How to rotate a frame in 4D Geometric algebra *Cl*4

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Abstract

The problem of base vectors rotation appeared in the formulation of the theory of AC circuits in the language of geometric algebra ([4]). In [4], it is shown how to get the desired rotor in a few steps. Here we show that it is possible to find a 4D rotor in a closed form.

Keywords: *geometric algebra*, *rotor*, *spinor*, *basis frame*, *AC circuits*

The rotor must have grades (0, 2, 4) in 4D

We want to rotate the basis vectors e_k to get a new basis c_k (not necessarily orthonormal). In geometric algebra, this means that we need to find a rotor R such that

$$c_i = Re_i R^\dagger, \; RR^\dagger = 1,$$

where $R^\dagger$ means the *reverse* of R. In [1], Ch. 10.3.2, we have (c^k is a *reciprocal frame*, see [2])

$$\sum_{k=1}^{n} c^k e_k = \sum_{k=1}^{n} Re^k R^\dagger e_k = n - 2R\left(2\left\langle R^\dagger\right\rangle_2 + 4\left\langle R^\dagger\right\rangle_4 + \cdots\right), \quad (1)$$

where $\left\langle R^\dagger\right\rangle_k$ means *grade* k of $R^\dagger$. In 3D, this formula gives

$$R \propto 1 + \sum_{k=1}^{3} c^k e_k \equiv r,$$

where the normalization factor $N = \sqrt{rr^\dagger}$ is easy to obtain (see [2]).

In 4D, the term $S = \sum_{k=1}^{4} c^k e_k$ has grades (0, 2); however, $SS^\dagger$ generally has grades (0, 4). This means that we need to make an extra effort to find a rotor.

Let us consider now the orthonormal frames in 4D Euclidean vector space. Then

$$S = \sum_{k=1}^{4} c_k e_k$$

and we can write

$$SS^\dagger = \alpha + \beta I, \quad \alpha, \beta \in \mathbb{R}, \quad I = e_1 e_2 e_3 e_4.$$

It is not hard to check that $\alpha \pm \beta > 0$. Note that the *pseudoscalar* I anticommutes with vectors, but commutes with 2-blades, like $e_i e_j$. We also have $I^2 = 1$, $I^\dagger = I$, and

$$(\alpha + \beta I)(\alpha - \beta I) = \alpha^2 - \beta^2.$$

The rotor in a closed form

With $R_i = \left\langle R^\dagger \right\rangle_i$, $R_4 = r_4 I$, $r_4 \in \mathbb{R}$, $R^\dagger = R_0 + R_2 + R_4$, we can write the expression (1) as

$$S = 4 - 4R\left(R_2 + 2R_4\right) = 4 - 4R\left(R^\dagger - R_0 + R_4\right) = 4R\left(R_0 - R_4\right),$$

which means

$$S\left(R_0 + R_4\right) = 4\left(R_0^2 - r_4^2\right)R,$$

$$R \propto S\left(R_0 + R_4\right) \equiv M .$$

Now we have (note that R_4 commutes with S)

$$MM^\dagger = SS^\dagger\left(R_0 + R_4\right)^2 = \left(\alpha + \beta I\right)\left(R_0^2 + r_4^2 + 2R_0 r_4 I\right).$$

Therefore, for $MM^\dagger$ to be a real number, we need

$$2R_0 r_4 = -\beta, \qquad R_0^2 + r_4^2 = \alpha .$$

The solutions are

$$r_4 = \pm\sqrt{\frac{\alpha \pm \sqrt{\alpha^2 - \beta^2}}{2}}, \quad R_0 = \mp\sqrt{\alpha - \frac{\alpha \pm \sqrt{\alpha^2 - \beta^2}}{2}},$$

and the real norm is

$$N = \pm\sqrt{MM^\dagger} = \pm\sqrt{\alpha^2 - \beta^2},$$

which means that the rotor is given by

$$R = S\left(R_0 + R_4\right)/N .$$

An example

Consider the orthonormal vectors c_i from [4]

$$c_1 = \left(e_1 - e_4\right)/\sqrt{2},$$

$$c_2 = \left(-e_1 + 2e_2 - e_4\right)/\sqrt{6},$$

$$c_3 = \left(-e_1 - e_2 + 3e_3 - e_4\right)/\sqrt{12},$$

$$c_4 = \left(e_1 + e_2 + e_3 + e_4\right)/2 .$$

The list of the rotor coefficients (generated in *Mathematica*) is

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{0.850475, -0.111967, -0.0943927, 0.352278, -0.07243, 0.270313, 0.227884, -0.0300015}
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This rotor is just the reverse of the rotor from [4]. The interested reader can find the *Mathematica* notebook *4Drotor.nb* at [3].

Rotor decomposition in 4D

Here we use $e_{ij} = e_i e_j$ and $j = e_{123}$.

In 4D, an even multivector has the form

$$M = m_0 + m_1 e_{12} + m_2 e_{13} + m_3 e_{14} + m_4 e_{23} + m_5 e_{24} + m_6 e_{34} + m_7 I \ ,$$

whence

$$MM^\dagger = \sum_{i=0}^{7} m_i^2 + 2\left(-m_3 m_4 + m_2 m_5 - m_1 m_6 + m_0 m_7\right) I \equiv \alpha + \beta I \ , \quad \alpha, \beta \in \mathbb{R} \ .$$

For $\beta = 0$, we have $MM^\dagger = \alpha \in \mathbb{R}$, which means that for $\alpha = 1$ we have a rotor, while for $\alpha \neq 1$ we have a *spinor* (a rotor with dilatation).

Let us define

$$M_1 = a_0 + a_1 e_{12} + a_2 e_{31} + a_3 e_{23} ,$$
$$M_2 = b_0 + b_1 e_{24} , \quad M_i M_i^\dagger \in \mathbb{R} .$$

Note that the product $M_2 M_1$ will recover all the six simple bivectors e_{ij} from *Cl*4, as well as the unit pseudoscalar I. Obviously, we have

$$M_2 M_1 \left(M_2 M_1\right)^\dagger = M_2 M_1 M_1^\dagger M_2^\dagger \in \mathbb{R} ,$$

which means that the product $M_2 M_1$ is a spinor. After normalization, we get a rotor. On the other hand, any rotor (spinor) M can be decomposed as $M_2 M_1$.

If we have two rotors

$$R_1 = a_0 + a_1 e_{12} + a_2 e_{31} + a_3 e_{23} ,$$
$$R_2 = b_0 + b_1 e_{24} , \quad R_i R_i^\dagger = 1 ,$$

the rotor $R = R_2 R_1$ means that we rotate about the axis

$$-j\left(a_1 e_{12} + a_2 e_{31} + a_3 e_{23}\right) = a_3 e_1 + a_2 e_2 + a_1 e_3$$

first (see [2]), and then we rotate all the vectors from the subspace spanned by the vectors e_2 and e_4 in their own plane. Note that the rotor R_1 is just a rotor in 3D.

In [4], the authors use the bivector e_{34} instead of e_{24}. However, it is easy to see that we can choose any combination of indices.

Such fluency of calculation and clarity of the geometric interpretation is difficult to imagine in any other formalism. Geometric algebra is the mathematics of the future.

Literature

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