Demonstration of a Technique to Construct a One-to-One Correspondence Between N and the Infinite Binary Decimals in (0,1)

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Abstract

In this paper we will see how by varying the initial conditions of the Cantor's demonstration we can use the Diagonal Method to produce a one-to-one correspondence between the set of natural numbers and the set of infinite binary decimals in the open interval (0, 1).

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I. Introduction

In 1891 Georg Cantor published his Diagonal Argument which, he asserted, proved that the real numbers cannot be put into a one-to-one correspondence with the natural numbers. Cantor's proof relies on the fact that a complete list of real numbers over a given interval say, (0, 1) cannot, in fact be realized. That is to say, the infinite set of real numbers in (0, 1) cannot be listed linearly in its entirety and thus is uncountable. Cantor's reasoning is that since the real number list must necessarily be incomplete then any attempt to put the numbers in (0, 1) in a one-to-one correspondence with the natural numbers is bound to fail. To demonstrate, we begin with a matched list of real infinite binary decimal numbers in (0, 1) which we assume is complete. Each number in the list is matched with a corresponding natural number. Then any real number found not to be contained in the list will have no natural number to pair with. Cantor's diagonal method produces numbers not contained in the list and from this he concludes that the set of real numbers in the interval (0, 1) must have more members than the set of all natural numbers.

In this paper we will see how by varying the initial conditions of the demonstration we can use Cantor's method to produce a one-to-one correspondence between the set of natural numbers and the set of infinite binary decimals in the open interval (0, 1). We concede that the initial list of infinite binary decimals is, in fact, incomplete and that the diagonal method does produce a number not contained in the list. Also, we'll agree that there are an infinite number of binary decimal numbers in the interval that aren't in the list. We will see how using the same diagonal method we can create infinitely many binary decimal numbers not initially contained in the interval and that each number we so create will correspond with one and only one natural number.

II. Initial Conditions, Cantor's 1891 Proof

1. The set B of infinite binary decimals on the interval (0, 1),

$$B = \{d \mid 0 < d < 1\}$$

2. A list L, of the elements of B.

B											
.	0	1	0	1	0	1	0	1	0	1	…
.	1	0	1	0	1	0	1	0	1	0	…
.	0	1	0	1	0	1	0	1	0	1	…
.	0	0	1	0	0	1	0	0	1	0	…
.	1	1	0	1	1	0	1	1	0	1	…
.	0	0	0	1	0	0	0	1	0	0	…
					…						

3. The set of natural numbers N.

$$N = \{1, 2, 3, \ldots\}$$

4. The following arrangement matching each item in B with an element of N:

N	B											
1	.	0	1	0	1	0	1	0	1	0	1	…
2	.	1	0	1	0	1	0	1	0	1	0	…

3	.	0	1	0	1	0	1	0	1	0	1	…
4	.	0	0	1	0	0	1	0	0	1	0	…
5	.	1	1	0	1	1	0	1	1	0	1	…
6	.	0	0	0	1	0	0	0	1	0	0	…
7	.	1	1	1	0	1	1	1	0	1	1	…
8	.	0	0	0	0	1	0	0	0	0	1	…
9	.	1	1	1	1	0	1	1	1	1	0	…
10	.	1	1	0	0	1	1	0	0	1	1	…
…							…					

III. Construct a number, Y, not contained in L

Next, we will use the diagonal argument to construct a binary decimal number that is not in the list by flipping the first digit of the number in the first row, the second digit of the number in the second row, the third digit in the number in the third row and so on. The resulting table will appear as follows:

N	B											
1	.	0	1	0	1	0	1	0	1	0	1	…
2	.	1	0	1	0	1	0	1	0	1	0	…
3	.	0	1	0	1	0	1	0	1	0	1	…
4	.	0	0	1	0	0	1	0	0	1	0	…
5	.	1	1	0	1	1	0	1	1	0	1	…
6	.	0	0	0	1	0	0	0	1	0	0	…
7	.	1	1	1	0	1	1	1	0	1	1	…
8	.	0	0	0	0	1	0	0	0	0	1	…
9	.	1	1	1	1	0	1	1	1	1	0	…
10	.	1	1	0	0	1	1	0	0	1	1	…
…							…					
Y	.	1	1	1	1	0	1	0	1	0	0	…

The number Y will differ at each n^{th} digit from the number in the n^{th} row of the table. Therefore, Y cannot be in L and cannot be matched with a natural number. Since Y cannot be paired with a natural number not already in L, Cantor reasoned that the set of numbers in (0, 1) must be larger than the set of natural numbers.

IV. An alternative arrangement to Cantor's

We will, in a finite number of steps, demonstrate that the real and natural numbers can be arranged in such a way as to match one natural number with one real number in the interval without exception.

Step 1 – Initial setup of the Construction

Given - The set N of natural numbers,

$$N = \{1, 2, 3, \dots\}$$

Given - The set B of infinite binary decimals in the interval (0, 1),

$$B = \{d \mid 0 < d < 1\}$$

Step 2 – List the elements of N and B

Arrange N and B in a list L as follows as follows:

N		B									
1	.	0	1	0	1	0	1	0	1	0	1 …
2											
3	.	1	0	1	0	1	0	1	0	1	0 …
4											
5	.	0	1	0	1	0	1	0	1	0	1 …
6											
7	.	0	0	1	0	0	1	0	0	1	0 …
8											
9	.	1	1	0	1	1	0	1	1	0	1 …
10											
11	.	0	0	0	1	0	0	0	1	0	0 …
12											
13	.	0	1	1	1	0	0	1	1	1	1 …
14											
15	.	1	1	1	1	1	0	1	0	0	0 …
16											
17	.	0	1	0	1	0	0	0	1	1	1 …
18											
19	.	0	1	0	1	0	1	1	1	1	1 …
20											

…

Our list is constructed slightly differently than Cantor's in that we are matching each real number in L with an odd element of N.

Step 3 – Construct a number, Y in B, that is not in L

We now begin as Cantor did, by constructing a number Y in B that is not in L. as shown below:

N		B									
1	.	0	1	0	1	0	1	0	1	0	1 …
2											
3	.	1	0	1	0	1	0	1	0	1	0 …
4											
5	.	0	1	0	1	0	1	0	1	0	1 …
6											
7	.	0	0	1	0	0	1	0	0	1	0 …
8											
9	.	1	1	0	1	1	0	1	1	0	1 …

10												
11	.	0	0	0	1	0	0	0	1	0	0	…
12												
13	.	0	1	1	1	0	0	1	1	1	1	…
14												
15	.	1	1	1	1	1	0	1	0	0	0	…
16												
17	.	0	1	0	1	0	0	0	1	1	1	…
18												
19	.	0	1	0	1	0	1	1	1	1	1	…
20												
						…						
Y	.	1	1	1	1	0	1	0	1	0	0	…

Step 4 – Enter Y into L

As before, Y is not in L. The next step requires that we enter Y into L. We put Y in at the first even element of the set N that contains no matching d, which is 2. See below.

N	B											
1	.	0	1	0	1	0	1	0	1	0	1	1	…
2	.	1	1	1	1	0	1	0	1	0	0	1	…
3	.	1	0	1	0	1	0	1	0	1	0	0	…
4													
5	.	0	1	0	1	0	1	0	1	0	1	1	…
6													
7	.	0	0	1	0	0	1	0	0	1	0	0	…
8													
9	.	1	1	0	1	1	0	1	1	0	1	0	…
10													
11	.	0	0	0	1	0	0	0	1	0	0	0	…
12													
13	.	0	1	1	1	0	0	1	1	1	1	1	…
14													
15	.	1	1	1	1	1	0	1	0	0	0	1	…
16													
17	.	0	1	0	1	0	0	0	1	1	1	1	…
18													
19	.	0	1	0	1	0	1	1	1	1	1	1	…
20													
						…							

Step 5 – Loop thru Steps 3 and 4

We will repeat Steps 3 and 4 with the new number added to the list to create another number Y, also not in the list and match this number with the next even element of N and add it to the list.

N	B											
1	.	0	1	0	1	0	1	0	1	0	1	1	…
2	.	1	1	1	1	0	1	0	1	0	0	1	…
3	.	1	0	1	0	1	0	1	0	1	0	0	…
4													
5	.	0	1	0	1	0	1	0	1	0	1	1	…
6													
7	.	0	0	1	0	0	1	0	0	1	0	0	…
8													
9	.	1	1	0	1	1	0	1	1	0	1	0	…
10													
11	.	0	0	0	1	0	0	0	1	0	0	0	…
12													
13	.	0	1	1	1	0	0	1	1	1	1	1	…
14													
15	.	1	1	1	1	1	0	1	0	0	0	1	…
16													
17	.	0	1	0	1	0	0	0	1	1	1	1	…
18													
19	.	0	1	0	1	0	1	1	1	1	1	1	…
20													
							…						
Y	.	1	0	0	0	1	1	1	0	1	0	0	…

N	B											
1	.	0	1	0	1	0	1	0	1	0	1	1	…
2	.	1	1	1	1	0	1	0	1	0	0	1	…
3	.	1	0	1	0	1	0	1	0	1	0	0	…
4	.	1	0	0	0	1	1	1	0	1	0	0	…
5	.	0	1	0	1	0	1	0	1	0	1	1	…
6													
7	.	0	0	1	0	0	1	0	0	1	0	0	…
8													
9	.	1	1	0	1	1	0	1	1	0	1	0	…
10													
11	.	0	0	0	1	0	0	0	1	0	0	0	…
12													
13	.	0	1	1	1	0	0	1	1	1	1	1	…
14													
15	.	1	1	1	1	1	0	1	0	0	0	1	…
16													

17	.	0	1	0	1	0	0	0	1	1	1	1 …
18												
19	.	0	1	0	1	0	1	1	1	1	1	1 …
20												

…

Discussion -

We now have a situation where every decimal number, d of L, at any moment in time is matched with one and only one natural number. And going forward, no matter how many numbers Y are added to the list, each will be matched with one and only one natural number.

Let N_e be defined as a subset of N such that each element n of N_e is an even natural number,

$$N_e \subseteq N \text{ and } N_e = \{n \mid n \text{ is even}\}.$$

Let N_o be defined as a subset of N such that each element m of N_o is an odd natural number,

$$N_o \subseteq N \text{ and } N_o = \{m \mid m \text{ is odd}\}.$$

Let B_1 be defined as a subset of B such that every element of B_1 is a member of L before any number Y has been created and inserted into L,

$$B_1 \subseteq B \text{ and } B_1 = \{d \mid d \in L\}.$$

Let B_2 be defined as a subset of B such that that every element of B_2 is not a member of L before any number Y has been created and inserted into L,

$$B_2 \subseteq B \text{ and } B_2 = \{d \mid d \notin L\}.$$

The union of B_1 and B_2 is the set B,

$$B_1 \cup B_2 = B$$

The union of N_e and N_o is the set N

$$N_e \cup N_o = N$$

Since for every $d \in B_1$ we have one and only one corresponding $m \in N_o$ we can say that

$$f: N_o \rightarrow B_1 \text{ exists.}$$

Since for every $d \in B_2$ we have one and only one corresponding $n \in N_e$ we can say that

$$f: N_e \rightarrow B_2 \text{ exists.}$$

We can now say that

$$f: (N_O \cup N_e) \rightarrow (B_1 \cup B_2) \text{ and}$$

$$\text{since } (N_O \cup N_e) = N \text{ and } (B_1 \cup B_2) = B \text{ then}$$

$$f: N \rightarrow B \text{ also exists}$$

and this completes the discussion.