Laplace limit constant

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Abstract. The Laplace limit is the maximum value of the eccentricity for which a solution to Kepler's equation, in terms of a power series in the eccentricity , converges. It is approximately $\lambda = 0.66274341$ ...

Introduction

Given real numbers M and ε , $|\varepsilon| < 1$, the accurate solution of Kepler's equation:

$$M = E - \varepsilon \, \sin E$$

is critical in celestial mechanics.

If M is not a multiple of π , then Kepler's equation has a unique solution , here given as a power series in ε (via the inversion method of Lagrange):

$$E = M + \sum_{n=1}^{\infty} a_n \, \varepsilon^n$$

where

$$a_n = \frac{1}{2^{n-1} n!} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} (n-2k)^{n-1} \sin\big((n-2k)M\big)$$

The power series diverge for $|\varepsilon| > 0.662$ ... as evidently first discovered by Laplace.

In fact , the power series for E converges like a geometric series with ratio

$$f(\varepsilon) = \frac{\varepsilon}{1 + \sqrt{1+\varepsilon^2}} \, exp\left(\sqrt{1+\varepsilon^2}\right)$$

The value $\lambda = 0.662743$ ... for which $f(\lambda) = 1$ is called the Laplace limit.

In this note we give some formulas for λ .

Sequences for Laplace limit constant

Entry 1.

$$x_{n+1} = \frac{1}{\sinh\sqrt{1+x_n^2}} \quad , \qquad x_0 = 0 \ , \lim_{n\to\infty} x_n = \lambda$$

Entry 2.

$$x_{n+1} = \frac{\sqrt{1+x_n^2}}{\cosh\sqrt{1+x_n^2}} \ , \qquad x_0 = 0 \ , \lim_{n\to\infty} x_n = \lambda$$

Entry 3.

$$x_{n+1} = \left(1+\sqrt{1+x_n^2}\right) e^{-\sqrt{1+x_n^2}} \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \lambda$$

Entry 4.

$$x_{n+1} = \coth x_n \ , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Remark:

$$\sqrt{1+\lambda^2} = \coth(\coth(\coth(\ldots \coth 1)))$$

$$\lambda = \operatorname{csch}(\coth(\coth(\coth(\ldots \coth 1))))$$

Entry 5.

$$x_{n+1} = 1 + (1+x_n)e^{-2x_n} \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 6.

$$x_{n+1} = exp\left(-2\left(\frac{1+x_n}{1-x_n}\right)\right) \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \frac{\lambda^2}{2+\lambda^2+2\sqrt{1+\lambda^2}}$$

Remarks:

$$u = \frac{\lambda^2}{2+\lambda^2+2\sqrt{1+\lambda^2}} = e^{-2\sqrt{1+\lambda^2}} \ , \qquad \lambda = \frac{2\sqrt{u}}{1-u}$$

$$x_{n+1} = exp\left(-2\left(\frac{1+x_n}{1-x_n}\right)\right) = exp\left(2 - \frac{4}{1-x_n}\right)$$

$$e^{-2\sqrt{1+\lambda^2}} = exp\left(2 - \frac{4}{1 - exp\left(2 - \frac{4}{1 - exp\left(2 - \frac{4}{1-\cdots}\right)}\right)}\right)$$

Entry 7.

$$x_{n+1} = \frac{2}{e^{2+2x_n} - 1} \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2} - 1$$

Entry 8.

$$x_{n+1} = \sinh^{-1}\left(\frac{\cosh x_n}{x_n}\right) \ , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 9.

$$x_{n+1} = 2\ \tan^{-1}(e^{-\sec x_n}) \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \tan^{-1}\lambda$$

Entry 10.

$$x_{n+1} = \sinh\sqrt{1+\left(\frac{1}{x_n}\right)^2} \ , \qquad x_0 = \frac{3}{2} \, , \lim_{n\to\infty} x_n = \frac{1}{\lambda}$$

Entry 11.

$$x_{n+1} = \frac{\sqrt{1+x_n^2}}{\sinh^{-1} x_n} \ , \qquad x_0 = \frac{3}{2} \, , \lim_{n\to\infty} x_n = \frac{1}{\lambda}$$

Entry 12.

$$x_{n+1} = -\ln\tanh\left(\frac{\cosh x_n}{2}\right) , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \sinh^{-1}\lambda$$

Entry 13.

$$x_{n+1} = \sinh^{-1}\left(\frac{e^{\coth x_n}}{1+\coth x_n}\right) \ , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 14.

$$x_{n+1} = \tan^{-1}\left(\frac{e^{\csc x_n}}{1+\csc x_n}\right) \ , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \tan^{-1}\frac{1}{\lambda}$$

Entry 15.

$$x_{n+1} = \sinh^{-1}\left(2\left(\cosh\frac{x_n}{2}\right)^2 e^{-\cosh x_n}\right) \ , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \sinh^{-1}\lambda$$

Entry 16.

$$x_{n+1} = \frac{3+2x_n}{2+3\sinh\sqrt{1+x_n^2}} \ , \qquad x_0 = 0 \ , \lim_{n\to\infty} x_n = \lambda$$

Entry 17.

$$x_{n+1} = \frac{1+x_n}{1+\tanh x_n} \ , \qquad x_0 = 0 \ , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

$$x_{n+1} = \frac{3 + x_n}{1 + 3\tanh x_n} \quad , \qquad x_0 = 0 \ , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 18.

$$x_{n+1} = \frac{3x_n + 7\coth x_n}{10} \quad , \qquad x_0 = 1 \ , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 19.

$$x_{n+1} = \sin^{-1}\left(\tanh\left(\frac{1}{\sin x_n}\right)\right) \quad , \qquad x_0 = 1 \ , \lim_{n\to\infty} x_n = \tan^{-1}\frac{1}{\lambda}$$

Entry 20.

$$x_{n+1} = \ln\left(\frac{2\cosh x_n}{x_n} + e^{-x_n}\right) \quad , \qquad x_0 = 1 \ , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 21.

$$x_{n+1} = \frac{2}{1 - e^{2-2x_n}} \quad , \qquad x_0 = 2 \, , \lim_{n\to\infty} x_n = 1 + \sqrt{1+\lambda^2}$$

Remark:

$$1 + \sqrt{1+\lambda^2} = 2\left(1 - e^{2-4\left(1-e^{2-4(1-\cdots)^{-1}}\right)^{-1}}\right)^{-1}$$

Entry 22.

$$x_{n+1} = \cosh^{-1}(\coth(\cosh x_n)) \quad , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \sinh^{-1}\lambda$$

Entry 23.

$$x_{n+1} = \cos^{-1}(\tanh(\sec x_n)) \quad , \qquad x_0 = 0 \, , \lim_{n\to\infty} x_n = \tan^{-1}\lambda$$

Entry 24.

$$x_{n+1} = \cot\left(\frac{\pi}{4} - \tan^{-1}(e^{-2x_n})\right) \, , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Remark:

$$\sqrt{1+\lambda^2} = \cot\left(\frac{\pi}{4} - \tan^{-1}\left(e^{-2\cot\left(\frac{\pi}{4}-\tan^{-1}\left(e^{-2\cdots}\right)\right)}\right)\right)$$

Entry 25.

$$x_{n+1} = \tanh\left(\frac{1}{2}\cosh(\ln x_n)\right) \quad , \qquad x_0 = 1 \, , \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2} - \lambda$$

Remark:

$$s = \sqrt{1+\lambda^2} - \lambda = \tanh\left(\frac{1}{2}\cosh\ln\tanh\left(\frac{1}{2}\cosh\ln\tanh\left(\frac{1}{2}\dots.\right)\right)\right)$$

$$\lambda = \frac{1}{2}\left(\frac{1}{s} - s\right)$$

Entry 26.

$$x_{n+1} = x_n + \left(\frac{1}{x_n} - \tanh x_n\right) \quad , \qquad x_0 = 1\,, \lim_{n\to\infty} x_n = \sqrt{1+\lambda^2}$$

Entry 27.

$$x_{n+1} = \sinh^{-1}(\operatorname{csch}(\cosh x_n)) \quad , \qquad x_0 = 0\,, \lim_{n\to\infty} x_n = \sinh^{-1}\lambda$$

Entry 28.

$$x_{n+1} = 2e^{-\sqrt{1+x_n^2}} + x_n\, e^{-2\sqrt{1+x_n^2}}\,, \qquad x_0 = 0\,, \lim_{n\to\infty} x_n = \lambda$$

Entry 29.

$$x_{n+1} = \frac{x_n}{\sqrt{1+x_n^2}}\,\sinh^{-1}\left(\frac{1}{x_n}\right), \qquad x_0 = 1\,, \lim_{n\to\infty} x_n = \lambda$$

Entry 30.

$$x_{n+1} = \left(\frac{1}{\sinh\sqrt{1+x_n}}\right)^2, \qquad x_0 = 0\,, \lim_{n\to\infty} x_n = \lambda^2$$

Entry 31.

$$x_{n+1} = \left(\sinh\sqrt{1+\frac{1}{x_n}}\right)^2, \qquad x_0 = 1\,, \lim_{n\to\infty} x_n = \frac{1}{\lambda^2}$$

Entry 32. For $c_0 = 1$, and

$$c_n = \sum_{k=1}^{n} \frac{2k-1}{(2k)!} c_{n-k} \quad , n = 1,2,3, \dots$$

we have

$$\lambda = \lim_{n\to\infty} \sqrt{\frac{c_n}{c_{n+1}} - 1}$$

some values

$$c_n = \left\{1, \frac{1}{2}, \frac{3}{8}, \frac{37}{144}, \frac{1031}{5760}, \frac{5569}{44800}, \frac{3761419}{43545600}, \dots\right\}$$

$$\frac{c_n}{c_{n+1}} = \left\{2, \frac{4}{3}, \frac{54}{37}, \frac{1480}{1031}, \frac{72170}{50121}, \frac{5413068}{3761419}, \frac{579258526}{402471917}, \dots\right\}$$

Related series

Entry 33. For $\lambda = \frac{1}{2}\left(\frac{1}{\sqrt{w}} - \sqrt{w}\right)$, we have

$$1 = 3w + \frac{4}{3}w^2 + \frac{4}{5}w^3 + \frac{4}{7}w^4 + \cdots = 3w + \sum_{n=0}^{\infty} \frac{4}{2n+3} w^{n+2}$$

and (via inversion of series)

$$w = \frac{1}{3} - \frac{4}{81} + \frac{52}{10935} - \frac{188}{413343} + \frac{15164}{279006525} - \frac{27076}{3945949425} + \frac{301123364}{339331920802875} - ..$$

Entry 34. For $\alpha = e^{-2\sqrt{1+\lambda^2}}$, we have

$$\alpha = exp\left(-\frac{2(1+\alpha)}{1-\alpha}\right) = e^{-2} \sum_{n=0}^{\infty} a_n \, \alpha^n$$

where

$$a_n = \frac{(2n-6)a_{n-1} - (n-2)a_{n-2}}{n} \quad , n = 2,3,4, \dots ; \; a_0 = 1, a_1 = -4$$

and (via inversion of series)

$$e^{-2\sqrt{1+\lambda^2}} = \frac{1}{4+e^2} + \frac{4}{(4+e^2)^3} + \frac{4(28+e^2)}{3(4+e^2)^5} + \frac{4(304+12e^2-e^4)}{3(4+e^2)^7} + \frac{4(18112+544e^2-184e^4-7e^6)}{15(4+e^2)^9} + \cdots$$

Integrals

Entry 35. For

$$h(x) = \frac{x\left(x\, e^{\sqrt{1+x^2}} - 1\right)}{e^{\sqrt{1+x^2}}\left(\sqrt{1+x^2} - 1\right) - x}$$

we have

$$\lambda = \frac{1}{8\pi}\int_0^{2\pi} e^{i\,x}\, h\left(\frac{1}{2}+\frac{1}{4}e^{i\,x}\right) dx$$

Entry 36. For

$$h(x) = i\,x + \frac{1}{\sinh\sqrt{1-x^2}}$$

we have

$$w = \int_0^{2\pi} \frac{i\,e^{i\,x}}{h(e^{i\,x})} dx$$

$$\lambda = \sqrt{\frac{2\pi}{w} - 1}$$

Remark: $i = \sqrt{-1}$.

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