compatibility of the Euclidean cosmology and the Leibnitzean approach can be based on the previous paragraphs:

- The first relate to the question of whether the concept of "point", as defined in the Euclidean cosmology, should be used in the Leibnitzean approach. To judge this, one should note that any number that is not of the form specified in expression 1.0.1 will eventually fall outside the intervals of any given infinitesimal as n becomes larger and larger. Therefore, if points as defined in the Euclidean approach should be introduced into the Leibnitzean approach, only points on the real line corresponding to numbers of the form 1.0.1 can be the limits of the infinitesimals. This implies that, should the limit of equation 1.2.1 as n tends to infinity be considered, the sum of the lengths of all points of this form will add up to one. But these points form a subset of the set of rational numbers and hence the set of these points is countable. Therefore, if space is assumed to be synthesised from points in the Leibnitzean approach, a countable number of zeros will sum to a non-zero total. This indicates that the assumption that space is formed from points should not be made in the Leibnitzean approach.

- The second relates to a comparison of equations 1.2.1 and 1.1.3:

$$1 = \lim_{n\to\infty} \sum_{i=1}^{3^n} d(a_i^n - \tfrac{1}{2}3^{-n} \, , \, a_i^n + \tfrac{1}{2}3^{-n}) \qquad [1.2.1]$$

And

$$1 = \sum_{\beta}^{\Gamma} \lim_{n\to\infty} \mathrm{d}(a_\beta - \tfrac{1}{2}3^{-n} \, , a_\beta + \tfrac{1}{2}3^{-n}) \qquad [1.1.3]$$

These two equations relate to the same analysis of the unit interval and, apart from some imaginative indexing, are identical but for the order of the sums and the limits. **But these two equations cannot be converted into each other because the sums and the limits cannot be interchanged in these equations.** (a conclusion underscored by the need for improvised indexing).

These two heuristic arguments suggest that the Euclidean- and the Leibnitzean approaches are not compatible. Thus two alternative and parallel sub-models for Mathematics, based on two non-compatible abductive assumptions about the nature of Space, follows from these two approaches.

## 1.4 The Leibnitzean Cosmology

There are two things to notice about expression 1.2.1:

The first is that the sum ranges over non-zero quantities – no matter how large the value of n. Therefore there is no need here to introduce the concept "more than countable".

The second is that the limit is taken for the sum, and not for the partitions. Therefore there is also no need to introduce the concept of "point" as a property of space.

Hence, if **Remark 1.3** above is taken into consideration, this leads to the

Leibnitzean Cosmology:

Any continuum, no matter how big or how small, is composed of smaller continua, no matter how small[17].

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17 It is fun to pun the well-known quip about fleas:
Big fleas have little fleas upon their backs to bite them, and little fleas have lesser fleas, and so *ad infinitum*
As: Big space has little space that sum to what is in it, and little space has lesser space and so on without limit.

This implies that a point, as a spatial entity, cannot form part of the Leibnitzean Cosmology because a point cannot be divided into two parts of non-zero extent.

Without the concept of “point” as a spatial entity, any discrete object has extent and is a continuum. Thus, on the philosophical level, in the Leibnitzean Cosmology whatever is discrete is formed from continuous entities.

## 1.5 Remark: Points and Infinitesimals

The concept of “point” is central to the Euclidean Cosmology but it has no role as a spatial entity in the Leibnitzean Cosmology. However, “point” is a handy word to use to refer to a “place in space”.

The way that the concept of “point” was introduced above, was done on purpose in such a way that the point was at the end of a given line. In Geometry the endpoints of lines, or vectors, are handy discontinuities to indicate places in space.

Note

While discussing the expression on the right in paragraph 1.1, a point (as per the Euclidean Cosmology) was identified with the following limit:

$$\lim_{n\to\infty}\left(a - \tfrac{1}{2}3^{-n}, a + \tfrac{1}{2}3^{-n}\right) = (a)$$

Where the right hand side, denoted by (a), is an interval of length zero consisting of the point a as a spatial entity. This point a, being a limit for the set of nested intervals of which the maximum diameters converge to zero, made space complete in the Euclidean Cosmology.

The intervals in the left hand side of this equation form a nested sequence of intervals, focussed at the endpoint a of the line (0;a) (as per the Leibnitzean Cosmology).

Thus, in the Euclidean Cosmology this limit exists as the spatial point a, but in the Leibnitzean Cosmology there is no corresponding spatial entity that can serve as a limit. Thus the sequence of nested intervals is a never ending sequence that is focussed at the endpoint of the line at a. Therefore space is not complete in the Leibnitzean cosmology.

Thus an infinitesimal is a never-ending sequence of volumes, surfaces or intervals for which no limit exists.

In the Leibnitzean Cosmology there is no harm in retaining the word “point” (as a shortening of “endpoint of a vector”) to have available for use when working with numbers; as long as it is used only to indicate the place in space where a line ends and is **not** used as a building block for a continuum. Therefore, in the Leibnitzean Cosmology, a point is simply the place in space where a line ends and thus a point is a property of a line and not a property of space.

Note that every infinitesimal is focussed at a place in space that can be indicated by a discontinuity like the end of a line.

## 1.6 Leibnitz, Riemann and L’Hospital

These three are the originators of all the ideas formulated here, but they never realised that they should move away from the Euclidean Cosmology. A look at a possible analysis of their thinking is in order. The ultimate goal of this is to motivate the extension of the number system that is in use in the Mathematical Sciences, to include the Cauchy Numbers of which the infinitesimal numbers form a subset.

### 1.6.1 Leibnitz

Leibnitz and Newton lived in the same era. That was the time when Newton’s Laws were formulated during a renewed interest in Mechanics due to the introduction of the heliocentric model for our solar system. The aim was to describe and predict the motion of particles and bodies. Both Newton and Leibnitz developed tools to study Mechanics. In their systems of describing the way that bodies moved, Newton used fluxions and Leibnitz used infinitesimals. Although Newton’s notation is still sometimes used in Mechanics, the notation introduced by Leibnitz survived in general use.

To describe the motion of a particle Leibnitz[18] had to find ways to determine the gradient of the tangent to the graph of the function $y = f(x)$ at the point $x=a$ in the XOY–plane and the area under the graph of $y=f(x)$ between the values $x=a$ and $x=b$.

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18 Gottfried Wilhelm (von) Leibniz 1/7/1646 – 14/11/1716

To find the gradient of the tangent, he started with a sequence of numbers

$\{\delta x_n$ ; n=1; 2; 3 …}

that converges to zero. He used this sequence to generate two sequences of nested intervals

{(a , a+$\delta x_n$) ; n=1; 2; 3 …}

each of length $\delta x_n$ and

{(f(a) , f(a+$\delta x_n$)) ; n=1; 2; 3 … }

each of length $\delta y_n$ = f(a+$\delta x_n$) - f(a) using the usual sign conventions.

The gradient of the tangent at x=a is then

$$\lim_{n\to\infty} \frac{\delta y_n}{\delta x_n}$$

This limit is of the indefinite form 0/0.

At this point Leibnitz must have realised that $\lim_{n\to\infty} \delta x_n$ and $\lim_{n\to\infty} \delta y_n$ cannot be the number zero of the Euclidean Cosmology. He therefore defined a new class of numbers to augment the number zero and called them the Infinitesimal numbers.

Leibnitz introduced the notation dx and dy to denote the infinitesimal numbers that are the limits of the above sequences $\{\delta x_n\}$ and $\{\delta y_n\}$. Thus

$$\lim_{n\to\infty} \frac{\delta y_n}{\delta x_n} = \frac{dy}{dx}$$

**Note**

This new kind of number was defined as a purely abstract concept by Leibnitz. In the present paradigm of the Mathematical Sciences there are no numerals for infinitesimal numbers, but they are visualised by some as numbers that are small enough to be the limits of sequences that converge to zero, but are still large enough to use in calculations. The limits of the sets of nested intervals above can then also not be points, but must be new spatial entities called infinitesimals which are visualised as intervals of almost zero length. However, with our awareness of the existence of transfinite numbers in the paradigm of Mathematics, these new infinitesimal numbers are immediately recognised as a separate class of cisfinite[19] numbers.

The introduction of infinitesimals and infinitesimal numbers created a structural discrepancy in the Euclidean Cosmology: While zero and infinitesimal numbers can comfortably co-exist in a number system, points and infinitesimals are two different types of spatial entities, and a sequence of nested intervals of which the lengths tend to zero cannot have two different spatial entities as limits[20].

The only way in which this conflict can be resolved is to formulate a second alternative model for Space in which a nested sequence of intervals of which the lengths converge to zero does not need to have a point as limit. This alternative model for Space is **the Leibnitzean Cosmology.**

Another reason why Leibnitz would have wanted to have an infinitesimal of length dx available to use as an interval of length almost zero at the point x on the X-axis, is to be able to calculate the area under the graph of the function y=f(x) between the values x=a and x=b. In this case, the product f(x)dx is an infinitesimal representing the area of a rectangle based on the infinitesimal and with height equal to the value of the function at that place on the axis. An "almost infinite" sum of these "almost zero" areas will then be the required area under the graph. Leibnitz used the notation

$$\int_a^b f(x)dx$$

for this process. The integral sign is an elongated "S" to indicate this "almost infinite" sum.

This operation is an indefinite form of type $\infty \cdot 0$.

### 1.6.2 Riemann

The repeated partitioning of an interval into subintervals to generate repeated approximations to the value of the integral was proposed by

19 A cisfinite number is defined as a non-negative number that is less than all positive numbers.
20 The proper study of Topology happened only after the death of Leibnitz so that he probably was not aware of this discrepancy.

Riemann. The limit of these approximations was the value of the integral.

The value of his insight is that he moved away from the sum of the limits, as Leibnitz did, to the limit of the sums. But he failed to notice the relationship between the various parts of consecutive partitions so that he never noticed the possibility of defining infinitesimals as sequences of intervals instead of as intervals like Leibnitz did.

Once the sequences of intervals that are called infinitesimals in the Leibnitzean Cosmology is recognised it is but a small step to realise that they can be organised to form a directed set by introducing a pre-order:

Let A and B be two infinitesimals. Then A > B means that all intervals forming the infinitesimal B are subintervals of an interval that is a part of the infinitesimal A

In the set of partitions of the basic construction of SECTION 1.0 the first infinitesimal of the directed set of infinitesimals is

$$A = \{(\tfrac{1}{2} - \tfrac{1}{2}3^{-n} , \tfrac{1}{2} + \tfrac{1}{2}3^{-n} ) ; n=0, 1, 2, \ldots.\} \qquad \text{(a)}$$

The second and third members of the directed set are

$$B1 = \{(\tfrac{1}{6} - \tfrac{1}{2}3^{-n} , \tfrac{1}{6} + \tfrac{1}{2}3^{-n} ) ; n=1, 2, 3, \ldots.\} \qquad \text{(b)}$$

and

$$B2 = \{(\tfrac{5}{6} - \tfrac{1}{2}3^{-n} , \tfrac{5}{6} + \tfrac{1}{2}3^{-n} ) ; n=1, 2, 3, \ldots.\} \qquad \text{(c)}$$

Thus A > B1 and A > B2 but B1 and B2 are not comparable, e.t.c.

The integral is then defined as a net defined on this directed set into the real numbers like in DIVISION 3 where the net is into the Cauchy numbers that are still to be defined here.

### 1.6.3 L'Hospital

L'Hospital studied the indefinite form 0/0 that is formed from the quotient of two functions h and g at the point x=a when both h(a)=0 and g(a)=0.

The rule of L'Hospital then states that

$$\lim_{x\to a}\frac{h(x)}{g(x)} = \frac{h'(a)}{g'(a)} \qquad \text{[A]}$$

Let $\{\delta_n\}$ be a sequence of rational numbers that converges to zero. Let $a_n = a + \delta_n$. Then the sequence of numbers X = $\{a_n\}$ is a Cauchy sequence that converges to a and the lines $(a, a_n)$ form a nested sequence of intervals of which the lengths converge to zero (Thus the set of intervals is an infinitesimal focussed at a).

Also, if the functions h and g are smooth enough, then H = $\{h(a_n)\}$ and G = $\{g(a_n)\}$ are two Cauchy sequences of numbers that converge to zero because both h and g are zero at x=a.

Using this notation, the Rule of L'Hospital [A] can be stated as

$$\lim_{n\to\infty}\frac{h(a_n)}{g(a_n)} = \frac{\lim_{n\to\infty}\frac{h(a_n)}{\delta_n}}{\lim_{n\to\infty}\frac{g(a_n)}{\delta_n}} \left[= \frac{\frac{dh}{dx}(a)}{\frac{dg}{dx}(a)}\right] \qquad \text{[B]}$$

Stated like this, the rule of L'Hospital points out a way to find numerals for infinitesimal numbers and a useful definition of infinitesimals by the following argument:

The equality [B] is an equality of real numbers. This means that the equivalence class of Cauchy sequences on the left hand side is the same as the equivalence class of Cauchy sequences on the right hand side.

The equivalence class formed by the left hand side is the limit of the Cauchy sequence

$$\{\frac{h(a_n)}{g(a_n)};\ \mathrm{n} = 1,2,\dots\}$$

and hence this Cauchy sequence belongs to this equivalence class.

But this Cauchy sequence is a comparison of the rates of convergence of the two Cauchy sequences

$$H = \{h(a_n): n = 1,2,3, \dots\}$$

And

$$G = \{g(a_n): n = 1,2,3, \dots\}$$

Which both converge to zero.

**Define** division of these two Cauchy sequences of rational numbers as term-wise division (provided that not more than a finite number of the values of g are zero):

$$\frac{H}{G} = \left\{\frac{h(a_n)}{g(a_n)} : n = 1,2,3,..\right\}$$

Let

$$X = \{\delta_n: n = 1,2,3, \dots\}$$

Then, twice using the same argument as above, the two limits appearing in then right hand side

$\lim_{n\to\infty} \frac{h(a_n)}{\delta_n}$ and $\lim_{n\to\infty} \frac{g(a_n)}{\delta_n}$

Become

$\frac{H}{X}$ and $\frac{G}{X}$

So that the rule of L'Hospital can be written as

$$\frac{H}{G} = \frac{H}{X} \Big/ \frac{G}{X}$$

In this form the rule of L'Hospital states that the rates of convergence of two Cauchy sequences that converge to zero can be obtained by comparing their rates of convergence relative to a "gauge" Cauchy sequence that converges to zero. He thus transformed the process of determining rates of convergence to algebraic operations on known rates of convergence.

Notice that comparison of the two forms of the right hand side implies that

$$\frac{H}{X} = \frac{dh}{dx} \text{ and } \frac{G}{X} = \frac{dg}{dx}$$

So that the infinitesimal numbers dh, dg and dx can be defined as the Cauchy sequences

dh = $\{h(a_n)\}$ ; dg = $\{g(a_n)\}$ and dx = $\{\delta_n\}$

which all converge to zero: i.e. they are all equivalent to 0 = {0; 0 ;0; ….}

The above motivates the following definition of the **Cauchy Numbers** (This is fully done in DIVISION THREE; PART TWO Section 3.1.1):

The Cauchy sequences that are the elements of the equivalence classes that form the real numbers, are defined as the Cauchy Numbers. The Cauchy numbers that are equivalent to zero are the infinitesimal numbers and the arithmetical operations for Cauchy numbers are performed component wise.

With this definition, the component-wise application of the arithmetical operations implies that the limiting process can be replaced with an arithmetical process and the rule of L'Hospital can be derived as follows:

It was shown above that because h(a)=0 and g(a)=0 the Cauchy numbers dh, dg as well as dx are all infinitesimal numbers. Therefore, at x=a

$$\frac{df}{dg} = \frac{df}{dg} \cdot \frac{dx}{dx} = \frac{df}{dx} \Big/ \frac{dg}{dx}$$

Note that the sign "=" means "is the same equivalence class" when used with real numbers and has the meaning "belong to the same equivalence class" when used with Cauchy numbers. With this and the definition of the arithmetical operations for Cauchy numbers in mind, the above operation is validated by the fact that the rational numbers that form two equivalent Cauchy numbers are asymptotically equal.

One should also note that this implies that these operations are not limited to infinitesimal numbers only, but are valid for all Cauchy numbers. For example, division of a rated number[21] by an infinitesimal number will have an infinite Cauchy number (a divergent sequence) as result.

21 A rated Cauchy number is a Cauchy sequence that has a finite non-zero real number as limit.

# 2. Numerals and the numbers they represent

## 2.0 Introduction

The ideas mentioned here about numerals are best explained when limited at first to natural numbers.

The word “numeral” has all but disappeared from the English language, and its place has been taken by misuse of the word “number”. The word “number” denotes an abstract[22] concept, best explained by referring to a natural number as a class of sets with the property that they can be mapped one-to-one onto each other. The word “numeral” denotes a physical entity, usually a symbol, which is associated with a number. For example the class of sets which can be mapped one-to-one onto the set {X ; X ; X} has as numerals the symbols 3 (a morph of the original symbol ≡), the roman numeral III, the binary numeral 11 and even FFFFFFTT as a byte in computer storage.

Addition, subtraction, multiplication, division and checking for equality of numbers can only be done by counting (mapping sets onto each other). When numerals are used certain actions (Arithmetic or computer operations etc.) for the manipulation of numerals can be performed to form new numerals and in so doing eliminate the necessity to count. (The Romans used tiles on tables as abaci. Therefore roman numerals are essentially a notation for abacus settings and are not amenable to arithmetic as we know it.)

In modern culture numerals for natural numbers, whole numbers and rational numbers have evolved over time so that these numerals are now standardised for the western decimal system. As a result the most important properties that we require that numerals should have now are (1) that each should identify the number that it represents uniquely, (2) that it should denote the value of the number unambiguously (3) that it allows a simple arithmetic and (4) that different numerals should be easily distinguishable. (Here the phrase “denote the value of a number” means that, given any two numerals, it can be decided which of the

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22 The word “abstract” is of Latin origin and its literal meaning is to “pull out of” - in this case it is pulled out of the reality that is experienced day to day – and thus it describes the nature of some essential property of reality.

numbers that they represent is the larger or whether they are equal. This is usually done using subtraction)

The numeral of a positive rational number is called a common fraction and it is formed by writing the numerals of two natural numbers one above the other with a line between them indicating division. The numeral on top is called the numerator and the one below the line is called the denominator.

Note that two rational numbers are equal if their difference is zero. Subtracting the two numerals $\frac{a}{b}$ and $\frac{c}{d}$ and setting the result to zero, shows that the set of rational numbers is divided into equivalence classes where these two numbers are equal iff they belong to the same equivalence class: i.e. if and only if ad = bc.

When the division used in the numeral is actually performed by using the arithmetical process of "long division" the result is a decimal fraction. This decimal fraction is either of finite length or it is never ending. In the latter case it is an "infinite decimal fraction".

Note that an infinite (never ending) decimal fraction cannot be a numeral, as described above, without additional information because the value of the number cannot be inferred unambiguously from such an infinite decimal fraction. This is illustrated by the following example as given in DIVISION THREE, PART ONE Section 1:

Example A

Let

$b = 0.b_1b_2b_3b_4\ldots$

Be an infinite (never ending) decimal fraction.

Consider a second infinite decimal fraction that is derived from b:

$b^* = 0.b^*_1b^*_2b^*_3b^*_4\ldots$

where

$b^*_n = 7$ if the two strings of digits "$b_1b_2b_3b_4\ldots.b_n$" and "$b_{n+1}b_{n+2}b_{n+3}b_{n+4}\ldots.b_{2n}$" are identical

and

$b^*_n = b_n$ if these two strings are not identical.

It is in principle possible to detect if b ≠ b* but it is impossible to detect if b = b*.

■

Thus it appears to be possible to change the numeral of a rational number into a non-numeral. But it is well known that an infinite decimal fraction generated from a rational number is a never ending repetition of an unchanging substring of digits, where the length of the repeating substring cannot exceed the value of the denominator. This allows the infinite decimal fraction to be changed back into a numeral by writing this repeating string only once, but with dots over the first and last digits of the string.

For example $7.\dot{1}2\dot{3}$ is a numeral for the never ending decimal fraction 7.123123123…..and therefore it is known that the string "123" is repeated indefinitely.

This numeral can be transformed back into a common fraction as follows:

let

$$a = 7.\dot{1}2\dot{3}$$

Then

$$1000a = 7123.\dot{1}2\dot{3}$$

Subtracting:

$$999a = 7116.$$

Thus

$$a = \frac{7116}{999} = \frac{2372}{333}$$

There are two important points underlined by this example.

The first is that two infinite decimal fractions were subtracted from each other contrary to the requirement that only numerals can be subtracted using arithmetic, even though it was argued above that an infinite decimal fraction cannot be a numeral as is required. This indicates that the important aspect of infinite decimal fractions is not the presence of infinitely many digits, but rather that the information about the number should be complete.

**Definition**

An object will be called **identified** if all relevant information about it are known.

The second is that, if the repetitive nature of “123” is not stated explicitly, this procedure cannot be done. Looking at the symbol 7.123123123…... without knowing that the string “123” is repetitive, adding the next digit to the symbol may well result in 7.1231231239….. This introduces an insurmountable obstacle in Mathematics: **it is not possible to identify the nature or the limit of a sequence by looking at the terms of the sequence**.[23]

Notice that this is the reason why equality cannot be decided in Example A.

The nature of this enigma is investigated in the next paragraph by looking at an example that illustrates on a fundamental level some aspects of its impact on Mathematics.

Note that when a numeral exist for a number, there is a one-to-one relationship between the words “number” and “numeral” up to equivalence of the numerals, hence nothing is lost using the word “number” for both, as has become common but may be misleading.

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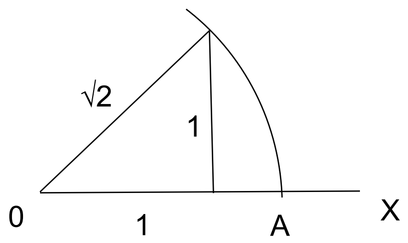
23 This is an inherent property of all inductive reasoning. In the medical profession it appears in diagnosis and in the legal profession circumstantial evidence can prove a case only “beyond reasonable doubt”.

## 2.1 Real Numbers

The study of real numbers is based here on an analysis of the following example:




Using Pythagoras' Theorem for a right triangle of which both catheti are of length one, the hypotenuse - and hence also the line OA - has the length $\sqrt{2}$.

However, it is easy to prove that the number $\sqrt{2}$, i.e. the value of the square root function $f(x)=\sqrt{x}$ at $x=2$, cannot be a rational number.

Thus it is necessary to extend the number system beyond the rational numbers in order to make it possible to assign a value to the length (extent) of the hypotenuse of this triangle. In this extension of the number system, the value assigned to $\sqrt{2}$ must be such that the square root function $f(x) = \sqrt{x}$ is continuous at the value $x=2$. This requires that the new number should be such that its square equals 2.

*Note that the value of the number assigned to $\sqrt{2}$ is a number and not a numeral.*

The quest for a numeral to represent this number can be investigated by looking at the following example, where the value of $\sqrt{2}$ is approximated to any required degree of accuracy from below by a sequence of rational numbers:

Since $1^2 = 1$ and $2^2 = 4$ it is clear that the value of the number that is to be used in this approximation is somewhere between 1 and 2. The following list of numbers can then be tried as candidates for the next approximation:

$$1.1^2 = 1.21 \; ; \; 1.2^2 = 1.44 \; ; \; 1.3^2 = 1.69 \; ; \; 1.4^2 = 1.96 \; ; \; 1.5^2 = 2.25$$

Since 2.25 is larger than 2, it is clear that the value of the number that is to be used is somewhere between 1.4 and 1.5 and the number 1.4 differs by less than $10^{-1}$ from the targeted value while still being less than the number itself.

This process can be repeated indefinitely and yields a sequence of numbers

1.4 ; 1.41 ; 1.414 ; 1.4142 ; 1.41424 ; …..

Of which the $n^{th}$ term deviates by less than $10^{-n}$ from the targeted value. These are all rational numbers and form a non-decreasing sequence. This sequence is a Cauchy sequence in the difference topology on the rational numbers.

On the given X-axis the points corresponding to each of the terms in the above sequence can be identified. The first term coincides with the point at x = 1, the second with the point at x = 1.4 and so forth. The length of the line OA is √2 and A is the point representing the number √2.

The difference topology for the rational numbers and the Euclidean topology for points on the line are identical under the mapping described above. Thus these points also form a Cauchy sequence in the Euclidean topology on the line and therefore converge to the endpoint A of the line. Hence, on the x-axis, there **exists** a limit point for the Cauchy sequence of points, and this limit is **identified** as the point A.

Note that in the Euclidean Cosmology this point is a spatial entity and in the Leibnitzean cosmology it is the place in space indicated by the endpoint of the line (vector) OA.

**Remark**: The possibility to identify the limit point for the Cauchy sequence is rare. In the case above it follows from the geometrical construction and not from numerical considerations.

In the above example the value of a function was given as the desired number and a Cauchy sequence of rational numbers (numerals) was generated to approximate it. But, according to Jacobi, “Mann muss immer umkehren”. Therefore one should also start from a given Cauchy sequence of rational numbers and from there progress to its unknown limit.

Thus, using the never-ending Fibonacci sequence

1 ; 1 ; 2 ; 3 ; 5 ; 8 ; 13 ; 21 ….

a never-ending string of digits

1123581321….

can be formed. This string can then be used to form the a never-ending sequence of numerals

1 ; 1.1 ; 1.12 ; 1.123 ; 1.1235 ; 1.12358 ; 1.123581 …

These numerals all represent rational numbers. They form a Cauchy sequence of non-decreasing rational numbers and, as before, the corresponding points on the x-axis can be identified. They also form a Cauchy sequence in the Euclidean topology and, as before, the sequence converges to a point. But although this point **exists** it cannot be **identified**.

This leads to the familiar

**Definition**: The limit of a Cauchy sequence of rational numbers is called a **real number**.

**To emphasise**: This is a number and not a numeral.

Note:

The Cauchy sequence

1.4 ; 1,41 ; 1.414 ; 1.4142 ; 1.41424 ; ….. [a]

is usually written in condensed form as

1.41424 ….. [b]

And the Cauchy sequence

1 ; 1.1 ; 1.12 ; 1.123 ; 1.1235 ; 1.12358 ; 1.123581 …

In condensed form as

1.123581321….

**Definition**

Either one of the above forms [a] or [b] will be called the **Cauchy Base** for the real number √2.

Thus, for a specific real number, the form of the Cauchy base is unique and is distinct from all other Cauchy sequences converging to that real number.

**Definition**

The vector

(1.4 ; 1,41 ; 1.414 ; 1.4142 ; 1.41424 ; …..)

will be called the **Cauchy form** for the number √2.

In the first case studied above, a required value √2 was given and a Cauchy base for this number was formed by constructing a Cauchy sequence that converges to this number.

**Definition**

When the value for a real number is specified the number is called **value specified**.

Note that this specified value is used to construct the Cauchy base. Therefore this is a deductive process.

In the case of the Fibonacci sequence a Cauchy base was constructed and this then defined the real number that is its limit.

**Definition**

When a real number is specified as the limit of a given Cauchy sequence, the real number is called **component-specified**.

Note that in the latter case the Cauchy base is constructed using the Cauchy sequence. The value of the limit is unknown and therefore this is an inductive process.

**To sum up:**

*For a value specified real number, the number is given and a Cauchy sequence of rational numbers is set up to have the number as a limit.*

And:

*For a component specified real number, a Cauchy sequence of rational numbers is specified and the real number is its (usually unknown) limit.*

**The real Line**

In the Euclidean cosmology the above definition of a real number causes the set of points corresponding to the rational numbers to be dense in the set of all points of the line. Thus a sequence of such points can be found to converge to any given point on the line. Hence there is a one-to-one order preserving homeomorphism between the real numbers and the points of a line. This correspondence is called the **real line**.

In the Leibnitzean cosmology it is assumed that every line has a definite length. Thus any given Cauchy sequence of rational numbers can be used to set up an infinitesimal that is focussed at the endpoint of some line. Conversely, any given line from the origin can be used to set up a Cauchy sequence of rational numbers to form an infinitesimal that is focussed at its endpoint, as was done in the introduction. Hence there is a one-to-one correspondence between the real numbers and all possible lines from the origin on the axis. This correspondence is also called the **real line** in the Leibnitzean cosmology.

**Remark:**

In the Euclidean Cosmology, there are more than countable many value specified real numbers. This is because, in this case, there are more than countable many points on the real line and for any given point a Cauchy sequence can be set up to converge to that point.

In the Leibnitzean Cosmology, Theorem 4.1 of DIVISION THREE, PART ONE Section 4 shows that there are countable many Cauchy Bases for real numbers. The following theorem extends this result to the set of all Cauchy sequences and hence that the set of the limits of all given Cauchy sequences is countable.

Not every Cauchy sequence that defines a real number is necessarily a Cauchy base for that number because a Cauchy base is required to be of a specific form - it has to be a never ending string of digits:

**Theorem:** Any Cauchy sequence of rational numbers is equivalent to the Cauchy base of some real number.

**Proof:**

Let $\{a_n\}$ be any Cauchy sequence of rational numbers.

In the definition of a Cauchy sequence, set $\varepsilon_n = 10^{-n}$. Then there exists a natural number $N_n$ such that $|a_r - a_s| < 10^{-n}$ for all $r,s \geq N_n$. Thus all the numbers in the subsequence $\{a_s : s \geq N_n\}$ can only differ after digit number n, and by an amount of at most $10^{-n}$. Let $b_n$ be the rational number formed by truncating any one of these fractions, say fraction number $N_n$, after digit number n. Then $\{b_n\}$ is a Cauchy sequence of non-decreasing rational numbers equivalent to $\{a_n\}$ and it is the Cauchy form of some real number b. Because of the ambiguity that may arise between the digits 9 and 0 at position n+1 the proof is easily adapted (remembering that a Cauchy Base is non-decreasing) by selecting the larger of the two possible forms of the number $b_n$ when required.

■

**Conclusion:**

In the Leibnitzean cosmology the set of equivalence classes of Cauchy sequences is countable.

**Proof:**

Let a be any equivalence class of Cauchy sequences and let $\{a_n\}$ be a Cauchy sequence in a. Then $\{a_n\}$ is equivalent to at least one Cauchy base. Thus every equivalence class of Cauchy sequences contains at least one Cauchy base, and this defines a component described real number. But two equivalence classes of Cauchy sequences cannot share a Cauchy base because of transitivity. Thus there exists a one-to-many mapping from the set of equivalence classes into the set of component specified real numbers, and the latter set is countable.

■

Hence theorem 4.1 of DIVISION THREE mentioned above shows algebraically that the set of real numbers in the Leibnitzean cosmology is countable.

# 3. Infinity

## 3.1 Infinity In The Leibnitzean Cosmology

In the Leibnitzean Cosmology the word “infinite”, used as an adverb or adjective, means “never ending” or “unbounded”. This is how infinitesimals and Cauchy numbers are defined in the Leibnitzean cosmology. The noun “infinity” refers to the imaginary ‘place’, indicated by the end of an axis, where infinite Cauchy numbers are focussed. This is geometrically shown in the compactification of the real numbers shown in DIVISION THREE: PART TWO: Section 3.1 in the Subsection 3.1.2.1.

This is in agreement with our primitive perception of “infinite” which stems from at least two relevant sources:

The first source is our perception of continuous space – a line of any length drawn in any direction can always be extended by any amount. This is also how we perceive the nature of time, which we experience to be without beginning and without end.

The second source is the existence of discrete symbols (like for the natural numbers) where it is possible to set up procedures whereby symbols can be systematically changed in such a way that a new symbol, which perceptibly differs from all previous symbols, can always be formed.

As introduced in DIVISION THREE and expanded in DIVISION ONE, in the Leibnitzean cosmology the phrases: ”the function f is zero” or “the function f is infinite” at the point x=a in general means that a is a Cauchy number and that f(a) is an infinitesimal number (possibly the infinitesimal number zero depending on the situation)[24] or f(a) is an infinite Cauchy number.

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24 I.e. if the fractional part of a is a finite string of digits, then f(a) has to be the infinitesimal number zero.

### 3.2 Infinity in the Euclidean Cosmology

In section 1.1 above it is argued that the assumption of the Euclidean Cosmology that space is synthesised from points leads to the necessary conclusion that an infinite number of operations can be performed and that the result of these operations is identified and discrete (or complete).

This then implies that the operation of finding ever increasing strings of digits to approximate $\sqrt{2}$ in section 2.1 above can be performed to completion. The string of digits that is so obtained will be of infinite length, where “infinite” means “a natural number larger than all other natural numbers”.

Such a string of digits cannot exist in the reality of our experience where it would be possible to manipulate it: for even if a digit could be etched on every atom in the galaxy, the string would still not be of infinite length. Therefore this string can only exist in an idealised reality, commonly referred to as “Abstract”.

Note that the ability to perform an infinite number of actions means that the restrictions of inductive logic does not apply to all aspects of Euclidean Cosmology. This makes it possible, in Cantor’s diagonal proof as given in DIVISION THREE PART ONE Section 1, to make the transition from “the infinite decimal fraction b differs from any given number in the list” to “the infinite decimal fraction b differs from all numbers in the list”. This validates Cantor’s proof in the Euclidean Cosmology.

But the list of strings of digits in Theorem 4.1 of DIVISION THREE mentioned above cannot be constructed to completion. Thus this Theorem does not form part of the Euclidean Cosmology.

Conversely, Cantor’s proof is invalid in the Leibnitzean Cosmology where the restrictions of inductive logic apply.

# DIVISION THREE: Cantor's Fallacy and the Leibnitzean Cosmology

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Dedicated to PARMENIDES of ELEA

## Foreword, Synopsis and Acknowledgement

When a crime is committed the existence of a criminal is automatically known. It is then the function of the forces of justice to identify the criminal.

In Mathematics this distinction between “To exist” and “To be identified” is often vague or ignored.

Whenever a Cauchy sequence of points is specified, it is known that its limit exists because of the completeness of the real line. But it is also known that this limit cannot necessarily be identified.

In part one the consequences of this vagueness between existence and identification of real numbers are studied. The analysis starts with an example showing that the equality/inequality of two real numbers can only be decided if both are identified. It is then concluded that the similarity of the infinite decimal fraction of the example to the fraction that is constructed in the diagonal proof indicates, in a simplistic way, that the argument of the diagonal proof is itself simplistic.

Some relevant concepts are then discussed and new concepts are introduced to facilitate the analysis. This discussion culminates in a proof that the set of equivalence classes of Cauchy sequences is countable.

In the final section of part one it is shown that the use of the axiom of choice in the construction of a real number leads only to the existence of the number and not to its identification. This is done by showing that, if the Euclidean topology is taken into account, a contradiction results when it is assumed that the real number which is constructed in the diagonal proof, is identified. As a consequence of this it is concluded that the diagonal proof, in the presence of limits, is invalid.

In part two the conundrum that the real numbers are countable according to number theory but more than countable according to the real line, is addressed. This is done by looking at the infinite divisibility of space according to Euclid and according to Leibnitz. It is concluded that these two approaches lead to two different models for Mathematics because the circumstances do not allow the order of sums and limits that occur to be inverted. These two models are then called the Euclidean- and the Leibnitzean Model for Mathematics, and they would lead ultimately to two different cosmologies.

The concept of infinitesimal is introduced and then a re-interpretation of standard Number Theory is used to generalise the concept of number to what is called here the Cauchy numbers. It is shown that the Cauchy numbers consist of three classes of numbers, namely the infinitesimal numbers, the rated numbers and the infinite numbers and that these three classes describe the real continuum. The relationship between these classes of numbers is then studied in the spirit of L'Hospital. The concept of differential is introduced. Finally cascades of infinitesimals are introduced and the fundamental Theorem of Calculus is studied in the Cauchy number context.

In part three the Leibnitzean cosmology is introduced. It is pointed out how the Leibnitzean cosmology fits in with the philosophy of Parmenides of Elea and Zeno. It is pointed out that the remark by Parmenides that there can be no motion (because everything is one) and the paradox of the arrow are explained by the Leibnitzean cosmology. It is also pointed out that some of the intractable problems of Physics - like the particle/wave duality and action at a distance - are consequences of the Euclidean cosmology that become tractable in the Leibnitzean cosmology.

In part four it is pointed out that once the Euclidean Cosmology is discarded, those results of the Euclidean model of Mathematics which are not related to the Euclidean cosmology, still form part of the Leibnitzean model; thus creating once again a single model for Mathematics, but without the Euclidean Cosmology.

Finally, I should point out that the ideas presented in this document were never subjected to proper scientific scrutiny. This is because these ideas are completely contrary to current group-thinking in the scientific community and thus no willing peers could be found. In that aspect this document should be looked at as a discussion document with the objective of soliciting criticism from the scientific community at large. However, I would like to thank my good friend Dr Anneke Roux of the Department of Civil Engineering at the University of Pretoria for supporting me in this by always having been willing to listen to me; no matter how weird my ideas were.

# PART ONE: The Fallacy in Cantor's Diagonal Proof for Real Numbers

## Section 1: Introduction

### Cantor's (well known) Theorem.

The real numbers are more than countable.

### Proof

Assume that the real numbers are countable. Hence a list containing all the real numbers larger than or equal to zero and less than or equal to one can be made. Each such real number is an infinite decimal fraction, so that a list of all the infinite decimal fractions between zero and one inclusive can be made:

$0.a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{16}\ a_{17}\ a_{18}$ .......

$0.a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}\ a_{26}\ a_{27}\ a_{28}$, .......

$0.a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ a_{36}\ a_{37}\ a_{38}$ .......

$0.a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45}\ a_{46}\ a_{47}\ a_{48}$ .......

$0.a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55}\ a_{56}\ a_{57}\ a_{58}$ .......

$0.a_{61}\ a_{62}\ a_{63}\ a_{64}\ a_{65}\ a_{66}\ a_{67}\ a_{68}$ .......

.

.

.

Where each $a_{ij}$ is one of the digits 0, 1, 2…, 9

Let b be that real number (infinite string of digits) $0,b_1b_2b_3b_4$.... which is such that

$b_i$=2 if $a_{ii} \neq 2$ and bi=5 if $a_{ii} = 2$. **[A]**

This real number b differs from any number in the list in at least one place. Thus b does not belong to the list. This contradicts the assumption that all real numbers larger than zero and less than one belong to the list. Thus all real numbers between 0 and 1 cannot be listed, and hence these numbers are more than countable.

■

The counterbalance to this theorem is the following example:

**Example A**

Let

$a = 0.a_1a_2a_3a_4a_5....$

be an irrational number between 0 and 1 as in Cantor's theorem.

Choose the digits of the infinite decimal fraction

$a^* = 0.a^*_1\, a^*_2\, a^*_3\, a^*_4\, a^*_5....$

as follows:

$a^*_i = a_i$ if the two strings of digits "$a_1a_2a_3a_4$ ….. $a_i$" and "$a_{(i+1)}a_{(i+2)}$ …..$a_{2i}$" are not identical

and

$a^*_i = 7$ if these strings are identical.

■

**Note**:

The fractional part of an irrational number is a pseudo-random string of digits – i.e. even though the order of the digits is fixed, there may be no regularity in their occurrence. However, for any random string of digits of length 2n, the probability that the string consisting of the first n digits is the same as the string consisting of the last n digits is $10^{-n}$. Thus, from a probabilistic point of view, the probability that $a^*_n$ differs from $a_n$ is $10^{-n}$. Therefore the probability that the fraction a* differs from the fraction a diminishes rapidly with the length of the substrings under consideration. **But this probability never becomes zero.**

In the light of this, it can be concluded that it is in principle possible to determine for a given infinite decimal fraction, a, whether a* is smaller or larger than a. This is done by generating the digits one after another until two is found that are not the same. But if no such digits appear, nothing at all about the equality/inequality of the numbers can be deduced. Hence it is not even in principle possible to determine whether a* is equal to a, nor is it possible to determine the value of a* should it differ from a.

This example shows that the equality/inequality of two infinite decimal fractions can only be decided once they are identified, i.e. all their digits are known. Or, to put it milder, specifying the finite digits of an infinite decimal fraction does not provide enough information to decide the question of their equality/inequality.

This example reveals the simplistic nature of the argument in Cantor's diagonal proof. It shows that the information contained in the strings b and b* is not sufficient to determine whether b and the associated fraction b*, which by assumption belongs to the list, are equal or different. Consequently the argument in the proof becomes suspect[25]. This is looked at in formal detail in section five.

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25 This example is enough to destroy the logical structure of the proof. See the addendum at the back of the monograph.

**A possible reason why this simplistic proof has been accepted without objection is because, according to the real line, the theorem is true.**

## Section 2: The Real Line

As an addition to the existence of volumes, areas and lines that extend in three, two and one directions, Euclid defined a point as 'That which has no extent'. This implies that a point is a thing (and consequently a piece of space) of which the volume, area and length are all zero. It is then required that a volume, area or line be formed by the combination of points. These assumptions will be called the **Euclidean Cosmology**.

For a line, the total length of a finite number of its points must be zero because it is a finite sum of zero's. But then the total length of a countable number of its points must also be zero because this length is the limit of the total lengths of the partial sums which are all zero. However, the total length of all the points on the unit interval of the line must be the length of the line, and hence it must be one. Thus the number of points on a line of unit length cannot be either finite or countable, and therefore it is concluded that there must be more than countable many points in this interval.

The concept of the real line extends this property to real numbers:

**The real line:**

There is an order-preserving one-to-one mapping of the real numbers onto the points of a line. This mapping is a homeomorphism in the standard topologies of the real numbers and the line.

This one-to-one mapping then implies that the real numbers must be more than countable too, and this gave support to Cantor's theorem.

## Section 3: Infinite Decimal Fractions

### 3.1 Note about Notation

From the way the phrases ‘infinite decimal fraction’ and ‘diagonal’ are used in the proof of Cantor’s theorem, follows that the following phrases refer to infinite strings of digits with different other symbols interspersed, and thus are assumed to be equivalent:

- ‘Infinite decimal fraction’
- ‘Infinite (dimensional) vector of digits’
- ‘Infinite sequence of digits’

In the same context, an infinite matrix is considered to be an infinite list of infinite vectors.

### 3.2 A synopsis of the logical history of infinite decimal fractions

When the division algorithm is used to convert a rational number into decimal form, it is found that the resulting decimal fraction is either of finite length or becomes a repeating sequence of digits of which the length of the repeating string is less than or equal to the denominator of the fraction. Conversely, a decimal fraction of finite length or with a repeating sequence of digits can be transformed back into a proper fraction.

Certain numbers, e.g. the function value √2, can be shown to be not rational by using the properties of the function. In a case like this the properties of the function and a tool like Taylor’s Theorem can be used to approximate the number to any desired accuracy by a decimal fraction. Because the given number is not rational, it cannot have the properties of the decimal representation of a rational number. Therefore this approximation cannot be either finite or have a repeating cycle of digits. Thus this approximation must be an infinite pseudo-random string of digits.

### 3.2.1 Notes

- In the above cases the digits of the resulting infinite decimal fraction are generated one-by-one. Thus the infinite decimal fraction a = $0.a_1a_2a_3a_4$…. is generated as the Cauchy sequence of numbers a = {$0.a_1$ ; $0.a_1a_2$ ; $0.a_1a_2a_3$ ; $0.a_1a_2a_3a_4$ ; ….}. This Cauchy sequence is the form in which infinite decimal fractions are studied in Number Theory, and will be called the **Cauchy form** or **Cauchy representation** of the number.

- In both these cases the word 'infinite' refers to calculations that are performed repeatedly, and hence it means 'never ending'

### 3.2.2 The real line is introduced

When a line and a suitable scale is chosen as axis, there exists a simple geometrical procedure that allows the construction of a line of length equal to any given rational number. This allows any rational number to be associated with a point of the line. Thus the terms of the Cauchy representation of a number maps onto an open set of points of the line. Because the numbers form a Cauchy sequence, these points form a Cauchy sequence in the Euclidean topology of the line.

But according to the definition of a point the line is complete in the Euclidean topology, and therefore there **exists** a limit point for the Cauchy sequence of points. This validates the introduction of a new kind of number as a limit for the Cauchy sequence formed by the Cauchy representation of a number. This new number is called a **real number**. If these two limits are mapped onto each other, the **real line** is defined and there is a one-to-one order preserving homeomorphism of the real numbers onto the points of a line.

Because a real number is the limit of an ever longer string of digits, it is called an **infinite decimal fraction** or a string of digits of infinite length. Note that in this scenario the term 'infinite' means 'a number larger than any natural number'. This has to be so because, if the number denoting the length of the infinite string is not larger than any natural number, it has to be a natural number itself and consequently the number then refers to a term of the Cauchy sequence and not to its limit.

There exist many Cauchy sequences of points converging to any given point on the real line. For any given point on the real line these sequences form an equivalence class in the set of all Cauchy sequences of points. This equivalence class is uniquely associated with the given point. The real line then ensures that there is a unique equivalence class of Cauchy sequences of numbers that is associated with any real number.

Therefore a real number has three avatars: (1) It is a point of a line, (2) it is an infinite sequence of digits (with 'infinite' meaning 'larger than any natural number') and (3) It is an equivalence class of Cauchy sequences of numbers.

**Remark**

Even though all the limits of these Cauchy sequences exist, they are not necessarily **identified**, as illustrated by Example A of section one.

### 3.2.3 Conclusion

**For two real numbers to be equal, they must both be identified and all three of their corresponding avatars must be identical.**

*Therefore, because an infinite decimal fraction is a single point on the real line, it is only identified if all its digits are known.*

### 3.2.4 Note

A Cauchy sequence forms an open set on the real line; but a real number, being a single point, is a closed set.

### 3.3 The specification of numbers

As pointed out above, numbers are shown to be irrational by default: the value of a function at a given argument is shown to be not rational by using the properties of the function – like for √2. A real number, specified in this way, will be called **value specified**. This is the number associated with the point on the real line onto which the argument of the function is mapped. Hence a real number specified in this way can also be said to be **value identified** because the point where it is located on the axis can be found by using the properties of the function. By implication both other avatars are then also value identified.

In Cantor's proof the constructed real number is described as an infinite decimal fraction by specifying the finite digits of its representation.

With the above note on notation in mind, a real number as well as an infinite matrix that is specified by stating their components, will be called **component specified**. Thus the real number constructed in Cantor's diagonal proof is an example of a component specified real number, as is the number constructed in Example A.

According to this example, although the irrational number √2 is both value specified and value identified, √2* is component specified but is not component identified. Hence the equality/inequality of these two numbers cannot be decided. This example emphasises that two real numbers cannot be compared unless both are identified.

Therefore the Cauchy form of a component-specified real number can be mapped onto an open set of points on the real line as a 'never-ending' sequence of points for which it is known that a limit exists, but for which the limit cannot necessarily be identified.

The same holds true for component specified infinite matrices.

## Section 4: Component Specified Real Numbers

### 4.1 Theorem

The set of component specified infinite decimal fractions is countable

### Proof

(Tis proof mimics the proof that the set of rational numbers is countable)

Write the digits 4, 5, 6, 7, 8, 9, 0, 1, 2, 3 on 10 consecutive lines

Repeat this group another 9 times so that 100 lines are filled with ten of these groups of ten digits.

Add second digits to the lines: a 1 to the first group, and from 2 to 0 respectively to every consecutive group. All 100 possible permutations of two digits are now listed, the top one being the first two digits of the decimal part of √2.

Take this 100x2 array and repeat it 9 times so that 1000 lines now have two digits. Append the next digit of √2 to the first 100, and then proceed cyclically as above, adding the other nine digits to the next nine groups. All 1000 possible permutations of three digits are now listed, the first line being the first three digits of √2.

Repeating this process, a component specified infinite two-dimensional array is constructed containing as rows all possible infinite permutations of the ten digits, the first row being the component specified fractional part of √2 i.e. the Cauchy form of √2.

*Thus an infinite list of all possible component-specified infinite decimal fractions between zero and one is constructed, and the theorem is proved.*

■

In the theory of numbers it is shown that any given Cauchy sequence can be converted to an equivalent infinite decimal fraction by using the definitions of equivalence and of Cauchy sequence. A number obtained in this way is a component specified real number.

**Corollary**

The set of equivalence classes of Cauchy sequences is countable.

**Proof**

Every equivalence class of Cauchy sequences contains at least one component specified infinite decimal fraction, as is pointed out above.

But transitivity of the equivalence relation prevents any two equivalence classes from sharing a component specified infinite decimal fraction. Thus there exists a one-to-many mapping of the set of equivalence classes into the set of component specified real numbers, and the latter is countable.

■

## Section 5: Infinite, Never-Ending and the Axiom Of Choice

### 5.1 Introduction

In the previous sections the attributes 'infinite' and 'never ending' were used in situations where the associated sets of points of the real line were respectively closed or open.

This is because, in the cases where these sets were closed, the term 'infinite' had properties of being a very large number – one that is 'larger than any other number'. Two examples of such sets are relevant:

First, consider the closed interval [0,1]. This interval is a continuum. If a single point, a, is removed from this interval, the continuum is destroyed and two half open intervals [0,a) and (a,1] are formed. This implies that the continuum can only exist if 'all' its points are present.

Next, consider the irrational number a, and let $a = \{a_n\}$ be its Cauchy form. The number 'a' maps onto a single point of the real line. This point is the limit of the points onto which $\{a_n\}$ maps and, being a single point, is a closed set. If a is referred to as an infinite decimal fraction, then the word 'infinite' must refer to a 'number larger that all natural numbers' because any natural number of digits would refer to a term of the Cauchy representation and not the limit (as was pointed out before).

### 5.2 Existence and Identification

The example of section 1 shows that component-specified real numbers, and thus also the number ‘b’ of the diagonal proof, can not necessarily be identified in practice – i.e. although any one of the finite digits can be identified, all of its digits cannot be found.

When Zeno stated the paradox of Achilles and the tortoise, the response to the paradox was, in essence, that the real line is complete and that, although an infinite number of steps is required by Zeno’s argument, a limit exists and that limit fixes the point and time when Achilles would pass the tortoise.

But Zeno was intent on creating a paradox and thus used the expression ‘Achilles can never pass the tortoise’. If he had instead said ‘We can never know where Achilles would pass the tortoise’ the answer to his statement would have been much more difficult; because then he would have conceded the existence of the limit point, but now required its identification.

In modern times it became part of Mathematics to assume that an infinite number of choices can in principle be made; this is known as the ‘Axiom of Choice’. However, in the present situation, it is not clear whether application of the axiom of choice will result in ‘existence’ or in the more specific ‘identification’.

Note that the diagonal proof becomes valid if, in the construction of the number b, the axiom of choice should lead to ‘identification’ of the number and not merely to its ‘existence’. This is because then ‘all’ digits of the number b would in principle be known a posteriori. All the digits of all the numbers in the list are assumed to be known a priori because of the logical structure of the argument. Because all digits of all numbers are known they can be compared and the number b would differ from all numbers in the list if it should differ from any given one at one digit.

### 5.2.1 The Siegfried Lemma

A component-specified infinite decimal fraction is component-identified if and only if a component-specified two-dimensional array of digits is component-identified.

**Proof**

Consider the component-specified infinite decimal fraction

$$0.a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{16}.....$$

This component-specified decimal fraction can be split into infinitely many component-specified decimal fractions as follows:

$$0.a_1\ a_2\ |\ a_5\ a_6\ |\ a_{11}\ a_{12}\ |\ a_{19}\ a_{20}\ |......$$

$$0.a_3\ a_4\ |\ a_9\ a_{10}\ |a_{17}\ a_{18}\ |......$$

$$0.a_7\ a_8\ |\ a_{15}\ a_{16}......$$

$$0.a_{13}\ a_{14}.......$$

Thus an infinite vector of digits can be used to construct an infinite two-dimensional array of digits (or a component-specified infinite matrix).

Because an infinite decimal fraction, and hence an infinite vector, is formed by stringing together digits from the left, an infinite two dimensional array of digits is derived from the infinite vector by filling it from the top left corner in some zigzag way with digits from the fraction.

Conversely, if any component-specified infinite matrix of digits is set up by filling it from its top left corner with digits according to some scheme, this two dimensional array can be transformed back into a single component-specified decimal fraction by stringing together the digits in a zigzag fashion.

Hence component-specifying an infinite decimal fraction (infinite string of digits) and component-specifying an infinite matrix of digits are equivalent because one can be transformed into the other. Thus, if it is possible to component-identify one of them, it is possible to component-identify the other.

■

### 5.2.2 Theorem

It is not possible to component-identify an infinite decimal fraction.

**Proof**

Assume that it is possible to component-identify an infinite decimal fraction. The lemma above then implies that the previously constructed list of all possible component-specified real numbers is a component-identified array of digits. But if a matrix of digits is component-identified, then each of its rows is also component identified. Thus the matrix is a list of all possible component-identified real numbers, and hence these numbers are countable.

But the assumption that it is possible to component-identify a real number validates Cantor's diagonal proof. So that these numbers are then also more than countable by his proof.

The contradiction proves the theorem.

■

Thus it is shown that, in the case of component-specified real numbers, the axiom of choice leads only to the **existence.** Hence Cantor's diagonal proof is invalid and therefore there can be no component-identified irrational numbers.

Therefore there are now two types of real numbers that should be distinguished; component specified real numbers (or equivalence classes of Cauchy sequences) which are countable, and value identified real numbers (or points on the real line) which are more than countable. Therefore a real number no longer has three avatars.

# PART TWO: The Infinite Divisibility of Space

# or

# WHAT COMES FIRST: The Limit or the Sum?

## Section 1: The Division of Space

Modern western civilisation is built on the philosophical foundations that were laid down in ancient Greece. The Greeks started what is now known as science – a deductive system of knowledge based on explicitly stated assumptions.

They settled on an atomic theory of matter. This meant that when a piece of matter is halved repeatedly, a piece of matter that cannot be divided again will be reached after a finite number of steps. This was called an ‘atom’ and continuous matter was assumed to be compounded of these discrete indivisible pieces of matter.

This is the explicit assumption that continuous matter is formed from discreet atoms, and is called the atomic theory.

But the Greeks accepted that space is infinitely divisible. Thus repeated division of a piece of space would lead to a never ending sequence of ever smaller pieces of space. For this sequence of ever smaller pieces of space they defined a discrete limit, called a point. The explicit assumption that continuous space is compounded of discrete points is called here the Euclidean cosmology

The ideas of the Euclidean cosmology was opposed by Parmenides of Elea and his eromenos Zeno. In their philosophy they concerned themselves with the concept of motion. Although they did not propose a cosmology of their own, their criticism of the Euclidean cosmology - and the resulting model of Mathematics - is today mostly known as Zeno’s paradoxes. Looking at these paradoxes, it is quite clear that they considered motion to be in essence continuous and thus incompatible with Euclidean cosmology which is in essence discrete. Some of these paradoxes will be referred to in the paragraphs to come.

## Section 2: The Infinite Divisibility of Space.

Strict rules apply today when limits and sums are to be interchanged in Mathematics. However, no such rules were followed when the infinite divisibility of space was considered at the time of laying down the foundations of Mathematics.

### 2.1.1(a) When the limit precedes the sum.

**First the limit:**

The Euclidean cosmology starts with a nested sequence of intervals of which the lengths of the intervals converge to zero. The limit of these intervals is then defined to be an entity of no extent, called a point.

Thus the length (or area or volume) of a point is zero.

**Followed by the sum:**

Next, the Euclidean cosmology states that a line of unit length is a string of points. This then requires that the length of the interval, which is non-zero, must be the sum of the lengths of all the constituent points, all of which are zero. This requires the line to be formed from more than countable many points, as was discussed in the first part.

The assumption that there are 'more than countable' many points is an ideological compromise that resolves the conflict that results when requiring that something which is continuous is to be formed by combining discrete entities - any discrete set can be counted one by one, but a continuum cannot be counted at all.

### 2.1.1(b) When the sum precedes the limit.

Two millennia later Leibnitz studied motion, volumes, areas and lengths. These are all continuous entities.

Looking at the area under the curve y=1:

**First the sum:**

Consider the unit interval on the X-axis and let d(a;b)=b-a denote the length of the interval (a;b). Assume that the unit interval has been repeatedly subdivided in such a way that at each new subdivision all previous intervals are subdivided into three equal intervals. After n such partitions there are $3^n$ subintervals (parts), each of length $3^{-n}$ and

$$1 = \sum_{i=0}^{3^n - 1} d\left(\frac{i}{3^n}; \frac{i+1}{3^n}\right)$$

After n steps the middle of interval i of these subintervals is at the point

$$x_{i+1} = \frac{2i-1}{2\cdot 3^n} : i = 0,1,..,3^n - 1$$

Notice that once a point is in the middle of a subinterval, it will be in the middle of a subinterval for all subsequent partitions.

**Followed by the limit:**

According to the theory of the Riemann integral:

$$1 = \int_0^1 1 \cdot dx = \lim_{\substack{n \to \infty \\ all\ \Delta x_i \to 0}} \sum_{i=0}^{3^{n-1}} 1 \cdot \Delta x_i$$

Where $\Delta x_i = 3^{-n}$ for all i. The parts of these partitions are intervals of which the lengths converge to zero as n becomes larger and larger and therefore any set of intervals with the same midpoint is a nested set that satisfies the requirements set out above for the definition of a point. Hence, according to the Euclidean cosmology, each set of nested intervals becomes a point in the limit. But only points that are in the middle of a subinterval for some value of n can be a limit point (because these intervals reduce symmetrically relative to their midpoints with each subsequent partition). Thus only rational points that are for some n and some i of the form

$$x_{n,i} = \frac{2i-1}{2 \cdot 3^n}$$

can qualify to be limit points. These points form a subset of the (countable) set of rational numbers.

Thus, if the sum precedes the limit, only countable many points, being the limits of monotonically decreasing nested intervals, are required to form the unit interval.

**Remarks**

- For any valid set of partitions these points are dense in the unit interval.
- In the Leibnitzean approach to the infinite divisibility of space, the need for the existence of more than countable many points disappears because only countable many points are needed to cover an interval. Thus, in the Leibnitzean approach, only countable many value specified real numbers need to exist in order to maintain the concept of the real line. Component specified real numbers and equivalence classes of Cauchy numbers have already been shown to be countable. Thus everything can become countable in a number system associated with the Leibnitz approach.

### 2.1.2 Conclusion

These two different ways in which the limits and the sums are considered when studying the infinite divisibility of space, leads to two completely different sets of assumptions about the nature of space and therefore of numbers. The first, the traditional one, will be called the **Euclidean model of Mathematics** which is based on discrete points, while the second will be called the **Leibnitzean model of Mathematics** which is based on continuous intervals. At this stage of the argument it looks as if these two models have contradicting properties and as such are not reconcilable.

Note that the order of the sums and the limits that occur here cannot be interchanged because the limit that occurs in the Euclidean model is zero.

## Section 3: Number Theory in the Leibnitzean Model

The study of Calculus inevitably ends in the use of infinitesimals. The word ‘infinitesimal’ is pidgin Latin that loosely translates into ‘that little thing at infinity’. Thus it is a valid question to ask whether points, as described in the Leibnitz model for the infinite divisibility of space, are indeed the elusive ‘infinitesimals’; seeing that only countable many of them are required to form the unit interval. However, this is not so. In part four it will be shown that points, as defined in the Euclidean Model, can be re-introduced as utilitarian entities that are associated with infinitesimals. In what follows the word ‘point’ will mean ‘a place in space’ – like the endpoint of a line (or vector) or the intersection of two arcs. In the Leibnitzean model a point will simply be a convenient word to describe the focus of a set of nested intervals.

In the Euclidean model the size of a point was defined as zero. In the Leibnitzean model the size of an infinitesimal (a spatial entity) will be defined as an infinitesimal number (a numerical entity).

### 3.1.1 Cauchy Numbers

Number theory is based on interpreting an infinite decimal fraction as a Cauchy sequence, for instance

$$\sqrt{2} = (1. \,;\, 1.4 \,;\, 1.41 \,;\, 1.414 \,;\, 1.4142 \,;\, \ldots\ldots)$$

In part one this representation of an infinite decimal fraction was called the **Cauchy Form** of the real number, and as such it is an element of some equivalence class of Cauchy sequences.

A Cauchy sequence that is associated with a positive infinite decimal fraction, like the one above, has non-decreasing components. To overcome this limitation the representation is generalised:

**Definition**

A Cauchy sequence of rational numbers is called a **Cauchy number**.

In general a Cauchy number will be specified as

$$a = (a_1 ; a_2 ; a_3 ; ....)$$

The rational numbers $a_n$ forming a Cauchy number are called its **components**.

The rules for the four basic operations on Cauchy numbers are those used for truncated decimal fractions:

Addition: $a+b = \{a_n\} + \{b_n\} = \{a_n + b_n\}$

Subtraction: $a - b = \{a_n\} - \{b_n\} = \{a_n - b_n\}$

Multiplication: $a x b = \{a_n\} x \{b_n\} = \{a_n x b_n\}$

Division: $\frac{a}{b} = \frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$ provided none of the numbers $\{b_n\}$ is zero.

Function values: $f(a) = \{f(a_n)\}$

A Cauchy number that is associated with a positive number can now have decreasing terms, e.g.

$$2 - \sqrt{2} = (1. ; 0.6 ; 0.59 ; 0.586 ....)$$

Thus a Cauchy number is but a different name for a Cauchy sequence of rational numbers as studied in number theory, but with an accompanying set of arithmetical operations defined. Thus all relevant results of Number Theory are Mutatis Mutandis applicable to Cauchy numbers. The principal results that are of interest here are the following:

- There is an equivalence relation defined between Cauchy numbers that causes the Cauchy numbers to be separated into equivalence classes. These equivalence classes are the **real numbers**.
- Each equivalence class contains at least one component described infinite decimal fraction, traditionally called the **main value**.
- Any two Cauchy numbers in the same equivalence class differ by a Cauchy number equivalent to zero.
- Zero is the Cauchy number (0 ; 0 ; 0 ; 0 …..).

**Remark**

It is necessary to extend the concept of Cauchy number further in order to make the Cauchy numbers closed under division by a Cauchy number that is equivalent to zero. This requires that the **infinite Cauchy numbers** be defined:

**Definition**

A sequence of numbers $\{a_n\}$ such that for any given number M there exists a number N such that $|a_n| \geq M$ for all $n \geq N$ is called an '**infinite Cauchy number**'.

Equivalent infinite Cauchy numbers differ by a finite Cauchy number.

### 3.1.2 Infinitesimal Numbers

**Definition:**

The elements of the class of Cauchy numbers that are equivalent to zero are called the **infinitesimal numbers**.

Infinitesimal numbers will be indicated using Greek letters e.g.

$$\alpha = (\alpha_1 ; \alpha_2 ; \alpha_3 ; \ldots.)$$

**Example:**

Once more using subtraction, it is possible to change an increasing sequence into a decreasing sequence, and thus the infinitesimal number:

$$\{ 0.1 ; 0.01 ; 0.001 ; 0.0001 ; \ldots.\}$$

Is the Cauchy representation for

$$1-0.999999\ldots..$$

Where the periods at the end indicate that the ‘9’ is a repeating digit.

The motivation for this definition is the observation that the lengths of the intervals of the partition of the unit interval, studied in section 2.1.1(b), form an infinitesimal number. But the reason becomes clearer when one notices that different infinitesimal numbers have different rates of convergence, even though they all converge to zero. This makes them suitable for the study of rates of change.

#### 3.1.2.1 L’Hospital: Classes of Cauchy Numbers

All four arithmetical operations can be performed as long as the Cauchy numbers involved do not have more than a finite number of zero components – i.e. from some point on they do not contain any zeroes.

Three classes of Cauchy numbers have been defined:

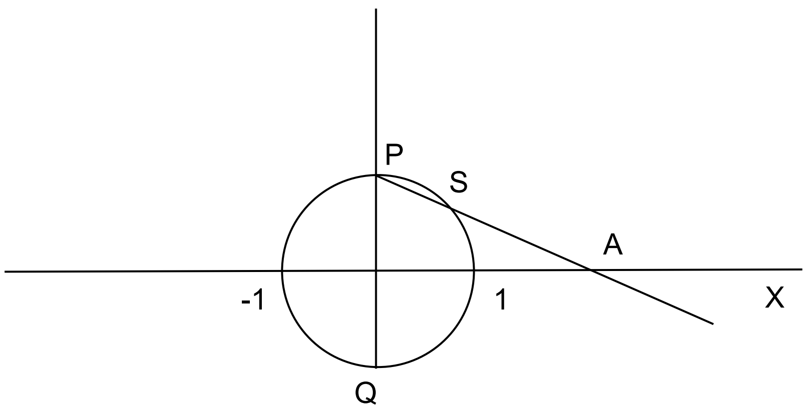
- The **infinitesimal numbers**. These are Cauchy numbers equivalent to zero, indicated as the class A.

- The **infinite numbers**. These are sequences of numbers of which the magnitude of the terms increases without limit, indicated as the class B.
- The **rated numbers**. These are Cauchy numbers belonging to the other equivalence classes of Cauchy numbers, indicated as the class C.

Any sequence of numbers that does not belong to any of these classes will be called a **meandering** sequence.

The rule of L'Hospital indicates that when two Cauchy numbers are multiplied or divided the result can move from class to class. The transition most often used is when the quotient of two infinitesimal numbers becomes a rated number. This is traditionally referred to as differentiation.

The nature of the class B of infinite numbers introduced here becomes clearer when one emulates the compactification of the complex plane as was done by Lars Ahlfors when he introduced his 'point at infinity'.



The horizontal line in the figure is the real line and the circle is a unit circle centred at the origin. A line drawn from the topmost point P of the circle to any point A on the real line then maps that point onto a point S of the circle. The rightmost and leftmost points of the circle are the points +1 and -1 and they map onto themselves. The origin maps onto the lowermost point Q while the topmost point P corresponds to the 'point at infinity'.

With a metric topology of 'length of arc' on the circle, the three classes of Cauchy numbers defined above correspond to classes of Cauchy sequences converging in this topology to points on this circle. Infinitesimals are Cauchy sequences that converge to Q, the image of zero. Rated Cauchy numbers are sequences that converge to all other points of the circle but the topmost, and infinite numbers correspond to Cauchy sequences converging to the topmost point P. In this sense the infinite numbers also form an equivalence class and this equivalence class will be called **infinity**[26]. This validates the term 'Infinite Cauchy Numbers', and allows 'infinity' to be considered as a real number.

The consequence of all this is that 'infinity' acquires a fine-structure and thus need not be avoided anymore (apart from division by the Cauchy number zero). For example, it will be shown later that the Dirac delta function (the derivative of the Heaviside function) is an ordinary piecewise function when using the Cauchy numbers and has an infinite number as value at the point of discontinuity.

Thus each point on this circle corresponds to an equivalence class of Cauchy numbers and is called, as is traditional, a real number. Thus the real line has been compacted to a real continuum.

### 3.1.3 Infinitesimals

**Definition**

A set of nested volumes, areas or lines of which the volumes, areas or lengths form infinitesimal numbers, is called an infinitesimal volume, -

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26 'Infinity' as a 'number larger than all other numbers' is not required in this model because the concept of limit is not essential.

area or –line provided that the focus of the set is a point. (If the context is clear, the traditional way is to just simply call it an infinitesimal.)

The volumes, areas or lengths that form an infinitesimal are called its **parts.** An infinitesimal will be indicated using capital letters:

$E = (E_1 ; E_2 ; E_3 ; \ldots.)$.

The partitions used in the example in section 2.1 consists of infinitesimals of which the lengths of the parts form the infinitesimal number

$\{3^{-n}; n= 0, 1, 2, 3\ldots\}$ from a value of n onwards.

The most used infinitesimals are the differentials:

### 3.1.4 Differentials

**Definition**

Let $d = \{\delta_n\}$ be an infinitesimal number, and let a be any point on the X-axis. Let r be any number such that $0 \leq r \leq 1$. Then a differential at the point a is the infinitesimal

$$D(d,a) = \{D_n\} = (a\text{-}rd ; a+[1\text{-}r]d)$$
$$= \{(a - r\delta_n ; a + [1\text{-}r]\delta_n) ; n=1,2,3\ldots\}$$

Thus a differential at the point a is an infinitesimal focused at a.

It is called a left-differential when r=1 and a right-differential when r=0.

Let y = g(x). Then the Cauchy number

$$dy= g(a+[1\text{-}r]d) – g(a\text{-}rd))$$
$$= \{g(a_n + [1\text{-}r]\delta_n) – g(a_n - r\delta_n) ; n=1,2,3\ldots\}$$

can be formed. Traditionally the function g is called **differentiable at a** if dy is an infinitesimal number. The ratio $\frac{dy}{dx}$ is called the **derivative** of g at the point a.

For Cauchy numbers the derivative can exist even if $dy$ is not an infinitesimal number.

**In general:**

$$\frac{dy}{dx} = \left\{ \frac{g(a_n + [1-r]\delta_n) - g(a_n - r\delta_n)}{\delta_n} \; ; n = 1,2,3.... \right\}$$

Consider the Heaviside function

H(a,x) = 0 if x< a and H(a,x) = 1 if x ≥ a

And let dx be an infinitesimal number. Choose r = 0.5 Then

dH(a,dx) = H(a+dx/2) – H(a-dx/2) = 1

Dividing by the infinitesimal number dx:

$$\frac{dH(a,dx)}{dx} = \begin{cases} 0 & \text{if } x \neq a \\ \frac{1}{dx} & \text{if } x = a \end{cases}$$

$$= \delta(a)$$

This is the Dirac-δ function of which the value at a is an infinite Cauchy number.

## Section 4: Cascades of Differentials

### 4.1 Cascades

The sequence of partitions used in section 2.1 of this part has the following properties:

- Each new partition of the interval (0;1) is a refinement of the preceding partition.
- From one partition to the next there is a part of the new partition that has the same midpoint as the part of which it is a refinement.
- The lengths of all the parts of a partition are the same.
- The lengths of the parts from one partition to the next form an infinitesimal number

$d^0 = \{3^{-n} ; n = 0; 1; 2; 3\ldots\}$

Therefore the interval (0;1) is the first part of the differential $D(d^0, \frac{1}{2})$.

The second part of $D(d^0, \frac{1}{2})$ is the interval $(\frac{1}{3} ; \frac{2}{3})$. The intervals $(0; \frac{1}{3})$ and $(\frac{2}{3} ;1)$ are the first parts of the differentials $D(d^1, \frac{1}{6})$ and $D(d^1, \frac{5}{6})$ where $d^1 = \{3^{-n} ; n = 1; 2 ;3\ldots\}$.

This pattern is repeated for every following partition, so that in the end a cascade of differentials is obtained.

The properties of the partitions that are required to generate this cascade, are (1) that each new partition should be a refinement of the previous partition and (2) that the lengths of the parts of all the partitions should converge to zero. This is a restatement of the infinite divisibility of space from the Leibnitz perspective.

A pre-order can be defined on any cascade using the two properties stated above:

**Definition.**

If all parts of the differential $D(d^r,a)$ are subsets of a part of the differential $D(d^s,b)$ then $D(d^s,b) \leq D(d^r,a)$.

This defines a pre-order on the cascade of differentials because it satisfies all the required properties of the ordering and has

$$D(d^0,c) \leq D(d^r,e) \text{ for all } r \geq 0$$

which makes $D(d^0,c)$ the first element of the pre-order.

### 4.1.1 Functions on the Cascade of Differentials.

The first function of interest is the mapping $D(d^s,a_m) \rightarrow a_m$.

Although the numbers $a_m$ in the example are all rational numbers, it is easy to construct a cascade of differentials for which $a_m$ is a component specified Cauchy number. The mapping only implies existence and not identification of the focus points of the differentials. However, this is good enough to validate the conclusion, made at the beginning of this part, namely that the lengths of a countable number of points, in the Euclidean sense, can add up to the length of the unit interval in the Leibnitzean model.

The second function of interest is the mapping $D(d^s,a_m) \rightarrow \frac{dF}{dx}(a_m)$

where F is a given function and $\frac{dF}{dx}(a_m)$ is the Cauchy number

$$\frac{dF}{dx}\left(a_m\right) = \left\{\frac{F(a_m + 0.5\delta_r) - F(a_m - 0.5\delta_r)}{\delta_r} ; r = s, s+1, ...\right\}$$

Where, in the case of the above example, $\delta_r = 3^{-r}$.

Note that

$$\frac{dF}{dx}\left(a_m\right) dx = \left\{\frac{F(a_m + 0.5\delta_r) - F(a_m - 0.5\delta_r)}{\delta_r} \delta_r ; r = s, s+1, ...\right\}$$

[A]

The third function is

#### 4.1.2 Nets: The Fundamental Theorem of Calculus

For the sake of simplicity and ease of notation, again consider the unit interval partitioned into $3^n = k$ intervals, all of the same length $\delta_n = 3^{-n}$ as in the example in section 2.1 of this part. Let F be a function defined on the unit interval.

Then

F(1) – F(0) = F($3^{-n}$) – F(0) + F(2x$3^{-n}$) – F($3^{-n}$) + ….+ F(1) – F([$3^n$ – 1]x$3^{-n}$)

$$= \sum_{i=0}^{k-1} \left(F\left((i+1).\delta_n\right) - F\left(i\delta_n\right)\right)$$

$$= \sum_{i=0}^{k-1} \left(F\left((i+1).\delta_n\right) - F\left(i\delta_n\right)\right)\frac{\delta_n}{\delta_n}$$

$$= \sum_{i=0}^{k-1} \frac{F((i+1)\delta_n - F(i\delta_n))}{\delta_n}\delta_n$$

[B]

The right hand side of [B] can be evaluated for each value of n, where n takes the values 0, 1 ,2, 3… Hence, if the right hand side of [B] is not a meandering sequence, it is a Cauchy number, and thus either belongs to an equivalence class of Cauchy sequences or is an infinite Cauchy number.

As was pointed out previously, when the right hand side of [B] is a finite Cauchy number, it follows from the theory of numbers that it is equivalent to a component specified infinite decimal fraction, called a main value of the equivalence class, plus an infinitesimal number.

If $\frac{dF}{dx}(x) = f(x)$ then this main value is called the "integral of f over the interval (0;1)" and is written as $\int_0^1 f(x)dx$. Thus

$$F(1) - F(0) = \int_0^1 f(x)dx + \alpha \qquad [C]$$

For any given function f this is a function from the cascade of directed differentials into the set of Cauchy numbers.

**Remarks**

1) In order for the right hand side of [C] to make sense, restrictions have to be placed on the properties of the function f that may appear in the integral. The fundamental restriction is mentioned above, namely that f should be such that the right hand side of [C] is not a meandering sequence. But interpretation of the integral as the area under the graph of the function f will require more drastic restrictions on the nature of f. Clearly, when f is the Dirac delta function, interpreting the integral as an area makes no sense.

2) There is nothing that prevents the right hand side of [C] to be an infinite Cauchy number. In that case α would be a finite Cauchy number.

3) Each infinitesimal in the cascade is component specified and therefore also the cascade as a whole. The right hand side of [C] is thus component specified too and the equivalence class to which it belongs cannot be identified. Hence each component of the number α is at most an indication of the accuracy of the value of the integral at that value of n in the sum [B].

4) When F is a known function, the left hand side of [C] is value identified. This implies that the right hand side is also identified, i.e. the equivalence class of Cauchy sequences to which it belongs is fixed - even though it is component specified. In this case it is usual to assume that the integral, as the main value of that equivalence class, is also value identified. In this case α becomes the Cauchy number zero. Traditionally this is called ‘taking the main value’ and corresponds to taking limits in the Euclidean model.

5) The nets as well as the integral that was defined here, are functions on the directed cascade of differentials. Thus the cascade is the fundamental entity present, and it consists of a never ending selection of differentials which are themselves never ending sets of intervals.

6) Like with fractals, there is no obvious definable limit for the cascade. As the number of refinements of the partition increases, it is only the scale that changes but the pattern in the cascade remains the same.

## PART THREE: The Case for Parmenides

In the Euclidean model of the universe, that which is continuous (space) is compounded of that which is discrete (points). Parmenides and Zeno were critical of the concepts of Euclidean cosmology and, as mentioned before, their criticism found its way into history mainly in the form of Zeno's paradoxes.

These paradoxes are all about the consequences of describing the motion of a body – or particle - in terms of its position at specific points of space at specific instants of time.

Their concern about the cosmological implications of the Euclidean model is clearly stated by the paradox of the arrow. This paradox states that if at some instant of time every point (particle) of an arrow is at some point of space then the motion of the arrow is "frozen" and the arrow cannot move out of this position. This concern about describing the position of a moving particle has been echoed in the middle of the twentieth century by Heisenberg's uncertainty principle which states that when the position of a particle is identified then nothing can be known about its speed and vice versa.

Parmenides did not put forward a detailed alternative cosmological model of his own, but the aspects of his philosophy of interest here are (a) nothing can come into being that has not existed before and (b) one object cannot move relative to another because everything is one; therefore motion does not exist and what we see is an illusion.

In the Euclidean model particles are defined as points. In the Leibnitzean model points are not spatial entities and hence a particle cannot be defined as a point but can only be defined as an infinitesimal with its focus at the position where the particle is perceived to be. This may be done as follows:

Assume that the universe is finite and has a radius R. Then a particle can be defined as an infinitesimal which is a set of nested spheres with radii $r_n = \frac{R}{n}$ (n=1, 2, 3, ….) and where the centre of each part of the infinitesimal has a suitable offset to focus the infinitesimal at the position of the particle.

This definition of a particle is completely in line with the requirements of the cosmology of Parmenides because:

(a) Everything exists throughout the universe and hence never come into being where it has not existed before.

(b) Everything is one because the infinitesimals of all particles share the whole universe as first part. Movement is an illusion because nothing moves – it is only the focus of the infinitesimal that shifts.

**Remarks**

1. In the Leibnitzean cosmology, space is modelled by infinitesimals which are component described. Although all infinitesimals are focussed at some place, the place where an infinitesimal is focused is in general not known. Therefore the description of space is fuzzy. Because of this, Heisenberg's uncertainty principle is to be expected in the Leibnitzean cosmology. What is disconcerting is that the uncertainty principle prescribes a lower limit for this inaccuracy. One possible reason for this may be that a third cosmology, the Heisenberg cosmology, can be described where space, like energy, is quantised!

2. Civilisation today is but the Greek civilisation two and a half millennia on. Therefore, except for parts of Calculus, even today the Euclidean cosmology underpins the whole of science. But some intractable problems of Physics are direct consequences of the fact that particles are considered to be points and therefore localised. Action at a distance and the particle/wave duality comes immediately to mind. In the Leibnitzean cosmology the nature of both these problems change and they all but disappear.

# PART FOUR: Conclusion and Inclusion

The results of part one shows that the Euclidean cosmology leads to inconsistencies in the accompanying number system. It was therefore abandoned in favor of the Leibnitzean cosmology which can accommodate a countable number system.

This countable number system was formed by a re-interpretation of existing number theory which led to an extension of the real numbers to the Cauchy numbers, which in turn form three classes of numbers, namely the infinitesimal numbers, the rated numbers and the infinite numbers. An arithmetic for these numbers was defined according to the everyday use of truncated numbers.

Some terms used in the Euclidean model had to be re-defined; the principal of which was that limits of numbers (infinite decimal fractions) were replaced by the concept of 'main value' and geometrical points lost their status of being spatial entities to become mere places. Many of the other terms remain valid, e.g. real numbers as equivalence classes of Cauchy numbers.

Other results of the Euclidean model had to be abandoned completely. These were results that follow directly from the Euclidean cosmology; the principal of these would most probably be the concept of open and closed sets in geometrical space. But results for non-geometrical spaces like function spaces, which are in essence discrete, should not be affected. Thus results from the Euclidean model should remain valid - perhaps with some adaptations - in such spaces.

The end result is that the Euclidean model, which is the smaller model of Mathematics, can easily - but with some alterations and omissions - become part of the Leibnitzean model which is the larger model. Thus the structure of Mathematics as a canonical model is not lost.

## ADDENDUM

Let

$$b = 0,b_1b_2b_3b_4....$$

be the number constructed in Cantor's diagonal proof.

There are infinitely many infinite decimal fractions in the list of which the first n digits are identical to $b_1b_2b_3b_4....b_n$, the first n digits of b. Select any one of them as the infinite decimal fraction $c_n$.

The sequence $\{c_n\}$ is a Cauchy sequence which is equivalent to b, and therefore its limit is equal to the real number b because both belong to the same equivalence class of Cauchy sequences.

Because of the real line, the assumption that all decimal fractions between zero and one (inclusive) belong to the list means that the list maps a priori onto the closed interval [0;1].

But $c_n$ is in the list for all n and $\{ c_n \}$ maps onto a Cauchy sequence in [0;1]. Thus its limit, the infinite decimal fraction b, belongs to the closure of the list which is [0;1]. Hence b belongs to the list.

But Cantor's proof shows that b does not belong to the list.

Thus two independent contradictory results follow from the same assumptions of the theorem. This constitutes a proof by contradiction. But the logic of a proof by contradiction requires that there must be a **single** identifiable false assumption.

Here there are at least two possible false assumptions:

- That a list of all possible infinite decimal fractions between 0 and 1 can be made. The current consensus is that this is the relevant false assumption.

- That it is possible to identify an infinite decimal fraction by choosing its finite digits according to some rule. Example A constructed in part one shows this is the relevant false assumption.

The fact that more than one possible false assumption exist when the real line is taken into account destroys the logical structure of the proof and the diagonal argument is void.

[i] Three concepts are central to the study of sequences in metric spaces. We are here only concerned with sequences of rational numbers in the metric space formed by the difference topology – i.e. d(a,b) = |a – b|.

The first is the concept of convergence. A convergent sequence is called a Cauchy sequence:

**Definition 1**: A sequence $\{a_n\}$ of rational numbers is called a **Cauchy sequence** if, for any given number ε, a natural number N can be found so that $|a_n - a_m| < \varepsilon$ for all m,n > N.

This defines convergence and is essentially a test whether a given sequence is convergent. This definition is valid irrespective of whether a limit exists or not.

When limits for convergent sequences exist, a test for whether a given number is a limit for a sequence is given by:

**Definition 2**: A given number l is called the **limit** of the converging sequence $\{a_n\}$ of rational numbers If, for any given number ε, a natural number N can be found so that $|a_n - l| < \varepsilon$ for all n > N.

Although this is referred to as the definition of a limit, it is essentially a test whether a given number l is the limit of the sequence.

In this document we are interested in Cauchy sequences with the same limit. Two such sequences are called:

**Definition 3**: Two sequences $\{a_n\}$ and $\{b_m\}$ are **equivalent** if, for any given number ε, a natural number N can be found so that $|a_n - b_m| < \varepsilon$ for all m,n > N.

This is essentially a test whether two sequences have the same limit.