Proof of Firoozbakht's conjecture

Kouji Takaki

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Abstract

We showed that the following inequality of Firoozbakht's conjecture holds when $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds.

$$\log(p_{n+1}) - \log(p_n) < \log(p_n)/n$$

Moreover, in other case, the following inequality holds because the derivative function of $\log(x)$ decreases monotonically for $x > 0$.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) < \log(n+1) - \log(n)$$

We showed that the inequality of this conjecture is obtained by this inequality when $p_{n+1} - p_n \geqq \log(p_n)$ holds. From the above, we proved that Firoozbakht's conjecture is true.

Contents

1. Introduction

In number theory, Firoozbakht's conjecture (or the Firoozbakht conjecture) is a conjecture about the distribution of prime numbers. It is named after the Iranian mathematician Farideh Firoozbakht from the University of Isfahan who stated it first in 1982. The conjecture states that $\sqrt[n]{p_n}$ (where p_n is the nth prime) is a strictly decreasing function of n, i.e.,

$$\sqrt[n+1]{p_{n+1}} < \sqrt[n]{p_n} \text{ for all } n \geqq 1$$

Equivalently,

$$p_{n+1} < {p_n}^{1+1/n} \text{ for all } n \geqq 1$$

(Quoted from Wikipedia)

2. Proof

When we write log in this paper, log refers to natural logarithm. We will prove that the following inequality holds.

$$p_{n+1} < {p_n}^{1+1/n} \quad \dots (1)$$

I When $\log(p_{n+1}) - \log(p_n) < \log(p_n)/p_n$ holds

Since $p_n > n$ holds,

$\log(p_{n+1}) - \log(p_n) < \log(p_n)/n$

$\log(p_{n+1})/\log(p_n) < 1 + 1/n$

holds. This inequality accords with inequality (1).

Ⅱ When $\log(p_{n+1}) - \log(p_n) \geqq \log(p_n)/p_n$ holds

ⅰ When $p_{n+1} - p_n < \log(p_n)$ holds

$p_{n+1}/p_n < 1 + \log(p_n)/p_n$

$\log(p_{n+1}) - \log(p_n) < \log(1 + \log(p_n)/p_n) < \log(p_n)/p_n$

The case of ⅰ does not exist since this inequality is contrary to the condition in the case of Ⅱ.

ⅱ When $p_{n+1} - p_n \geqq \log(p_n)$ holds

Let $f(x) = \log(x)$. $f'(x) = 1/x$ and $f''(x) = -1/x^2$ hold. The derivative function of $f(x)$ is a monotonically decreasing function for $x > 0$ since $f''(x) < 0$ holds for $x > 0$. The following inequalities hold for all n where $n \geqq 1$ holds because $f'(x) > 0$ and $f''(x) < 0$ hold for $x > 0$ and $n + 1 \leqq p_n$ holds.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) < f'(p_n) < \log(n+1) - \log(n) \quad \dots (2)$$

From the case ⅰ, the lower bound of $p_{n+1} - p_n$ is $\log(p_n)$ and the minimum of p_{n+1} becomes $p_n + \log(p_n)$. When $p_{n+1} = p_n + \log(p_n)$ holds, the left side value of the inequalities (2) becomes the maximum and is lesser than the value of $f'(p_n)$ since the left side inequality in (2) holds for any value that $p_{n+1} - p_n$ can be. Therefore, the following inequalities hold for $n \geqq 1$.

$$(\log(p_{n+1}) - \log(p_n))/(p_{n+1} - p_n) \leqq (\log(p_{n+1}) - \log(p_n))/\log(p_n) < f'(p_n) \quad \dots (3)$$

By the inequalities (2) and (3),

$$(\log(p_{n+1}) - \log(p_n))/\log(p_n) < \log(n+1) - \log(n)$$

$$\log(p_{n+1})/\log(p_n) < 1 + \log(1 + 1/n) < 1 + 1/n$$

holds for $n \geqq 1$. This inequality coincides with inequality (1).

From the above, it is proved that Firoozbakht's conjecture is true since inequality (1) holds for all n where $n \geqq 1$ holds. (Q.E.D.)

3. Acknowledgement

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4. References

[1] Wikipedia https://en.wikipedia.org/wiki/Firoozbakht%27s_conjecture

[2] Wikipedia https://en.wikipedia.org/wiki/Prime_number_theorem