Seven Archimedean circles with six-fold symmetry for the arbelos

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Abstract. We show that there are seven Archimedean circles with 6-fold symmetry for the arbelos.

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1. Introduction

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$, we consider an arbelos formed by the three semicircles of diameters BC, CA and AB constructed on the same side of AB. The perpendicular to AB at C is called the axis. Let D and E be the points such that ACD and BCE are equilateral triangles. If the segments AE and CD meet in a point F and the segments BD and CE meet in a point G, then E. A. J. Garcìa shows that the circles of diameters CF, CG and FG are Archimedean [2] (see the circles in yellow in Figure 1), i.e., they have radius $r_{\text{A}} = \frac{ab}{a+b}$. In this note we show that there are two more Archimedean circles indicated in red in Figure 1.

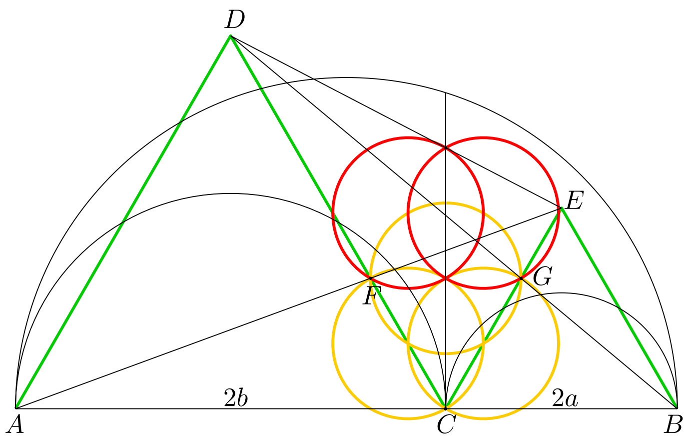

Figure 1.

2. Result

Our results can be obtained as a special case of the next theorem. A similar result to the part (i) can be found in [4].

Theorem 1. *Assume that ACD and BCE are similar isosceles triangles with bases AC and BC and the segments AE and CD meet in a point F and the segments BD and CE meet in a point G. Then the following statements are true.*
(i) *The line FG is parallel to AB.*
(ii) *If ax, bx and d are the distances from the points E, D and G to the line AB, respectively, then $d = xr_{\text{A}}$.*
(iii) *If the segment DE meets the axis in a point H, then $FCGH$ is a rhombus.*

Proof. Considering the figure in the real projective plane, we can assume that $AD \cap CE = L_3$ and $CD \cap BE = M_1$ (see Figure 2). Let $L_1 = A$, $L_2 = D$, $M_2 = E$ and $M_3 = B$. Then the points L_1, L_2 and L_3 are collinear, and the points M_1, M_2 and M_3 are also collinear. Therefore by Pappus theorem, the three points $F = (12) = L_1M_2 \cap L_2M_1$, $G = (23) = L_2M_3 \cap L_3M_2$ and $(31) = L_3M_1 \cap L_1M_3$ are collinear. Since the last point lies on the line at infinity, the lines AB and FG meet on this line, i.e., they are parallel. This proves (i). We denote the foot of perpendicular from a point P to AB by P_f. Let $t = |CF_f| = |CG_f|$. From the similar triangles BDD_f and BGG_f, we have

$$\frac{bx}{2a+b} = \frac{d}{2a-t}.$$

Similarly, from the similar triangles AEE_f and AFF_f, we have

$$\frac{ax}{a+2b} = \frac{d}{2b-t}.$$

Eliminating t from the equations, we have $d = xr_{\mathrm{A}}$. This proves (ii). We can assume $a < b$. From the similar triangles formed by DE, the line parallel to AB passing through E, and the lines DD_f and HC, we have

$$\frac{|CH| - ax}{a} = \frac{bx - ax}{a+b}.$$

This implies $|CH| = 2xr_{\mathrm{A}}$. Therefore (iii) is proved by (i) and (ii). □

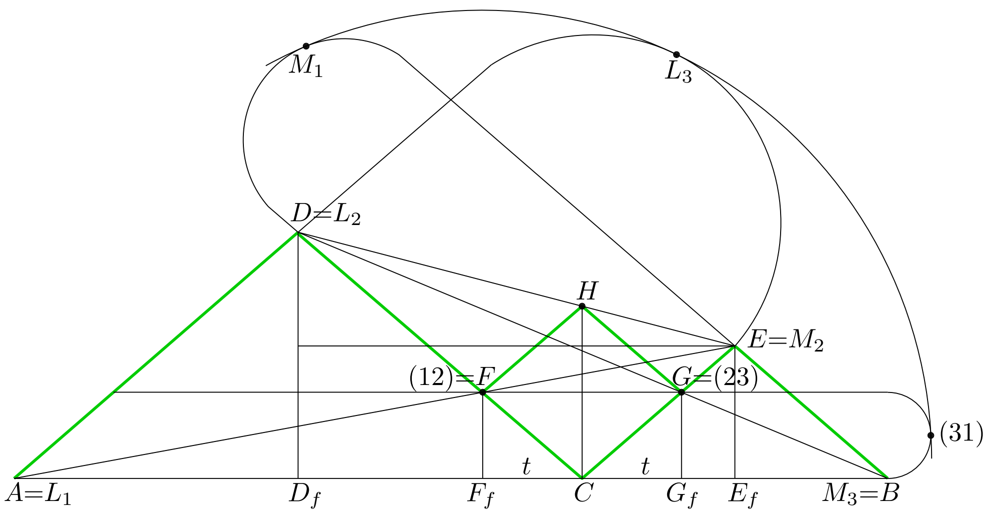

Figure 2.

Theorem 2. *If ACD and BCE are equilateral triangles constructed on the same side of AB as the arbelos, and the segment DE meets the axis in a point H, then the circles of diameters FH and GH are Archimedean.*

Proof. Since the point H is the reflection of the point C in the line FG by Theorem 1(iii), the circles of diameters FH and GH are the reflections of the Archimedean circles of diameters CF and CG, respectively (see Figure 3). □

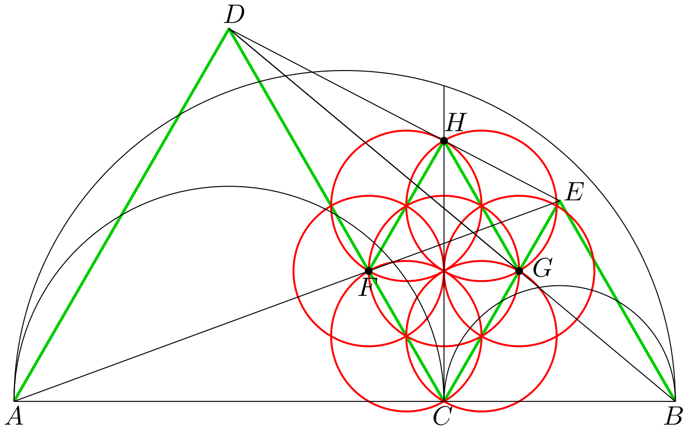

Figure 3: Seven Archimedean circles.

Since the circles with centers F and G touching the axis are Archimedean, we get the seven Archimedean circles in total with 6-hold symmetry. Archimedean circles with four-hold symmetry can be found in [3].

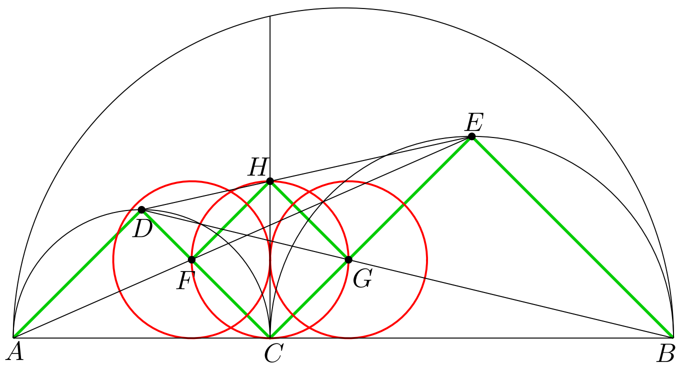

Figure 4: Three Archimedean circles.

If the points D and E lie on the semicircles of diameters AC and BC, then we have $x = 1$ in Theorem 1. Hence the rhombus $FCGH$ is a square and the circumcircle of this and the circles with centers F and G touching the axis are Archimedean (see Figure 4). The circumcircle of $FCGH$ is the Bankoff circle [1].

REFERENCES

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, Math. Mag., **72** (1974) 214–218.

[2] E. A. J. García, Three Archimedean circles arising from equilateral triangles, Sangaku J. Math., **4** (2020) 86–87.

[3] H. Okumura, Archimedean circles passing through a special point, Int. J. Geom., **4** (2015) 5–10.

[4] H. Okumura, Two similar triangles, Mathematical Gazette, **79** no. 486, (1995) 569–571.