Proof of Riemann hypothesis

By Toshihiko Ishiwata

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $Re(s) > 0$ and 2 formulas ($1/2 + a + bi$, $1/2 - a - bi$) which show zero point of $\zeta(s)$. 2. We find that the value of $F(a)$ (that is the infinite series regarding a) must be zero from the above infinite number of infinite series. 3. We find that $F(a) = 0$ has the only solution of $a = 0$. 4. Zero point of $\zeta(s)$ must be $1/2 \pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+ \cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \le a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 14$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 [Conjugate complex number of S_0] $= 1/2 + a - bi$ is also zero point of $\zeta(s)$. Therefore $b \ge 0$ is necessary and sufficient range for investigation.

1.2 The range of b of zero points found until now is $b > 14$.

The following (3) also shows zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. And the value of (11) is always zero at any value of x.

$$\begin{aligned}
0 \equiv{} & \cos x\{\text{right side of (9)}\} + \sin x\{\text{right side of (10)}\} \\
={} & \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots\} \\
& + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots\} \\
={} & f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\
& - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots
\end{aligned} \tag{11}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned}
0 ={} & f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\
& - f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots
\end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned}
0 ={} & f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\
& - f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots
\end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$0 = f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)$$

$$- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \quad (12\text{-}3)$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, 8, \cdots\cdots)$

$$\begin{aligned} 0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N) \\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned} \quad (12\text{-N})$$

3. Verification of $F(a) = 0$

We define $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned} g(k, N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cdots + \cos(b\log k - b\log N) \\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cdots + \cos(b\log N - b\log k) \\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \end{aligned} \quad (13)$$

We can have the following (14) from the equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N) with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned} 0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log N)\} \\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log N)\} \\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log N)\} \\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log N)\} \\ &+ \cdots\cdots \\ =& f(2)g(2, N) - f(3)g(3, N) + f(4)g(4, N) - f(5)g(5, N) + \cdots\cdots \end{aligned} \quad (14)$$

Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \quad (15)$$

We can have the following (16) by dividing the above (14) by $g(2, N)$. Because $g(2, N) \neq 0$ is true in $N_0 \leq N$ as shown in [Appendix 2: Proof of $g(2, N) \neq 0$]. N_0 is the large natural number that holds (29) in [Appendix 2].

$$0 = f(2) - \frac{f(3)g(3, N)}{g(2, N)} + \frac{f(4)g(4, N)}{g(2, N)} - \frac{f(5)g(5, N)}{g(2, N)} + \cdots\cdots \quad (N_0 \leq N) \quad (16)$$

We can have the following (17) from the above (16) by performing $N \to \infty$. Because $\lim_{N\to\infty} \frac{g(k, N)}{g(2, N)} = 1$ $(k = 3, 4, 5, 6, 7\cdots\cdots)$ is true as shown in [Appendix 3: Proof of $\lim_{N\to\infty} \frac{g(k, N)}{g(2, N)} = 1$].

$$\begin{aligned} 0 &= \lim_{N\to\infty} \{f(2) - \frac{f(3)g(3, N)}{g(2, N)} + \frac{f(4)g(4, N)}{g(2, N)} - \frac{f(5)g(5, N)}{g(2, N)} + \cdots\cdots\} \\ &= f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots = F(a) \quad (N_0 \leq N) \end{aligned} \quad (17)$$

4. Conclusion

$F(a) = 0$ has the only solution of $a = 0$ as shown in [Appendix 4: Solution for $F(a) = 0$]. a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero because a is the solution for $F(a) = 0$. Due to $a = 0$ non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$ and other zero point does not exist. Therefore Riemann hypothesis which says " All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s) = 1/2$. " is true.

Appendix 1. : Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) are true.

$$\text{(Series 1)} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A$$
$$\text{(Series 2)} = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B$$
$$\text{(Series 3)} = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B$$
$$\text{(Series 4)} = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B$$

1.1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 1 \qquad (6)$$

$$\text{(Series 2)} = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 1 \qquad (4)$$

$$\text{(Series 4)} = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots = 1 - 1 = 0 \qquad (9)$$

Here $f(n)$ is defined as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \qquad (8)$$

1.2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 0 \qquad (7)$$

$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 0 \qquad (5)$$

$$(\text{Series } 4) = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots = 0 - 0 \qquad (10)$$

1.3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}(\text{Series } 1) &= \cos x\{\text{right side of } (9)\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series } 2) &= \sin x\{\text{right side of } (10)\}\\ &= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series } 3) &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &\quad - f(5)\cos(b\log 5 - x) + \cdots\cdots \equiv 0 + 0 \qquad (11)\end{aligned}$$

1.4. Construction of (14)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 1) = &f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0 \qquad (12\text{-}1)\\ (\text{Series } 2) = &f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0 \qquad (12\text{-}2)\\ (\text{Series } 3) = &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &+ \cdots\cdots = 0 + 0 \qquad (12\text{-}1{*}2)\end{aligned}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) = &f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ &+ f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ &+ f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad (12\text{-}3)\\ (\text{Series } 3) = &f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\end{aligned}$$

$$+\cdots\cdots = 0+0 \qquad (12\text{-}1{*}3)$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series 2}) = {} & f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4)\\ & + f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4)\\ & + f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \qquad (12\text{-}4)\end{aligned}$$

$$\begin{aligned}&(\text{Series 3})\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log 4)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log 4)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log 4)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log 4)\}\\ &\quad + \cdots\cdots = 0+0 \qquad (12\text{-}1{*}4)\end{aligned}$$

1.4.4 In the same way as above we can have the following (12-1*N)=(14) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. ($N = 5, 6, 7, 8, \cdots\cdots$) $g(k, N)$ is defined in page 3. ($k = 2, 3, 4, 5, \cdots\cdots$)

$$\begin{aligned}& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log N)\}\\ & - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log N)\}\\ & + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log N)\}\\ & - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log N)\}\\ & + \cdots\cdots\\ & = f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + f(6)g(6,N) - \cdots\cdots\\ & = 0+0 \qquad (12\text{-}1{*}N)\end{aligned}$$

Appendix 2. : Proof of $g(2, N) \neq 0$

2.1. Investigation of $g(k, N)$

2.1.1 We define G and H as follows.

$$\begin{aligned}G &= \lim_{N\to\infty} \frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}\\ &= \int_0^1 \cos(b\log x)dx \qquad (20\text{-}1)\\ H &= \lim_{N\to\infty} \frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}\\ &= \int_0^1 \sin(b\log x)dx \qquad (20\text{-}2)\end{aligned}$$

We calculate G and H by Integration by parts.

$$G = [x\cos(b\log x)]_0^1 + bH = 1 + bH$$

$$H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \tag{21}$$

2.1.2 We define as follows.

$$\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G = E_c(N) \tag{22-1}$$

$$\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H = E_s(N) \tag{22-2}$$

From the definition of (20-1), (20-2), (22-1) and (22-2) we have the following (23).

$$\lim_{N\to\infty} E_c(N) = 0 \qquad \lim_{N\to\infty} E_s(N) = 0 \tag{23}$$

2.1.3 From (13) we can calculate $g(k,N)$ as follows.

$$g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)$$

$$= N\frac{1}{N}\{\cos(b\log\frac{1}{N}\frac{N}{k}) + \cos(b\log\frac{2}{N}\frac{N}{k}) + \cos(b\log\frac{3}{N}\frac{N}{k}) + \cdots + \cos(b\log\frac{N}{N}\frac{N}{k})\}$$

$$= N\frac{1}{N}\{\cos(b\log\frac{1}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{2}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{3}{N} + b\log\frac{N}{k})$$
$$+ \cdots\cdots + \cos(b\log\frac{N}{N} + b\log\frac{N}{k})\}$$

$$= N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$- N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$

$$= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}\{\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G\}$$
$$- N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}\{\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H\} \tag{24-1}$$

$$= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}E_c(N)$$
$$- N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}E_s(N) \tag{24-2}$$

$$= N\{\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + N\{\cos(b\log\frac{N}{k})\}E_c(N)$$
$$+ N\{\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2} - N\{\sin(b\log\frac{N}{k})\}E_s(N) \tag{24-3}$$

$$= \frac{N\sin(b\log N/k + \tan^{-1}1/b)}{\sqrt{1+b^2}} + N\{\cos(b\log\frac{N}{k})\}E_c(N) - N\{\sin(b\log\frac{N}{k})\}E_s(N) \tag{24-4}$$

$$= \frac{N\sin(b\log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - N\sqrt{E_c(N)^2 + E_s(N)^2}\sin\{b\log\frac{N}{k} - \tan^{-1}\frac{E_c(N)}{E_s(N)}\} \tag{24-5}$$

$$= NR(1)\sin\{b\log N/k + \theta(1)\} - NR(2)\sin\{b\log N/k - \theta(2)\} \tag{24-6}$$

$$= NR(3)\sin\{b\log N/k + \theta(3)\} \tag{24-7}$$

2.1.4 From (22-1), (22-2) and (24-1) we have (24-2). From (21) and (24-2) we have (24-3).

2.1.5 We define as follows. From (24-5), the following (26-1) and (26-2) we have (24-6).

$$R(1) = 1/\sqrt{1+b^2} \qquad \theta(1) = \tan^{-1} 1/b \tag{26-1}$$

$$R(2) = \sqrt{E_c(N)^2 + E_s(N)^2} \qquad \theta(2) = \tan^{-1}\frac{E_c(N)}{E_s(N)} \tag{26-2}$$

2.1.6 We can calculate the following (27-1) and (27-2) from the following (FIgure 1). $R(3)$ can be calculated by Cosine theorem. We have (24-7) from (24-6), (27-1) and (27-2).

$$R(3) = \sqrt{R(1)^2 + R(2)^2 - 2R(1)R(2)\cos\{\theta(1)+\theta(2)\}} \tag{27-1}$$

$$\theta(3) = \tan^{-1}\frac{R(1)\sin\theta(1) + R(2)\sin\theta(2)}{R(1)\cos\theta(1) - R(2)\cos\theta(2)} \tag{27-2}$$

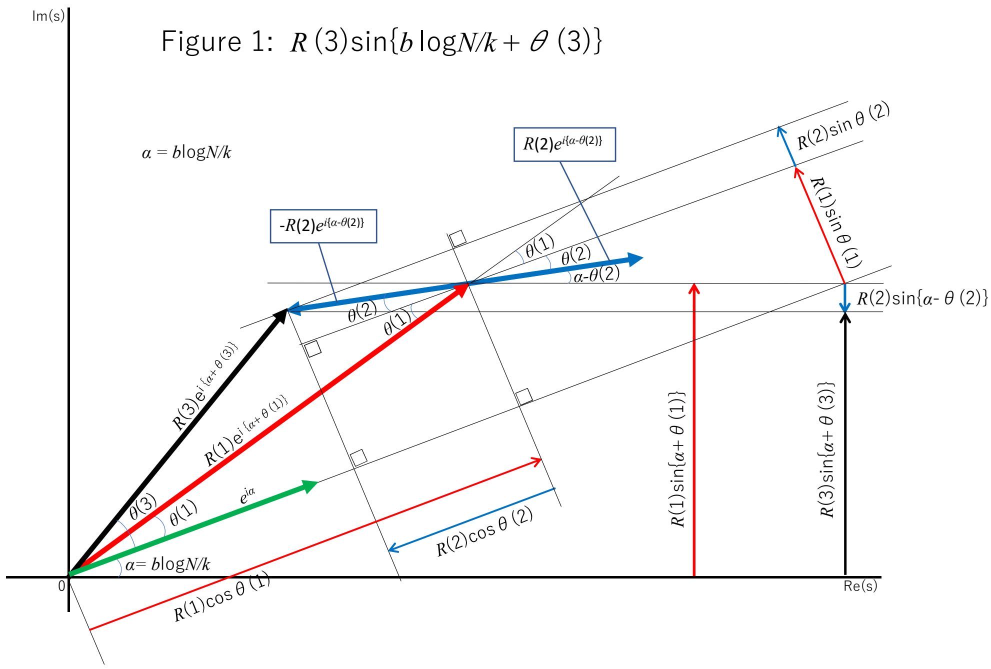


Figure 1: $R(3)\sin\{b\log N/k + \theta(3)\}$

2.1.7 The condition of $R(3) = 0$ is as follows.

$$R(1) = 1/\sqrt{1+b^2} = \sqrt{E_c(N)^2 + E_s(N)^2} = R(2) \tag{28-1}$$
$$\theta(1) = \tan^{-1} 1/b = -\tan^{-1} E_c(N)/E_s(N) = -\theta(2) \tag{28-2}$$

There is the large natural number N_0 that holds the following (29) because of $\lim_{N\to\infty} \sqrt{E_c(N)^2 + E_s(N)^2} = 0$.

$$1/\sqrt{1+b^2} > \sqrt{E_c(N)^2 + E_s(N)^2} > 0 \qquad (N_0 \leq N) \tag{29}$$

From the above (28-1) and (29) the following (30) holds.

$$R(3) \neq 0 \qquad (N_0 \leq N) \tag{30}$$

2.2. Verification of $\sin\{b \log N/2 + \theta(3)\} \neq 0$

2.2.1 If we assume the following (31) is true, the following (32) is also supposed to be true.

$$\sin\{b \log N/2 + \theta(3)\} = 0 \qquad (N = 3, 4, 5, 6, 7, \cdots\cdots) \tag{31}$$
$$b \log N/2 + \theta(3) = K\pi \qquad (K = 2, 3, 4, 5, 6, \cdots\cdots) \tag{32}$$

The range of b is $14 < b$ as shown in page 1. We have $\log 3/2 = 0.405$ and $-\pi/2 < \theta(3) < \pi/2$ from (27-2). Then we have $K > 1.3$ from $[14*0.405 - \pi/2 = 4.09 < K\pi]$. Therefore $(K = 2, 3, 4, \cdots\cdots)$ holds.

2.2.2 We define as follows.

Type1 irrational number : Irrational number which consists of singular or plural irrational terms such as $2\sqrt{2}/e,\ \sqrt{2}/e + \sqrt{3},\ \sqrt{2}/e + \sqrt{3} + \sqrt{5}$, etc.

Type2 irrational number : Irrational number which has the formation of (rational number)+(type1 irrational number) such as $1 + \sqrt{2},\ 2 + 2\sqrt{2}/e + \sqrt{3},\ 3 + 2\sqrt{2}/e + \sqrt{3} + \sqrt{5}$, etc.

2.2.3 The above (32) holds in the following cases.

Case 1 : $b \log N/2 = A\pi \quad \theta(3) = B\pi \quad A + B = K \quad A, B$: (rational number)

Case 2 : $b \log N/2 = (A + C)\pi \quad \theta(3) = (B - C)\pi \quad C$: (type1 irrational number) $A + C$: (type2 irrational number)

2.2.4 From $b \log N/2 = D\pi$ we have the following (33).

$$D = \frac{b \log N/2}{\pi} \tag{33}$$

The formation of D becomes (type1 irratioal number) regardless of the formation of b as follows.

Case 3 : $b =$(rational number)

$$D = (\text{rational number})\frac{\log N/2}{\pi} = (\text{type1 irratioal number-1})$$

Case 4 : $b =$(type1 irrational number)

$$D = (\text{type1 irrational number-2})\frac{\log N/2}{\pi} = (\text{type1 irratioal number-3})$$

Case 5 : $b =$(type2 irrational number)=(rational number)+(type1 irrational number)

$$\begin{aligned} D =& \{(\text{rational number})+(\text{type1 irrational number-4})\}\frac{\log N/2}{\pi} \\ =&(\text{type1 irrational number-5})+(\text{type1 irratioal number-6}) \\ =&(\text{type1 irrational number-7}) \end{aligned}$$

2.2.5 As shown in the above item 2.2.4 D is not (rational number) or (type2 irrational number) but (type1 irrational number). Therefore (case 1) and (case 2) do not hold i.e. the above (32) does not hold. Now we can confirm the following (34).

$$\sin\{b \log N/2 + \theta(3)\} \neq 0 \qquad (N = 3, 4, 5, 6, 7, \cdots\cdots) \tag{34}$$

2.3. Verification of $g(2, N) \neq 0$

We have the following (35) from (24-7) in item 2.1.3, (30) and (34). We can confirm that $g(2, N)$ does not have the value of zero in $N_0 \leq N$.

$$g(2, N) = NR(3) \sin\{b \log N/2 + \theta(3)\} \neq 0 \qquad (N_0 \leq N) \tag{35}$$

Appendix 3. : Proof of $\lim_{N\to\infty} \frac{g(k,N)}{g(2,N)} = 1$

From (24-4) in item 2.1.3 we have the following (36).

$$\begin{aligned} &\frac{g(k,N)}{g(2,N)} \\ =& \frac{\frac{N\sin(b\log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} + N\{\cos(b\log \frac{N}{k})\}E_c(N) - N\{\sin(b\log\frac{N}{k})\}E_s(N)}{\frac{N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} + N\{\cos(b\log \frac{N}{2})\}E_c(N) - N\{\sin(b\log\frac{N}{2})\}E_s(N)} \\ =& \frac{\sin(b\log\frac{N}{k} + \tan^{-1}\frac{1}{b}) + \sqrt{1+b^2}\{\cos(b\log\frac{N}{k})E_c(N) - \sin(b\log\frac{N}{k})E_s(N)\}}{\sin(b\log\frac{N}{2} + \tan^{-1}\frac{1}{b}) + \sqrt{1+b^2}\{\cos(b\log\frac{N}{2})E_c(N) - \sin(b\log\frac{N}{2})E_s(N)\}} \\ =& \frac{\sin\{\frac{b\log N/k + \tan^{-1}1/b}{b\log N/2+\tan^{-1}1/b}(b\log\frac{N}{2} + \tan^{-1}\frac{1}{b})\} + \sqrt{1+b^2}\{\cos(b\log\frac{N}{k})E_c(N) - \sin(b\log\frac{N}{k})E_s(N)\}}{\sin(b\log\frac{N}{2} + \tan^{-1}\frac{1}{b}) + \sqrt{1+b^2}\{\cos(b\log\frac{N}{2})E_c(N) - \sin(b\log\frac{N}{2})E_s(N)\}} \end{aligned} \tag{36}$$

We can have the following (37) from the above (36), the following (38) and the following (23) shown in item 2.1.2. Then we can confirm $\lim_{N\to\infty}\dfrac{g(k,N)}{g(2,N)} = 1$.

$$\lim_{N\to\infty}\frac{g(k,N)}{g(2,N)} = \frac{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})} = 1 \qquad (N_0 \le N) \tag{37}$$

$$\lim_{N\to\infty}\frac{b\log\frac{N}{k}+\tan^{-1}\frac{1}{b}}{b\log\frac{N}{2}+\tan^{-1}\frac{1}{b}} = \lim_{N\to\infty}\frac{1-\frac{\log k}{\log N}+\frac{\tan^{-1}1/b}{b\log N}}{1-\frac{\log 2}{\log N}+\frac{\tan^{-1}1/b}{b\log N}} = 1 \tag{38}$$

$$\lim_{N\to\infty}E_c(N)=0 \qquad \lim_{N\to\infty}E_s(N)=0 \tag{23}$$

Appendix 4. : Solution for $F(a) = 0$

4.1. Preparation for verification of $F(a) > 0$

4.1.1. Investigation of $f(n)$

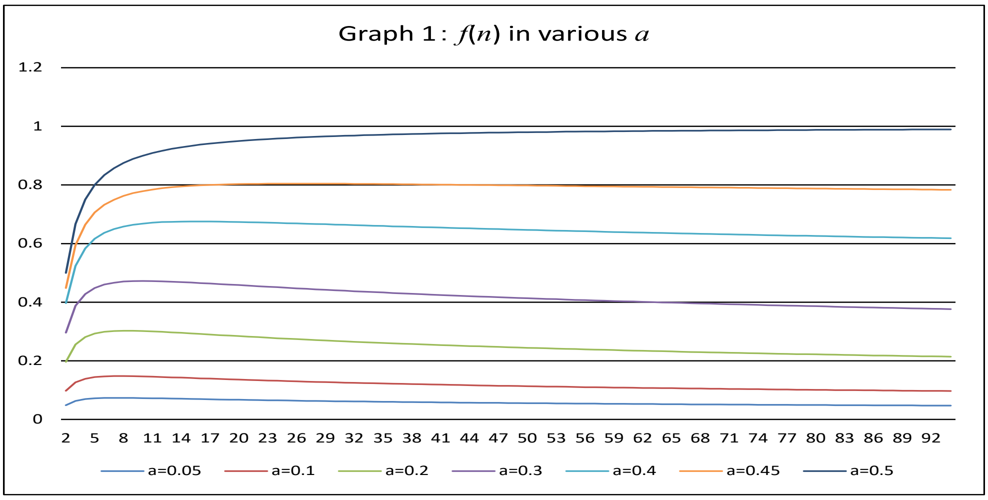
$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \ge 0 \qquad (n = 2,3,4,5,\cdots\cdots) \tag{8}$$

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{15}$$

$a = 0$ is the solution for $F(a) = 0$ due to $f(n) \equiv 0$ at $a = 0$. Hereafter we define the range of a as $0 < a < 1/2$ to verify $F(a) > 0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty} f(n) = 0$.
We have the following (51) by differentiating $f(n)$ regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\{1-(\frac{1/2-a}{1/2+a})n^{2a}\} \tag{51}$$

The value of $f(n)$ increases with increase of n and reaches the maximum value $f(n_{max})$ at $n = n_{max}$. Afterward $f(n)$ decreases to zero with $n \to \infty$. n_{max} is one of the 2 consecutive natural numbers that sandwich $(\frac{1/2+a}{1/2-a})^{\frac{1}{2a}}$. (Graph 1) shows $f(n)$ in various value of a. At $a = 1/2$ $f(n)$ does not have $f(n_{max})$ and increases to 1 with $n \to \infty$ due to $n_{max} = \infty$.



4.1.2. Verification method for $F(a) > 0$

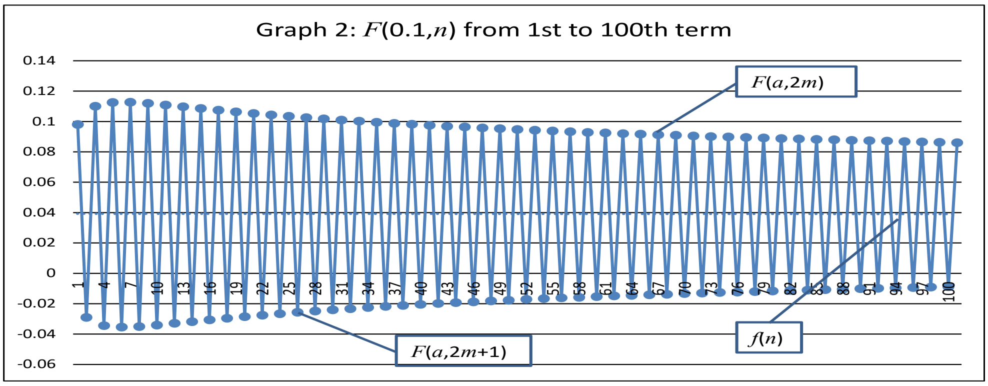
We define $F(a,n)$ as the following (52).

$$F(a,n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \qquad (n = 2,3,4,5,\cdots\cdots) \tag{52}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{53}$$

$F(a)$ is an alternating series. So $F(a,n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in (Graph 2). In (Graph 2) upper points mean $F(a,2m)$ $(m = 1,2,3,\cdots\cdots)$ and lower points mean $F(a,2m+1)$. $F(a,2m)$ decreases and converges to $F(a)$ with $m \to \infty$. $F(a,2m+1)$ increases and also converges to $F(a)$ with $m \to \infty$ due to $\lim_{n\to\infty} f(n) = 0$. We can have the following (54).

$$\lim_{m\to\infty} F(a,2m) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{54}$$



We define $F1(a)$ and $F1(a,2m+1)$ as follws.

$$F1(a) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \{f(6) - f(7)\} + \cdots\cdots \tag{55}$$

$$F1(a,2m+1) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(2m) - f(2m+1)\}$$

$$= f(2) - f(3) + f(4) - f(5) + \cdots + f(2m) - f(2m+1) = F(a, 2m+1) \quad (56)$$

$$\lim_{m\to\infty} F1(a, 2m+1) = F1(a) \quad (57)$$

From the above (54), (56) and (57) we have $F(a) = F1(a)$. We can use $F1(a)$ instead of $F(a)$ to verify $F(a) > 0$.

We enclose 2 terms of $F(a)$ each from the first term with { } as follows. If n_{max} is p or $p+1$ (p: odd number), the inside sum of { } from $f(2)$ to $f(p)$ has negative value and the inside sum of { } after $f(p+1)$ has positive value.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - f(7) + \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(p-1) - f(p)\} + \{f(p+1) - f(p+2)\} + \cdots\cdots$$

(inside sum of { }) $< 0 \longleftarrow|\longrightarrow$ (inside sum of { }) > 0

(total sum of { }) $= -B \longleftarrow|\longrightarrow$ (total sum of { }) $= A$

We define as follows.

[the partial sum from $f(2)$ to $f(p)$] $= -B < 0$

[the partial sum from $f(p+1)$ to $f(\infty)$] $= A > 0$

$$F(a) = A - B \quad (58)$$

So we can verify $F(a) > 0$ by verifying $A > B$.

4.1.3. Investigation of $\{f(n) - f(n+1)\}$

We have the following (59) by differentiating $\{f(n) - f(n+1)\}$ regarding n.

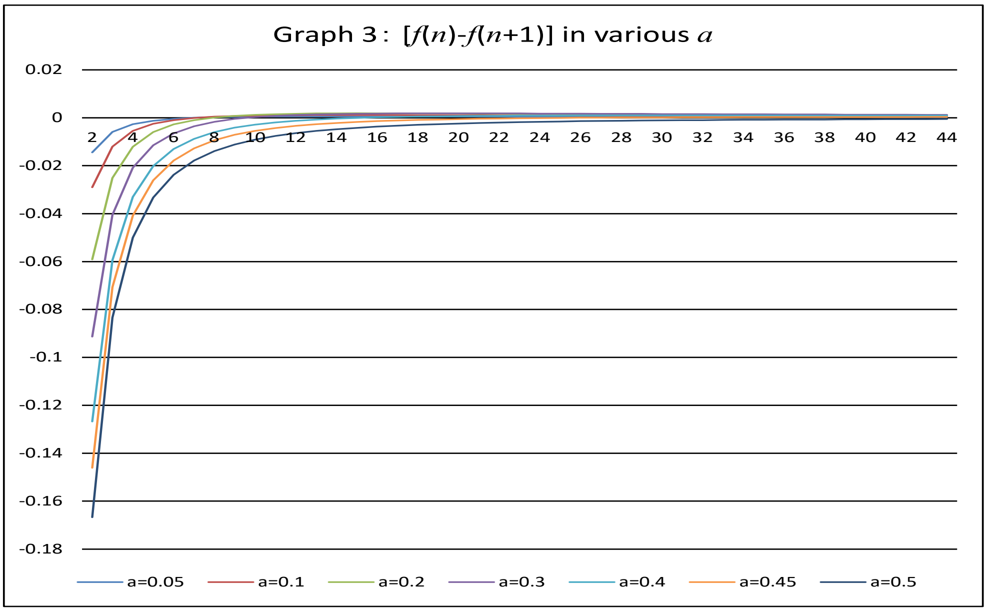
$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\{1 - (\frac{n}{n+1})^{3/2+a}\} - \frac{1/2-a}{n^{3/2-a}}\{1 - (\frac{n}{n+1})^{3/2-a}\}$$
$$= C(n) - D(n) \quad (59)$$

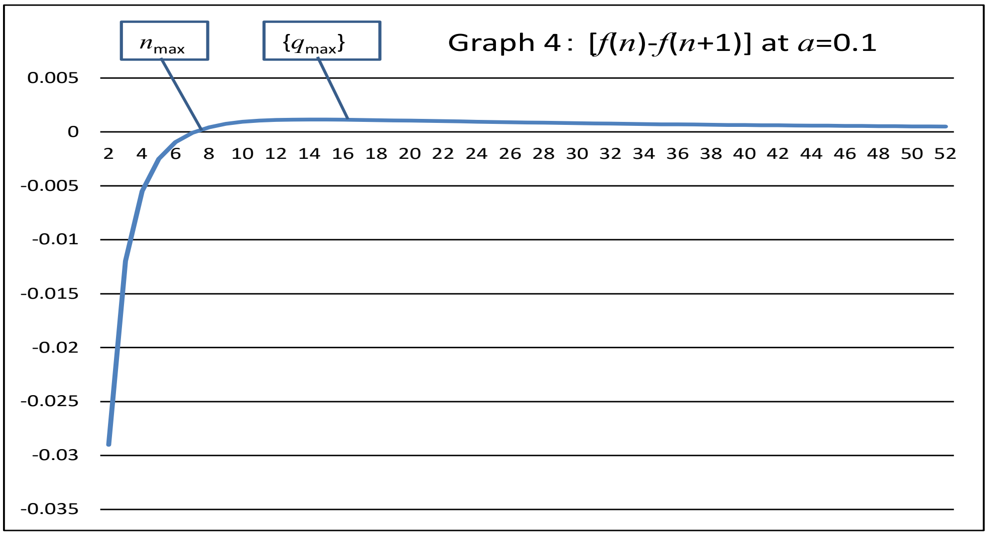
When n is a small natural number the value of $\{f(n)-f(n+1)\}$ increases with increase of n due to $C(n) > D(n)$. With increase of n the value reaches the maximum value $\{q_{max}\}$ at $C(n) \fallingdotseq D(n)$. (n is a natural number. The situation cannot be $C(n) = D(n)$.) After that the situation changes to $C(n) < D(n)$ and the value decreases to zero with $n \to \infty$. (Graph 3) shows the value of $\{f(n) - f(n+1)\}$ in various value of a. (Graph 4) shows the value of $\{f(n) - f(n+1)\}$ at $a = 0.1$. We can find the following from (Graph 3) and (Graph 4).

4.1.3.1 When $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value $|f(n) - f(n+1)|$ also becomes the maximum value at same value of a. From (Graph 1) we can find that $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value at $n = 2$. Therefore the maximum value of $|f(n) - f(n+1)|$ is $\{f(3) - f(2)\}$ at same value of a as shown in (Graph 3).

4.1.3.2 With increase of n the sign of $\{f(n) - f(n+1)\}$ changes from minus to plus at $n = n_{max}$ $(n = n_{max} + 1)$ when n_{max} is even(odd) number as shown in (Graph 4).

4.1.3.3 After that the value reaches the maximum value $\{q_{max}\}$ and the value decreases to zero with $n \to \infty$ as shown in (Graph 4).





4.2. Verification of $A > B$ (n_{max} is odd number.)

n_{max} is odd number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 2)\} + \{f(n_{max} - 1) - f(n_{max})\}$$
$$+\{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 2) - f(n_{max} - 3)\} + \{f(n_{max}) - f(n_{max} - 1)\}$$
$$A = \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} + \{f(n_{max} + 5) - f(n_{max} + 6)\} + \cdots\cdots$$

4.2.1. Condition for B

We define as follows.

{ } : the term which is included within B.

{ } : the term which is not included within B.

We have the following (60).

$$f(n_{max}) - f(2) = \{f(n_{max}) - f(n_{max}-1)\} + \{f(n_{max}-1) - f(n_{max}-2)\} + \{f(n_{max}-2) - f(n_{max}-3)\} + \cdots + \{f(7)-f(6)\} + \{f(6)-f(5)\} + \{f(5)-f(4)\} + \{f(4)-f(3)\} + \{f(3)-f(2)\} \quad (60)$$

And we have the following inequalities from (Graph 3) and (Graph 4).

$$\{f(3)-f(2)\} > \{f(4)-f(3)\} > \{f(5)-f(4)\} > \{f(6)-f(5)\} > \{f(7)-f(6)\} > \cdots\cdots > \{f(n_{max}-2)-f(n_{max}-3)\} > \{f(n_{max}-1)-f(n_{max}-2)\} > \{f(n_{max})-f(n_{max}-1)\} > 0$$

From the above (60) we have the following (61).

$$\begin{aligned}
&f(n_{max}) - f(2) + \{f(3)-f(2)\} \\
&= \{f(3)-f(2)\} + \{f(5)-f(4)\} + \{f(7)-f(6)\} + \cdots + \{f(n_{max}-2)-f(n_{max}-3)\} + \{f(n_{max})-f(n_{max}-1)\} \\
&\qquad \| \qquad\qquad \wedge \qquad\qquad \wedge \qquad\qquad\qquad \wedge \quad \leftarrow\text{Value comparison}\rightarrow \quad \wedge \\
&+\{f(3)-f(2)\} + \{f(4)-f(3)\} + \{f(6)-f(5)\} + \cdots + \{f(n_{max}-3)-f(n_{max}-4)\} + \{f(n_{max}-1)-f(n_{max}-2)\} \\
&> 2B
\end{aligned} \quad (61)$$

Due to [Total sum of upper row of the above (61) $= B <$ Total sum of lower row of (61)] we have the following (62).

$$f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \quad (62)$$

4.2.2. Condition for A ($\{q_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max} + q) - f(n_{max} + q + 1)\}$ to $\{q\}$ for easy description. ($q = 0, 1, 2, 3, \cdots\cdots$) All $\{q\}$ has positive value as shown in item 4.1.2.

We define as follows.

{ } : the term which is included within A.

{ } : the term which is not included within A.

$\{q_{max}\}$ has the maximum value in all $\{q\}$. And $\{q_{max}\}$ is included within A. Then value comparison of $\{q\}$ is as follows.

$$\{1\} < \{2\} < \{3\} < \cdots < \{q_{max}-3\} < \{q_{max}-2\} < \{q_{max}-1\} < \{q_{max}\} > \{q_{max}+1\} > \{q_{max}+2\} > \{q_{max}+3\} > \cdots\cdots$$

We have the following (63).

$$\begin{aligned}
f(n_{max}+1) &= \{f(n_{max}+1) - f(n_{max}+2)\} + \{f(n_{max}+2) - f(n_{max}+3)\} + \{f(n_{max}+3) - f(n_{max}+4)\} \\
&\quad + \{f(n_{max}+4) - f(n_{max}+5)\} + \cdots\cdots \\
&= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max}-3\} + \{q_{max}-2\} + \{q_{max}-1\} + \{q_{max}\} + \{q_{max}+1\} + \{q_{max}+2\} + \{q_{max}+3\} + \cdots\cdots
\end{aligned} \quad (63)$$

From the above (63) we have the following (64).

$$f(n_{max}+1)-\{q_{max}-1\}$$
$$=\{1\}+\{2\}+\{3\}+\{4\}+\cdots+\{q_{max}-3\}+\{q_{max}-2\}+\{q_{max}\}+\{q_{max}+1\}+\{q_{max}+2\}+\{q_{max}+3\}+\cdots\cdots \quad (64)$$

← ··············· Range 1 ··············· →|← ············ Range 2 ············

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\}<\{2\}<\{3\}<\{4\}<\cdots<\{q_{max}-4\}<\{q_{max}-3\}<\{q_{max}-2\}$$

And we can find the following.

$$\text{Total sum of }\{\ \}=\{1\}+\{3\}+\{5\}+\{7\}+\cdots+\{q_{max}-4\}+\{q_{max}-2\}$$
$$\vee \quad \vee \quad \vee \quad \vee \quad \vee \quad \leftarrow\text{Value comparison}$$
$$\text{Total sum of }\{\ \}=\{2\}+\{4\}+\{6\}+\cdots+\{q_{max}-5\}+\{q_{max}-3\}$$

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}\}>\{q_{max}+1\}>\{q_{max}+2\}>\{q_{max}+3\}>\{q_{max}+4\}>\{q_{max}+5\}>\{q_{max}+6\}>\cdots\cdots$$

And we can find the following.

$$\text{Total sum of }\{\ \}=\{q_{max}\}+\{q_{max}+2\}+\{q_{max}+4\}+\{q_{max}+6\}+\cdots\cdots$$
$$\vee \quad \vee \quad \vee \quad \vee \quad \leftarrow\text{Value comparison}$$
$$\text{Total sum of }\{\ \}=\{q_{max}+1\}+\{q_{max}+3\}+\{q_{max}+5\}+\{q_{max}+7\}+\cdots\cdots$$

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 1)+(Range 2) we have [Total sum of { } = A > Total sum of { }].
We have the following (65).

$$f(n_{max}+1)-\{q_{max}-1\}<2A \tag{65}$$

4.2.3. Condition for A ($\{q_{max}\}$ is not included within A.)

We have the following (66). $\{q_{max}\}$ is not included within A.

$$f(n_{max}+1)=\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$$
$$+\{f(n_{max}+4)-f(n_{max}+5)\}+\cdots\cdots$$
$$=\{1\}+\{2\}+\{3\}+\{4\}+\cdots+\{q_{max}-3\}+\{q_{max}-2\}+\{q_{max}-1\}+\{q_{max}\}+\{q_{max}+1\}+\{q_{max}+2\}+\{q_{max}+3\}+\cdots\cdots \tag{66}$$

From the above (66) we have the following (67).

$$f(n_{max}+1)-\{q_{max}\}$$
$$=\{1\}+\{2\}+\{3\}+\{4\}+\cdots+\{q_{max}-3\}+\{q_{max}-2\}+\{q_{max}-1\}+\{q_{max}+1\}+\{q_{max}+2\}+\{q_{max}+3\}+\cdots\cdots \quad (67)$$

← ····················· Range 1 ··················· →|← ········· Range 2 ·········

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\}<\{2\}<\{3\}<\{4\}<\cdots<\{q_{max}-4\}<\{q_{max}-3\}<\{q_{max}-2\}<\{q_{max}-1\}$$

And we can find the following.

$$\begin{array}{lccccccccc} \text{Total sum of } \{\ \} = & \{1\} & + \{3\} & + \{5\} & + \{7\} & + \cdots + & \{q_{max}-3\} & + \{q_{max}-1\} & \\ & & \vee & \vee & \vee & & \vee & \vee & \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\ \} = & & \{2\} & + \{4\} & + \{6\} & + \cdots + & \{q_{max}-4\} & + \{q_{max}-2\} & \end{array}$$

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}+1\} > \{q_{max}+2\} > \{q_{max}+3\} > \{q_{max}+4\} > \{q_{max}+5\} > \{q_{max}+6\} > \{q_{max}+7\} > \cdots\cdots$$

And we can find the following.

$$\begin{array}{lccccc} \text{Total sum of } \{\ \} = & \{q_{max}+1\} & + \{q_{max}+3\} & + \{q_{max}+3\} & + \{q_{max}+7\} & + \cdots\cdots \\ & \vee & \vee & \vee & \vee & \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\ \} = & \{q_{max}+2\} & + \{q_{max}+4\} & + \{q_{max}+6\} & + \{q_{max}+8\} & + \cdots\cdots \end{array}$$

Therefore [Total sum of { } > Total sum of { }] holds.
In (Range 1)+(Range 2) we have [Total sum of { } $= A >$ Total sum of { }].
We have the following (68).

$$f(n_{max}+1) - \{q_{max}\} < 2A \tag{68}$$

4.2.4. Condition for $A > B$

From (65) and (68) we have the following inequality.

$$f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] < 2A$$

As shown in item 4.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n+1)|$. Then the following holds.

$$\begin{aligned} \{f(3) - f(2)\} >& [\{q_{max}\} \text{ or } \{q_{max}-1\}] \\ \{f(3) - f(2)\} >& f(n_{max}) - f(n_{max}+1) \end{aligned}$$

We have the following inequality from the above 3 inequalities.

$$\begin{aligned} 2A >& f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}+1) - \{f(3) - f(2)\} \\ >& f(n_{max}) - \{f(3) - f(2)\} - \{f(3) - f(2)\} = f(n_{max}) - 2\{f(3) - f(2)\} \end{aligned} \tag{69}$$

We have the following (70) for $A > B$ from (62) and (69).

$$2A > f(n_{max}) - 2\{f(3) - f(2)\} > f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \tag{70}$$

From (70) we can have the final condition for $A > B$ as follows.

$$(4/3)f(2) > f(3) \tag{71}$$

(Graph 5) shows $(4/3)f(2) - f(3) = (4/3)(\frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}}) - (\frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}})$.

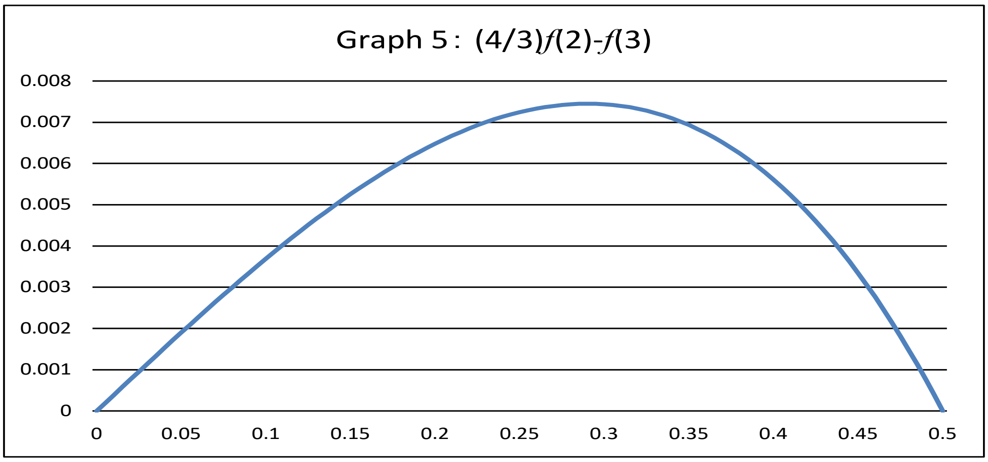


Table 1 : The values of $(4/3)f(2) - f(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

(Graph 6) shows [differentiated $\{(4/3)f(2) - f(3)\}$ regarding a] i.e. $(4/3)f'(2) - f'(3) = (4/3)\{\log 2(\frac{1}{2^{1/2-a}} + \frac{1}{2^{1/2+a}})\} - \{\log 3(\frac{1}{3^{1/2-a}} + \frac{1}{3^{1/2+a}})\}$.

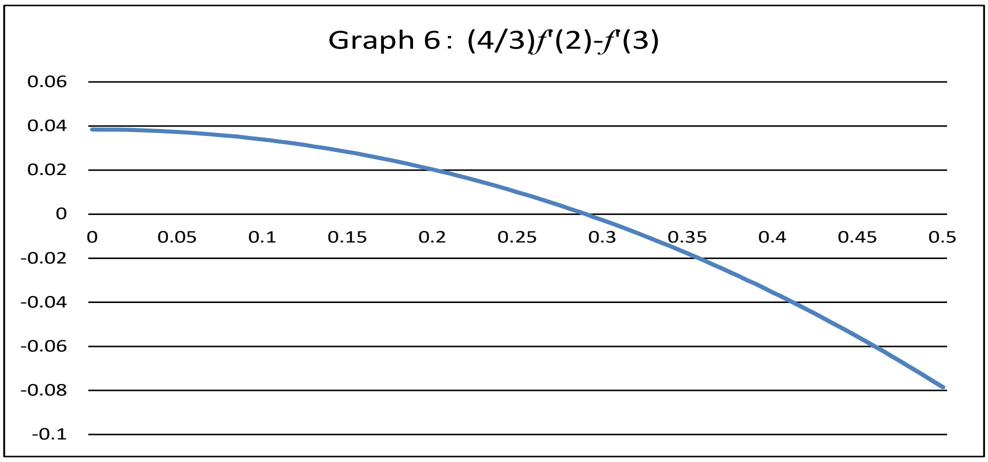


Table 2 : The values of $(4/3)f'(2) - f'(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)-f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	-0.00272	-0.01785	-0.03547	-0.05567	-0.07852

From (Graph 5) and (Graph 6) we can find $[(4/3)f(2) - f(3) > 0$ in $0 < a < 1/2]$ that means $A > B$ i.e. $F(a) > 0$ in $0 < a < 1/2$.

4.3. Verification of $A > B$ (n_{max} is even number.)

n_{max} is even number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 4) - f(n_{max} - 3)\} + \{f(n_{max} - 2) - f(n_{max} - 1)\}$$
$$+\{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$
$$A = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$f(n_{max}) = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\}$$
$$+\{f(n_{max} + 3) - f(n_{max} + 4)\} + \cdots\cdots$$
$$= \{0\} + \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots$$

After the same process as in item 4.2.1 we can have the following (72).

$$f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{72}$$

As shown in item 4.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n + 1)|$. Then the following holds.

$$\{f(3) - f(2)\} >[\{q_{max}\} \text{ or } \{q_{max} - 1\}]$$
$$f(n_{max}) >f(n_{max} - 1)$$

We have the following (73) from the above inequalities and the same process as in item 4.2.2 and item 4.2.3.

$$2A >f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max} - 1\}] > f(n_{max}) - \{f(3) - f(2)\}$$
$$>f(n_{max} - 1) - \{f(3) - f(2)\} \tag{73}$$

We have the following (74) for $A > B$ from (72) and (73).

$$2A > f(n_{max} - 1) - \{f(3) - f(2)\} > f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{74}$$

From (74) we can have the final condition for $A > B$ as follows.

$$(3/2)f(2) > f(3) \tag{75}$$

In the inequality of $[(3/2)f(2) > (4/3)f(2) > f(3) > 0]$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 4.2.4 we already confirmed that the following (71) was true in $0 < a < 1/2$.

$$(4/3)f(2) > f(3) \tag{71}$$

Therefore the above (75) is true in $0 < a < 1/2$. Now we can confirm $F(a) > 0$ in $0 < a < 1/2$.

4.4. Conclusion

As shown in item 4.2 and item 4.3 $F(a) = 0$ has the only solution of $a = 0$ due to $[0 \leq a < 1/2], [F(0) = 0]$ and $[F(a) > 0$ in $0 < a < 1/2]$.

4.5. Graph of $F(a)$

We can approximate $F(a)$ with the average of $\{F(a, n-1) + F(a, n)\}/2$. But we approximate $F(a)$ by the following (76) for better accuracy. (Graph 7) shows $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$.

$$\frac{\frac{F(a,n-1)+F(a,n)}{2} + \frac{F(a,n)+F(a,n+1)}{2}}{2} = F(a)_n \tag{76}$$

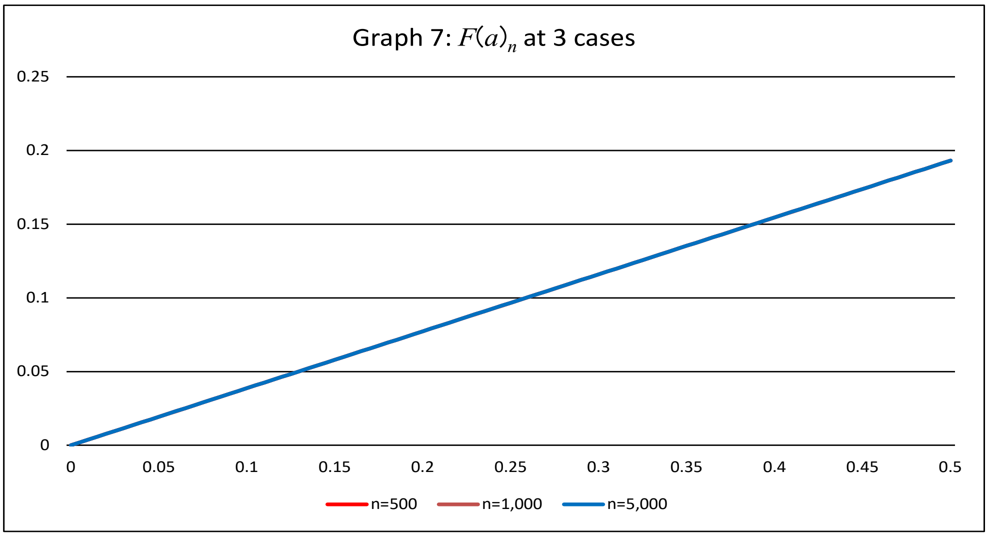


Table 3 : The values of $F(a)_n$ at 3 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
n=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
n=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

3 line graphs overlapped. Because $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$ are equal to 4 digits after the decimal point. The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason. $[f(n)$ at $a = 1/2]$ is $(1-1/n)$ and $F(1/2)$ fluctuates due to $\lim_{n\to\infty} f(n) = 1$. But the value of the above (76) converges to the fixed value on the condition of $\lim_{n\to\infty} \{f(n+1) - f(n)\} = 0$. The condition holds due to $f(n+1) - f(n) = 1/(n + n^2)$.

$F(a)$ is a monotonically increasing function as shown in (Graph 7). So $F(a) = 0$ has the only solution and the solution must be $a = 0$ due to the following facts. Therefore Riemann hypothesis must be true.

4.5.1 In 1914 G. H.Hardy proved that there are infinite zero points on the line of $Re(s) = 1/2$.

4.5.2 All zero points found until now exist on the line of $Re(s) = 1/2$.

Data availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

Toshihiko Ishiwata
E-mail: toshihiko.ishiwata@gmail.com