Exotic Integral

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ABSTRACT

We give an integral related to $\frac{\Gamma(3/5)\Gamma(4/5)}{\Gamma(7/5)}$

Introduction

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$I = \int_0^1 (1-x)^{-1/5} x^{-2/5}\, dx = \frac{\Gamma(3/5)\Gamma(4/5)}{\Gamma(7/5)} = \frac{5\Gamma(3/5)^2\Gamma(4/5)}{4\pi\sqrt{2}}\sqrt{5+\sqrt{5}}$$

$$I = \frac{5}{3}\, F\left(\frac{1}{5}, \frac{3}{5}, \frac{8}{5}, 1\right) = \frac{5}{4}\, F\left(\frac{2}{5}, \frac{4}{5}, \frac{9}{5}, 1\right)$$

$$I = \frac{5}{3} a^{3/5}\, F\left(\frac{1}{5}, \frac{3}{5}, \frac{8}{5}, a\right) + \frac{5}{4}(1-a)^{4/5}\, F\left(\frac{2}{5}, \frac{4}{5}, \frac{9}{5}, 1-a\right) \quad , 0 \le a \le 1$$

Remark 1: $\Gamma(x)$ is the Gamma function.

Remark 2: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Exotic Integral

If $u = \left(\frac{27}{4}\right)^{1/5}$, then

$$I = u - \int_u^\infty \frac{\sqrt{3}}{2x^2\sqrt{x}} \sec\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\frac{3\sqrt{3}}{2x^2\sqrt{x}}\right) dx + \int_u^\infty \left(1 + \frac{\sqrt{3}}{2x^2\sqrt{x}} \sec\left(\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}\frac{3\sqrt{3}}{2x^2\sqrt{x}}\right)\right) dx$$

If $0 \leq z \leq \frac{2}{3}$, $w = z^{-2/5}(1-z)^{-1/5}$, then

$$\frac{5}{3} z^{3/5} F\left(\frac{1}{5}, \frac{3}{5}, \frac{8}{5}, z\right) = \left(\frac{z^3}{1-z}\right)^{1/5} - \int_w^\infty \frac{\sqrt{3}}{2x^2\sqrt{x}} \sec\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\frac{3\sqrt{3}}{2x^2\sqrt{x}}\right) dx$$

References

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