# FORMULA FOR THE PRIME-COUNTING FUNCTION

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**Abstract**

This paper presents an exact elementary formula for the prime-counting function and an exact formula for counting pairs of twin prime numbers.

## Prime-counting function *π(n)*

For *n>3*

$$\pi(n)=2+\sum_{k=1}^{\left\lfloor\frac{n+1}{6}\right\rfloor}\left\lfloor\left[1+\sum_{a=1}^{\left\lfloor\frac{1+\sqrt{-1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{-1+6\cdot k}{-1+6\cdot a}\right\rfloor-\left\lfloor\frac{-2+6\cdot k}{-1+6\cdot a}\right\rfloor\right)+\sum_{a=1}^{\left\lfloor\frac{-1+\sqrt{-1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{-1+6\cdot k}{1+6\cdot a}\right\rfloor-\left\lfloor\frac{-2+6\cdot k}{1+6\cdot a}\right\rfloor\right)\right]^{-1}\right\rfloor$$
$$+\sum_{k=1}^{\left\lfloor\frac{n-1}{6}\right\rfloor}\left\lfloor\left[1+\sum_{a=1}^{\left\lfloor\frac{1+\sqrt{1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{1+6\cdot k}{-1+6\cdot a}\right\rfloor-\left\lfloor\frac{6\cdot k}{-1+6\cdot a}\right\rfloor\right)+\sum_{a=1}^{\left\lfloor\frac{-1+\sqrt{1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{1+6\cdot k}{1+6\cdot a}\right\rfloor-\left\lfloor\frac{6\cdot k}{1+6\cdot a}\right\rfloor\right)\right]^{-1}\right\rfloor$$

where $\lfloor . \rfloor$ is the floor function. The formula can be derived from the following:

for *m,p* integer

$$\left\lfloor\frac{m}{p}\right\rfloor-\left\lfloor\frac{m-1}{p}\right\rfloor=\begin{cases}0 & if \quad p\nmid m\\ 1 & if \quad p\mid m\end{cases}$$

indeed $\frac{m}{p}=q+\frac{r}{p}$ with $r<p$ and $\left\lfloor\frac{m}{p}\right\rfloor=q$

if $p\nmid m$ we have $r>0$

$\frac{m-1}{p}=q+\frac{r-1}{p}$ and $\left\lfloor\frac{m}{p}\right\rfloor=\left\lfloor\frac{m-1}{p}\right\rfloor=q$ therefore: $\left\lfloor\frac{m}{p}\right\rfloor-\left\lfloor\frac{m-1}{p}\right\rfloor=0$

if $p\mid m$

$\frac{m}{p}=q$ and $\frac{m-1}{p}=q+\frac{-1}{p}=(q-1)+\frac{p-1}{p}$ therefore: $\left\lfloor\frac{m}{p}\right\rfloor-\left\lfloor\frac{m-1}{p}\right\rfloor=q-(q-1)=1$

That said, we have:

$$1+\sum_{p=2}^{\lfloor\sqrt{m}\rfloor}\left(\left\lfloor\frac{m}{p}\right\rfloor-\left\lfloor\frac{m-1}{p}\right\rfloor\right)=\begin{cases}1 & if\ \ m\ is\ prime\\ i>1 & if\ \ m\ is\ composite\end{cases}$$

therefore:

$$\pi(n)=2+\sum_{m=4}^{n}\left\lfloor\left[1+\sum_{p=2}^{\lfloor\sqrt{m}\rfloor}\left(\left\lfloor\frac{m}{p}\right\rfloor-\left\lfloor\frac{m-1}{p}\right\rfloor\right)\right]^{-1}\right\rfloor$$

given that the prime numbers except *2* and *3* are congruent to $\pm 1 (mod\, 6)$ eliminating the values of *m* and *p* multiples of *2* and *3* we have $m=\pm 1+6\cdot k$ and $p=\pm 1+6\cdot a$ you get the given formula.

## Twin prime counting function $\pi_{twin}(n)$

$$\pi_{twin}(n)=1+\sum_{k=1}^{\left\lfloor\frac{n-1}{6}\right\rfloor}\left\lfloor\left[1+\sum_{a=1}^{\left\lfloor\frac{1+\sqrt{-1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{-1+6\cdot k}{-1+6\cdot a}\right\rfloor-\left\lfloor\frac{-2+6\cdot k}{-1+6\cdot a}\right\rfloor\right)+\sum_{a=1}^{\left\lfloor\frac{-1+\sqrt{-1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{-1+6\cdot k}{1+6\cdot a}\right\rfloor-\left\lfloor\frac{-2+6\cdot k}{1+6\cdot a}\right\rfloor\right)+\sum_{a=1}^{\left\lfloor\frac{1+\sqrt{1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{1+6\cdot k}{-1+6\cdot a}\right\rfloor-\left\lfloor\frac{6\cdot k}{-1+6\cdot a}\right\rfloor\right)+\sum_{a=1}^{\left\lfloor\frac{-1+\sqrt{1+6\cdot k}}{6}\right\rfloor}\left(\left\lfloor\frac{1+6\cdot k}{1+6\cdot a}\right\rfloor-\left\lfloor\frac{6\cdot k}{1+6\cdot a}\right\rfloor\right)\right]^{-1}\right\rfloor$$

## Implementation Using *SciLab*

```
n=10000;
Pi=2;
for k=1:floor((n+1)/6)
    Sk=1;
    for a=1:floor((1+sqrt(-1+6*k))/6)
        Sk = Sk + floor((-1+6*k)/(-1+6*a)) - floor((-2+6*k)/(-1+6*a));
    end
    for a=1:floor((-1+sqrt(-1+6*k))/6)
        Sk = Sk + floor((-1+6*k)/(1+6*a)) -  floor((-2+6*k)/(1+6*a));
    end
    Pi=Pi+floor(1/Sk);
end
for k=1:floor((n-1)/6)
    Sk =1;
    for a=1:floor((1+sqrt((1+6*k)))/6)
        Sk = Sk + floor((1+6*k)/(-1+6*a)) - floor((6*k)/(-1+6*a));
    end
    for a=1:floor((-1+sqrt((1+6*k)))/6)
        Sk = Sk + floor((1+6*k)/(1+6*a)) -  floor((6*k)/(1+6*a)) ;
    end
    Pi=Pi+floor(1/Sk);
end

Pi_twin=1;
for k=1:floor((n-1)/6)
    Sk=1;
    for a=1:floor((1+sqrt(-1+6*k))/6)
        Sk = Sk + floor((-1+6*k)/(-1+6*a)) - floor((-2+6*k)/(-1+6*a));
    end
    for a=1:floor((-1+sqrt(-1+6*k))/6)
        Sk = Sk + floor((-1+6*k)/(1+6*a)) -  floor((-2+6*k)/(1+6*a));
    end
    for a=1:floor((1+sqrt((1+6*k)))/6)
        Sk = Sk + floor((1+6*k)/(-1+6*a)) - floor((6*k)/(-1+6*a));
    end
    for a=1:floor((-1+sqrt((1+6*k)))/6)
        Sk = Sk + floor((1+6*k)/(1+6*a)) -  floor((6*k)/(1+6*a)) ;
    end
    Pi_twin=Pi_twin+floor(1/Sk);
end
```