Proof of Riemann hypothesis

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We make the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $Re(s) > 0$ and 2 formulas ($1/2 + a + bi$, $1/2 - a - bi$) which show zero point of $\zeta(s)$. 2. We find that the value of $F(a)$ (that is the infinite series regarding a) must be zero from the above infinite number of infinite series. 3. We find that $F(a) = 0$ has the only solution of $a = 0$. 4. Zero point of $\zeta(s)$ must be $1/2 \pm bi$ because a cannot have any value but zero.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+ \cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 14$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 [Conjugate complex number of S_0] $= 1/2 + a - bi$ is also zero point of $\zeta(s)$. Therefore $b \geq 0$ is necessary and sufficient range for investigation.

1.2 The range of b of zero points found until now is $b > 14$.

The following (3) also shows zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

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$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1.1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 1.2 of [Appendix 1].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 1.3 of [Appendix 1]. And the value of (11) is always zero at any value of x.

$$\begin{aligned}
0 &\equiv \cos x\{\text{right side of (9)}\} + \sin x\{\text{right side of (10)}\} \\
&= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots\} \\
&\quad + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots\} \\
&= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\
&\quad - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots
\end{aligned} \tag{11}$$

We have the following (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned}
0 =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\
&- f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots
\end{aligned} \tag{12-1}$$

We have the following (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned}
0 =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\
&- f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots
\end{aligned} \tag{12-2}$$

We have the following (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \end{aligned} \tag{12-3}$$

In the same way as above we can have the following (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, 8, \cdots\cdots)$

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned} \tag{12-N}$$

3. Verification of $F(a) = 0$

We define $g(k)$ and $g(k, N)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned} g(k) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + \cdots\cdots\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + \cdots\cdots\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cos(b\log 4/k) + \cdots\cdots \quad (13)\\ g(k, N) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cdots + \cos(b\log k - b\log N)\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \quad (14)\\ \lim_{N\to\infty} & g(k, N) = g(k) \quad (15)\end{aligned}$$

We can have the following (16) from the infinite equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N), $\cdots\cdots$ with the method shown in item 1.4 of [Appendix 1].

$$\begin{aligned}0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots\cdots\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots\cdots\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots\cdots\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots\cdots\}\\ &+ f(6)\{\cos(b\log 6 - b\log 1) + \cos(b\log 6 - b\log 2) + \cos(b\log 6 - b\log 3) + \cdots\cdots\}\\ &- \cdots\cdots\\ =& f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + \cdots\cdots \end{aligned} \tag{16}$$

Here we define $F(a)$ as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{17}$$

We have $F(a) = 0$ from the above (16) as shown in the following (18) because of the following reasons.

$$\begin{aligned}0 =& f(2) - \frac{f(3)g(3)}{g(2)} + \frac{f(4)g(4)}{g(2)} - \frac{f(5)g(5)}{g(2)} + \frac{f(6)g(6)}{g(2)} - \frac{f(7)g(7)}{g(2)} + \cdots\cdots\\ =& f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots\\ =& F(a) \end{aligned} \tag{18}$$

3.1 $g(2,N)$ fluctuates between $+\infty$ and $-\infty$ with $N\to\infty$ but does not have the value of zero in $N_0 < N$ as shown in [Appendix 2: Proof of $g(2)\neq 0$]. (N_0 is a large natural number.) Therefore $\lim_{N\to\infty} g(2,N) = g(2) \neq 0$ is true because $g(2,N)$ does not converge to zero with $N\to\infty$. We can divide the rightmost side of (16) by $g(2)$.

3.2 $g(k)/g(2) = 1 \quad (k = 3,4,5,6,7\cdots\cdots)$ is true as shown in [Appendix 3: Proof of $g(k)/g(2) = 1$].

4. Conclusion

$F(a) = 0$ has the only solution of $a = 0$ as shown in [Appendix 4: Solution for $F(a) = 0$]. a has the range of $0 \leq a < 1/2$ by the critical strip of $\zeta(s)$. However, a cannot have any value but zero because a is the solution for $F(a) = 0$. Due to $a = 0$ non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by (2) and (3) must be $1/2 \pm bi$ and other zero point does not exist. Therefore Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s) = 1/2$." is true.

Appendix 1. Equation construction

We can construct (9), (10), (11) and (16) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge respectively, the following (Series 3) and (Series 4) are true.

$$\begin{aligned}
&(\text{Series 1}) = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots\cdots = A\\
&(\text{Series 2}) = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots\cdots = B\\
&(\text{Series 3}) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + \cdots\cdots = A + B\\
&(\text{Series 4}) = (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + (a_4 - b_4) + \cdots\cdots = A - B
\end{aligned}$$

1.1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 1 \qquad (6)$$

$$(\text{Series 2}) = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 1 \qquad (4)$$

$$\begin{aligned}(\text{Series 4}) = {} & f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ & + \cdots\cdots = 1 - 1 = 0 \end{aligned} \qquad (9)$$

Here $f(n)$ is defined as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2,3,4,5,\cdots\cdots) \qquad (8)$$

1.2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 0 \qquad (7)$$

$$(\text{Series 2}) = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 0 \qquad (5)$$

$$(\text{Series 4}) = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots = 0 - 0 \qquad (10)$$

1.3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}(\text{Series 1}) &= \cos x\{\text{right side of (9)}\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series 2}) &= \sin x\{\text{right side of (10)}\}\\ &= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad + \cdots\cdots\} \equiv 0\\ (\text{Series 3}) &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &\quad - f(5)\cos(b\log 5 - x) + \cdots\cdots \equiv 0 + 0 \qquad (11)\end{aligned}$$

1.4. Construction of (16)

1.4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series 1}) &= f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &\quad + f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &\quad + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0 \qquad (12\text{-}1)\\ (\text{Series 2}) &= f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &\quad + f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &\quad + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0 \qquad (12\text{-}2)\\ (\text{Series 3}) &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\\ &\quad + \cdots\cdots = 0 + 0 \qquad (12\text{-}1{*}2)\end{aligned}$$

1.4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ &+ f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ &+ f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad \text{(12-3)}\\ \text{(Series 3)} =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\\ &+ \cdots\cdots = 0 + 0 \qquad \text{(12-1*3)}\end{aligned}$$

1.4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4)\\ &+ f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4)\\ &+ f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \qquad \text{(12-4)}\end{aligned}$$

$$\begin{aligned}&\text{(Series 3)}\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log 4)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log 4)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log 4)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log 4)\}\\ &\quad + \cdots\cdots = 0 + 0 \qquad \text{(12-1*4)}\end{aligned}$$

1.4.4 In the same way as above we can have the following (12-1*N) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively. $(N = 5, 6, 7, 8, \cdots\cdots)$ $g(k, N)$ is defined in page 3. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned}&f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log N)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log N)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+ \cdots\cdots\\ &= f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + f(6)g(6,N) - \cdots\cdots\\ &= 0 + 0 \qquad \text{(12-1*N)}\end{aligned}$$

1.4.5 Performing the operation in the above item 1.4.4 once increases all N in (12-1*N) by 1. If we repeat this operation infinitely i.e. we perform $N \to \infty$, we can have the following (12-1*∞) = (16). $g(k)$ is defined in page 3.

$$f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + \cdots\cdots = 0 \qquad \text{(12-1*}\infty\text{)}$$

Appendix 2. Proof of $g(2) \neq 0$

2.1. Investigation of $g(k, N)$

We define G and H as follows.

$$G = \lim_{N\to\infty} \frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$= \int_0^1 \cos(b\log x)dx \quad (20\text{-}1)$$
$$H = \lim_{N\to\infty} \frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= \int_0^1 \sin(b\log x)dx \quad (20\text{-}2)$$

We calculate G and H by Integration by parts.

$$G = [x\cos(b\log x)]_0^1 + bH = 1 + bH$$
$$H = [x\sin(b\log x)]_0^1 - bG = -bG$$

Then we can have the values of G and H from the above equations as follows.

$$G = \frac{1}{1+b^2} \qquad H = \frac{-b}{1+b^2} \quad (21)$$

From (14) we can calculate $g(k, N)$ as follows.

$$g(k,N) = \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)$$
$$= N\frac{1}{N}\{\cos(b\log\frac{1}{N}\frac{N}{k}) + \cos(b\log\frac{2}{N}\frac{N}{k}) + \cos(b\log\frac{3}{N}\frac{N}{k}) + \cdots + \cos(b\log\frac{N}{N}\frac{N}{k})\}$$
$$= N\frac{1}{N}\{\cos(b\log\frac{1}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{2}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{3}{N} + b\log\frac{N}{k})$$
$$+ \cdots\cdots + \cos(b\log\frac{N}{N} + b\log\frac{N}{k})\}$$
$$= N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$- N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$
$$= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}\{\frac{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})}{N} - G\}$$
$$- N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}\{\frac{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})}{N} - H\}$$
$$(22\text{-}1)$$
$$= N\{\cos(b\log\frac{N}{k})\}G + N\{\cos(b\log\frac{N}{k})\}E_c(N)$$
$$- N\{\sin(b\log\frac{N}{k})\}H - N\{\sin(b\log\frac{N}{k})\}E_s(N) \quad (22\text{-}2)$$
$$= N\{\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + N\{\cos(b\log\frac{N}{k})\}E_c(N)$$

$$+ N\{\sin(b \log \frac{N}{k})\}\frac{b}{1+b^2} - N\{\sin(b \log \frac{N}{k})\}E_s(N) \tag{22-3}$$

$$= \frac{N \sin(b \log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - N\sqrt{E_c(N)^2 + E_s(N)^2}\sin\{b \log N/k - \tan^{-1} E_c(N)/E_s(N)\} \tag{22-4}$$

$$= NR(1)\sin\{b \log N/k + \theta(1)\} - NR(2)\sin\{b \log N/k - \theta(2)\} \tag{22-5}$$

$$= NR(3)\sin\{b \log N/k + \theta(3)\} \tag{22-6}$$

2.1.1 We define as follows.

$$\frac{\cos(b \log \frac{1}{N}) + \cos(b \log \frac{2}{N}) + \cdots + \cos(b \log \frac{N}{N})}{N} - G = E_c(N) \tag{23-1}$$

$$\frac{\sin(b \log \frac{1}{N}) + \sin(b \log \frac{2}{N}) + \cdots + \sin(b \log \frac{N}{N})}{N} - H = E_s(N) \tag{23-2}$$

From the definition of (20-1), (20-2), (23-1) and (23-2) we have the following (24).

$$\lim_{N\to\infty} E_c(N) = 0 \qquad \lim_{N\to\infty} E_s(N) = 0 \tag{24}$$

From (22-1), (23-1) and (23-2) we have (22-2).

2.1.2 From (21) and (22-2) we have (22-3). From (22-3) and (24) we have the following (25).

$$g(k, N) = \text{(22-3)}$$

$$= N\{\cos(b \log \frac{N}{k})\frac{1}{1+b^2} + \cos(b \log \frac{N}{k})E_c(N) + \sin(b \log \frac{N}{k})\frac{b}{1+b^2} - \sin(b \log \frac{N}{k})E_s(N)\}$$

$$\sim \quad N\{\cos(b \log \frac{N}{k})\frac{1}{1+b^2} + \sin(b \log \frac{N}{k})\frac{b}{1+b^2}\}$$

$$= \frac{N \sin(b \log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \qquad (N \to \infty) \tag{25}$$

2.1.3 We define as follows. From (22-4) and the following (26) we have (22-5).

$$R(1) = 1/\sqrt{1+b^2} \qquad \theta(1) = \tan^{-1} 1/b$$

$$R(2) = \sqrt{E_c(N)^2 + E_s(N)^2} \qquad \theta(2) = \tan^{-1} E_c(N)/E_s(N) \tag{26}$$

2.1.4 We can calculate the following (27-1) and (27-2) from the following (FIgure 1). $R(3)$ can be calculated by Cosine theorem. We have (22-6) from (22-5), (27-1) and (27-2).

$$R(3) = \sqrt{R(1)^2 + R(2)^2 - 2R(1)R(2)\cos\{\theta(1) + \theta(2)\}} \tag{27-1}$$

$$\theta(3) = \tan^{-1} \frac{R(1)\sin\theta(1) + R(2)\sin\theta(2)}{R(1)\cos\theta(1) - R(2)\cos\theta(2)} \tag{27-2}$$

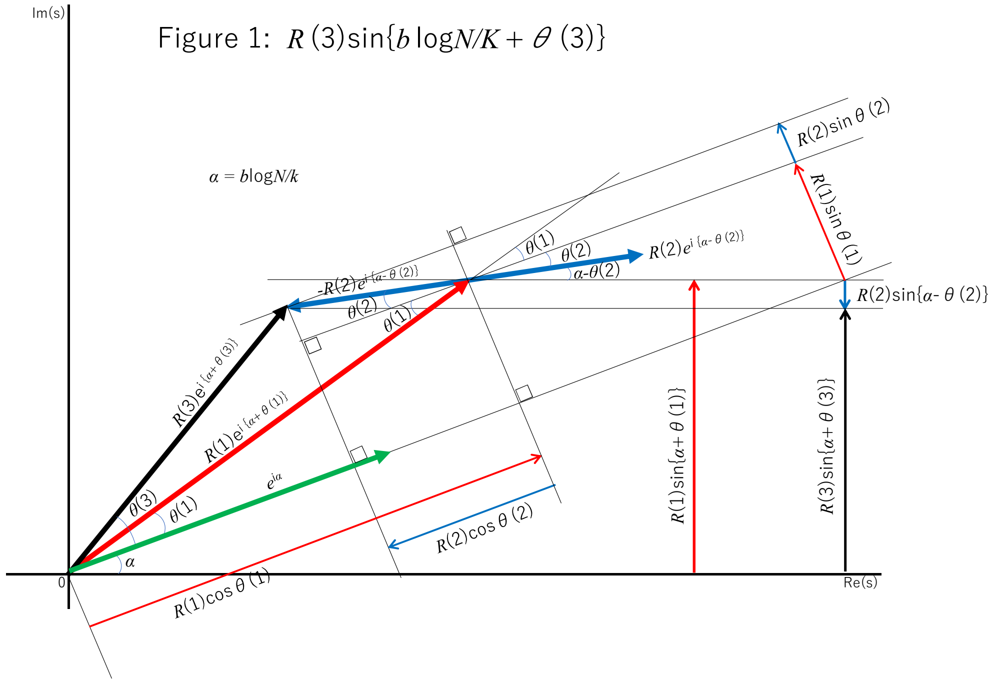


Figure 1: $R\,(3)\sin\{b\log N/K + \theta\,(3)\}$

2.1.5 The condition of $R(3) = 0$ is as follows.

$$R(1) = 1/\sqrt{1+b^2} = \sqrt{E_c(N)^2 + E_s(N)^2} = R(2) \tag{28-1}$$

$$\theta(1) = \tan^{-1} 1/b = -\tan^{-1} E_c(N)/E_s(N) = -\theta(2) \tag{28-2}$$

There is a large natural number N_0 that holds the following (29) because of $\lim_{N\to\infty} \sqrt{E_c(N)^2 + E_s(N)^2} = 0$.

$$1/\sqrt{1+b^2} > \sqrt{E_c(N)^2 + E_s(N)^2} > 0 \qquad (N_0 < N) \tag{29}$$

From the above (28-1) and (29) the following (30) holds.

$$R(3) \neq 0 \qquad\qquad (N_0 < N) \tag{30}$$

2.2. Verification of $\sin\{b\log N/2 + \theta(3)\} \neq 0$

If we assume that $\sin\{b\log N/2 + \theta(3)\} = 0 \quad (N = 3, 4, 5, 6, 7, \cdots\cdots)$ is true, the following (31) is supposed to be true.

$$b\log N/2 + \theta(3) = K\pi \qquad (K = 2, 3, 4, \cdots\cdots) \tag{31}$$

The range of b is $14 < b$ as shown in page 1. We have $\log 3/2 = 0.405$ and $-\pi/2 < \theta(3) < \pi/2$ from (27-2). Then we have $K > 1.3$ from $14 * 0.405 - \pi/2 = 4.09 < K\pi$. Therefore $(K = 2, 3, 4, \cdots\cdots)$ holds.

From (31) we have the following (32).

$$\log N/2 = \frac{K\pi - \theta(3)}{b} = M > 0$$
$$N = 2e^{M} \tag{32}$$

We have $M > 0$ from $K \geq 2$ and $\theta(3) < \pi/2$. (32) has an impossible formation like (natural number) = (irrational number). Therefore (32) is false and (31) (which is the original equation of (32)) is also false. Now we can have the following (33).

$$\sin\{b \log N/2 + \theta(3)\} \neq 0 \qquad (N = 3, 4, 5, 6, 7, \cdots\cdots) \tag{33}$$

2.3. Verification of $g(2) \neq 0$

We have the following (25-1) from (25) in item 2.1.2 and the following (34) from (30) and (33).

$$g(2, N) \sim \frac{N \sin(b \log N/2 + \tan^{-1} 1/b)}{\sqrt{1 + b^2}} \qquad (N \to \infty) \tag{25-1}$$
$$g(2, N) \neq 0 \qquad (N_0 < N) \tag{34}$$

We can confirm that $g(2, N)$ fluctuates between $+\infty$ and $-\infty$ with $N \to \infty$ but does not have the value of zero in $N_0 < N$. (N_0 is a large natural number.) Therefore $\lim_{N\to\infty} g(2, N) = g(2) \neq 0$ is true because $g(2, N)$ does not converge to zero with $N \to \infty$.

Appendix 3. Proof of $g(k)/g(2) = 1$

We can confirm $g(k)/g(2) = 1$ according to the following process. $(k = 3, 4, 5, \cdots\cdots)$

3.1 We can have the following (35) from (15) in page 3.

$$\frac{g(k)}{g(2)} = \lim_{N\to\infty} \frac{g(k, N)}{g(2, N)} \tag{35}$$

3.2 We can have the following (36) from (25) in [Appendix 2].

$$\frac{g(k, N)}{g(2, N)} \sim \frac{\frac{N}{\sqrt{1+b^2}} \sin(b \log \frac{N}{k} + \tan^{-1} \frac{1}{b})}{\frac{N}{\sqrt{1+b^2}} \sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})} = \frac{\sin(b \log \frac{N}{k} + \tan^{-1} \frac{1}{b})}{\sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})} \quad (N \to \infty) \tag{36}$$

3.3 We can have the following (37) from the following (38).

$$\frac{\sin(b \log \frac{N}{k} + \tan^{-1} \frac{1}{b})}{\sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})} = \frac{\sin\{\frac{b \log N/k + \tan^{-1} 1/b}{b \log N/2 + \tan^{-1} 1/b}(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})\}}{\sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})}$$
$$\sim \frac{\sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})}{\sin(b \log \frac{N}{2} + \tan^{-1} \frac{1}{b})} = 1 \qquad (N \to \infty) \tag{37}$$

$$\lim_{N\to\infty} \frac{b \log \frac{N}{k} + \tan^{-1} \frac{1}{b}}{b \log \frac{N}{2} + \tan^{-1} \frac{1}{b}} = \lim_{N\to\infty} \frac{1 - \frac{\log k}{\log N} + \frac{\tan^{-1} 1/b}{b \log N}}{1 - \frac{\log 2}{\log N} + \frac{\tan^{-1} 1/b}{b \log N}} = 1 \tag{38}$$

3.4 $\frac{g(k,N)}{g(2,N)}$ approaches to $\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}$ infinitely with $N\to\infty$ as shown in the above (36). And $\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})}$ converges to 1 with $N\to\infty$ as shown in the above (37). Therefore $\frac{g(k,N)}{g(2,N)}$ also converges to 1 with $N\to\infty$. From (35), (36) and (37) we have the following (39).

$$\frac{g(k)}{g(2)} = \lim_{N\to\infty}\frac{g(k,N)}{g(2,N)} = \lim_{N\to\infty}\frac{\sin(b\log\frac{N}{k}+\tan^{-1}\frac{1}{b})}{\sin(b\log\frac{N}{2}+\tan^{-1}\frac{1}{b})} = 1 \tag{39}$$

Appendix 4. Solution for $F(a) = 0$

4.1. Preparation for verification of $F(a) > 0$

4.1.1. Investigation of $f(n)$

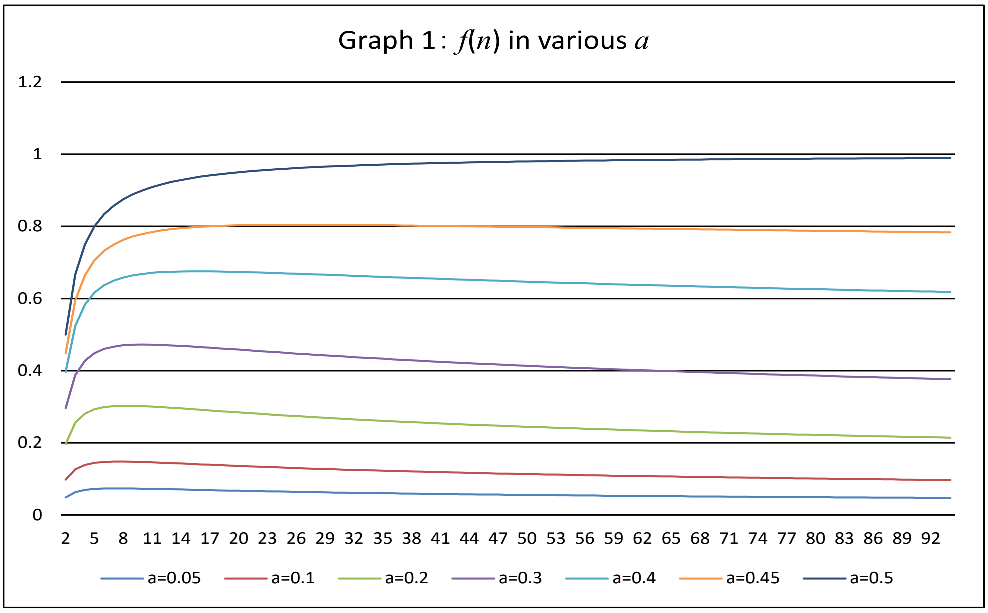
$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2,3,4,5,\cdots\cdots) \tag{8}$$

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots \tag{17}$$

$a = 0$ is the solution for $F(a) = 0$ due to $f(n) \equiv 0$ at $a = 0$. Hereafter we define the range of a as $0 < a < 1/2$ to verify $F(a) > 0$. The alternating series $F(a)$ converges due to $\lim_{n\to\infty} f(n) = 0$.
We have the following (41) by differentiating $f(n)$ regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\{1-(\frac{1/2-a}{1/2+a})n^{2a}\} \tag{41}$$

The value of $f(n)$ increases with increase of n and reaches the maximum value $f(n_{max})$ at $n = n_{max}$. Afterward $f(n)$ decreases to zero with $n\to\infty$. n_{max} is one of the 2 consecutive natural numbers that sandwich $(\frac{1/2+a}{1/2-a})^{\frac{1}{2a}}$. (Graph 1) shows $f(n)$ in various value of a. At $a = 1/2$ $f(n)$ does not have $f(n_{max})$ and increases to 1 with $n\to\infty$ due to $n_{max} = \infty$.



4.1.2. Verification method for $F(a) > 0$

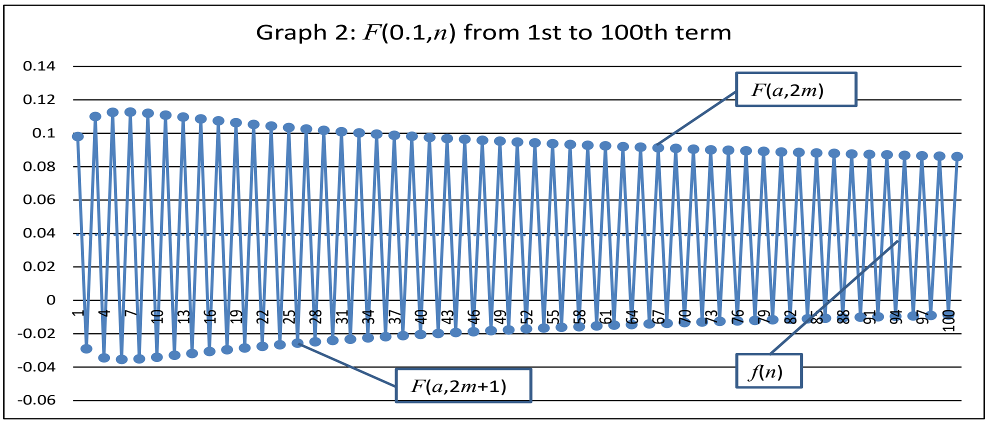
We define $F(a, n)$ as the following (42).

$$F(a,n) = f(2) - f(3) + f(4) - f(5) + \cdots + (-1)^n f(n) \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{42}$$

$$\lim_{n\to\infty} F(a,n) = F(a) \tag{43}$$

$F(a)$ is an alternating series. So $F(a, n)$ repeats increase and decrease by $f(n)$ with increase of n as shown in (Graph 2). In (Graph 2) upper points mean $F(a, 2m)$ $(m = 1, 2, 3, \cdots\cdots)$ and lower points mean $F(a, 2m+1)$. $F(a, 2m)$ decreases and converges to $F(a)$ with $m \to \infty$. $F(a, 2m+1)$ increases and also converges to $F(a)$ with $m \to \infty$ due to $\lim_{n\to\infty} f(n) = 0$. We can have the following (44).

$$\lim_{m\to\infty} F(a,2m) = \lim_{m\to\infty} F(a,2m+1) = F(a) \tag{44}$$



We define $F1(a)$ and $F1(a, 2m+1)$ as follws.

$$F1(a) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \{f(6) - f(7)\} + \cdots\cdots \quad (45)$$

$$F1(a, 2m+1) = \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(2m) - f(2m+1)\}$$

$$= f(2) - f(3) + f(4) - f(5) + \cdots + f(2m) - f(2m+1) = F(a, 2m+1) \quad (46)$$

$$\lim_{m\to\infty} F1(a, 2m+1) = F1(a) \quad (47)$$

From the above (44), (46) and (47) we have $F(a) = F1(a)$. We can use $F1(a)$ instead of $F(a)$ to verify $F(a) > 0$.
We enclose 2 terms of $F(a)$ each from the first term with { } as follows. If n_{max} is p or $p+1$ (p: odd number), the inside sum of { } from $f(2)$ to $f(p)$ has negative value and the inside sum of { } after $f(p+1)$ has positive value.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - f(7) + \cdots\cdots$$

$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(p-1) - f(p)\} + \{f(p+1) - f(p+2)\} + \cdots\cdots$$

(inside sum of { }) < 0 ⟵|⟶(inside sum of { }) > 0
(total sum of { }) $= -B$ ⟵|⟶(total sum of { }) $= A$

We define as follows.
[the partial sum from $f(2)$ to $f(p)$] $= -B < 0$
[the partial sum from $f(p+1)$ to $f(\infty)$] $= A > 0$

$$F(a) = A - B \quad (48)$$

So we can verify $F(a) > 0$ by verifying $A > B$.

4.1.3. Investigation of $\{f(n) - f(n+1)\}$

We have the following (49) by differentiating $\{f(n) - f(n+1)\}$ regarding n.

$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\{1 - (\frac{n}{n+1})^{3/2+a}\} - \frac{1/2-a}{n^{3/2-a}}\{1 - (\frac{n}{n+1})^{3/2-a}\}$$

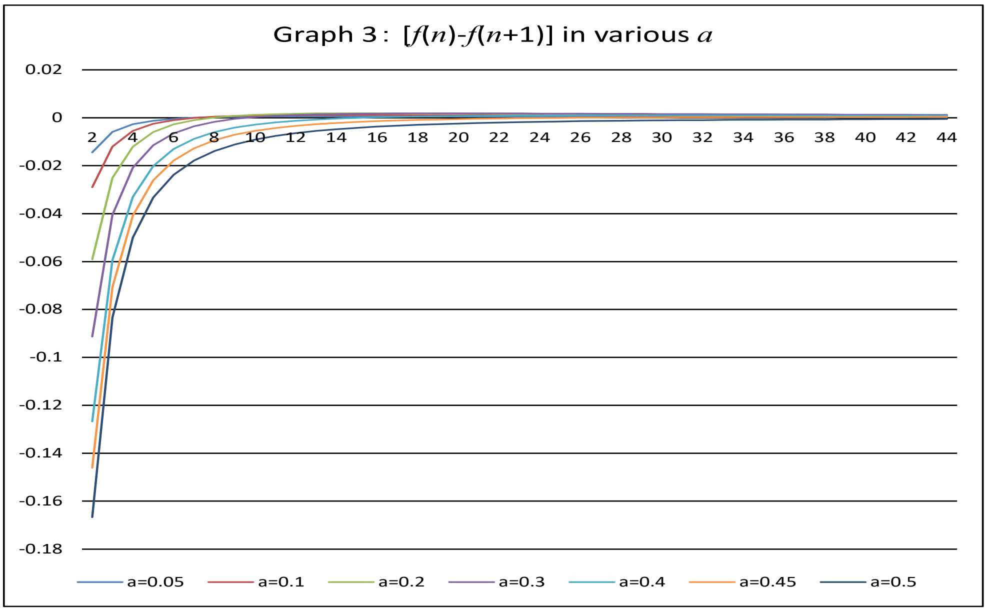
$$= C(n) - D(n) \quad (49)$$

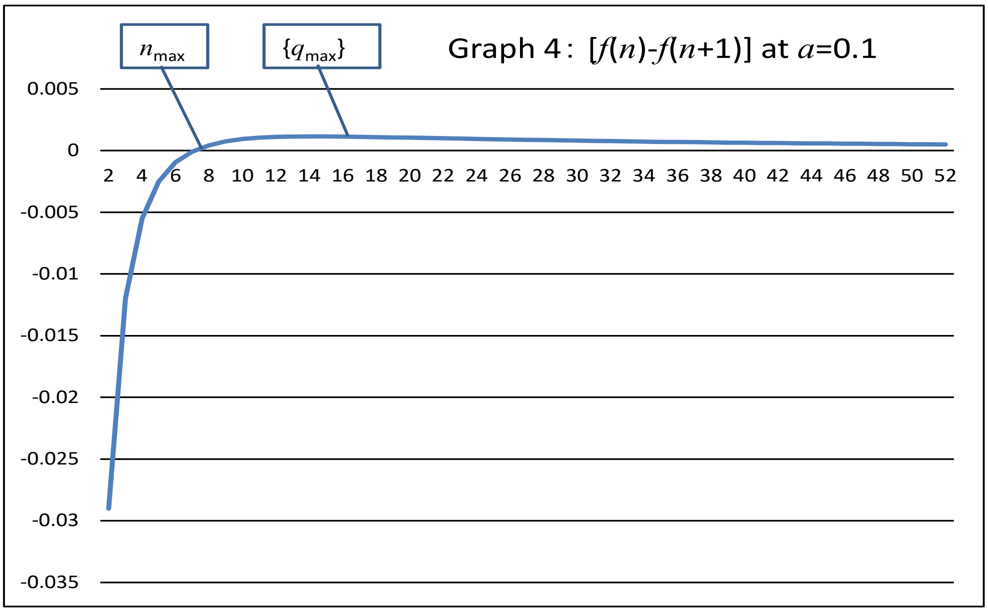
When n is a small natural number the value of $\{f(n) - f(n+1)\}$ increases with increase of n due to $C(n) > D(n)$. With increase of n the value reaches the maximum value $\{q_{max}\}$ at $C(n) \fallingdotseq D(n)$. (n is a natural number. The situation cannot be $C(n) = D(n)$.) After that the situation changes to $C(n) < D(n)$ and the value decreases to zero with $n \to \infty$. (Graph 3) shows the value of $\{f(n) - f(n+1)\}$ in various value of a. (Graph 4) shows the value of $\{f(n) - f(n+1)\}$ at $a = 0.1$. We can find the following from (Graph 3) and (Graph 4).

4.1.3.1 When $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value $|f(n) - f(n+1)|$ also becomes the maximum value at same value of a. From (Graph 1) we can find that $\left|\frac{df(n)}{dn}\right|$ becomes the maximum value at $n = 2$. Therefore the maximum value of $|f(n) - f(n+1)|$ is $\{f(3) - f(2)\}$ at same value of a as shown in (Graph 3).

4.1.3.2 With increase of n the sign of $\{f(n) - f(n+1)\}$ changes from minus to plus at $n = n_{max}$ ($n = n_{max} + 1$) when n_{max} is even(odd) number as shown in (Graph 4).

4.1.3.3 After that the value reaches the maximum value $\{q_{max}\}$ and the value decreases to zero with $n \to \infty$ as shown in (Graph 4).





4.2. Verification of $A > B$ (n_{max} is odd number.)

n_{max} is odd number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 2)\} + \{f(n_{max} - 1) - f(n_{max})\}$$

$$+\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+\cdots\cdots$$

We can have A and B as follows.

$$B=\{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+\cdots+\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$$
$$A=\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+\cdots\cdots$$

4.2.1. Condition for B

We define as follows.

{ [yellow] } : the term which is included within B.

{ [gray] } : the term which is not included within B.

We have the following (50).

$$f(n_{max})-f(2)=\{f(n_{max})-f(n_{max}-1)\}+\{f(n_{max}-1)-f(n_{max}-2)\}+\{f(n_{max}-2)-f(n_{max}-3)\}$$
$$+\cdots+\{f(7)-f(6)\}+\{f(6)-f(5)\}+\{f(5)-f(4)\}+\{f(4)-f(3)\}+\{f(3)-f(2)\} \quad (50)$$

And we have the following inequalities from (Graph 3) and (Graph 4).

$$\{f(3)-f(2)\}>\{f(4)-f(3)\}>\{f(5)-f(4)\}>\{f(6)-f(5)\}>\{f(7)-f(6)\}>\cdots\cdots$$
$$>\{f(n_{max}-2)-f(n_{max}-3)\}>\{f(n_{max}-1)-f(n_{max}-2)\}>\{f(n_{max})-f(n_{max}-1)\}>0$$

From the above (50) we have the following (51).

$$f(n_{max})-f(2)+\{f(3)-f(2)\}$$
$$=\{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+\cdots+\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$$
$$\qquad \| \qquad\qquad \wedge \qquad\qquad \wedge \qquad\qquad\qquad \wedge \quad \leftarrow\text{Value comparison}\rightarrow \quad \wedge$$
$$+\{f(3)-f(2)\}+\{f(4)-f(3)\}+\{f(6)-f(5)\}+\cdots+\{f(n_{max}-3)-f(n_{max}-4)\}+\{f(n_{max}-1)-f(n_{max}-2)\}$$
$$>2B \quad (51)$$

Due to [Total sum of upper row of the above (51) $=B<$ Total sum of lower row of (51)] we have the following (52).

$$f(n_{max})-f(2)+\{f(3)-f(2)\}>2B \quad (52)$$

4.2.2. Condition for A ($\{q_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max}+q)-f(n_{max}+q+1)\}$ to $\{q\}$ for easy description. ($q=0,1,2,3,\cdots\cdots$) All $\{q\}$ has positive value as shown in item 4.1.2.

We define as follows.

{ [yellow] } : the term which is included within A.

{ [gray] } : the term which is not included within A.

$\{q_{max}\}$ has the maximum value in all $\{q\}$. And $\{q_{max}\}$ is included within A. Then value comparison of $\{q\}$ is as follows.

$$\{1\}<\{2\}<\{3\}<\cdots<\{q_{max}-3\}<\{q_{max}-2\}<\{q_{max}-1\}<\{q_{max}\}>\{q_{max}+1\}>\{q_{max}+2\}>\{q_{max}+3\}>\cdots\cdots$$

We have the following (53).

$$f(n_{max}+1)=\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$$
$$+\{f(n_{max}+4)-f(n_{max}+5)\}+\cdots\cdots$$

$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{53}$$

From the above (53) we have the following (54).

$$f(n_{max} + 1) - \{q_{max} - 1\}$$
$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{54}$$
$$\leftarrow \cdots\cdots\cdots\cdots\cdots \text{ Range 1 } \cdots\cdots\cdots\cdots\cdots \rightarrow | \leftarrow \cdots\cdots\cdots\cdots \text{ Range 2 } \cdots\cdots\cdots\cdots$$

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$$\{1\} < \{2\} < \{3\} < \{4\} < \cdots < \{q_{max} - 4\} < \{q_{max} - 3\} < \{q_{max} - 2\}$$

And we can find the following.

$$\begin{aligned}
&\text{Total sum of } \{\colorbox{yellow}{\ \ }\} = \{1\} + \{3\} + \{5\} + \{7\} + \cdots + \{q_{max} - 4\} + \{q_{max} - 2\} \\
&\qquad\qquad\qquad\qquad\quad \vee \qquad \vee \qquad \vee \qquad\qquad\quad \vee \qquad\qquad\quad \vee \ \leftarrow \text{Value comparison} \\
&\text{Total sum of } \{\colorbox{lightgray}{\ \ }\} = \qquad \{2\} + \{4\} + \{6\} + \cdots + \{q_{max} - 5\} + \{q_{max} - 3\}
\end{aligned}$$

Therefore [Total sum of $\{\colorbox{yellow}{\ \ }\}$ > Total sum of $\{\colorbox{lightgray}{\ \ }\}$] holds.
In (Range 2) value comparison is as follows.

$$\{q_{max}\} > \{q_{max} + 1\} > \{q_{max} + 2\} > \{q_{max} + 3\} > \{q_{max} + 4\} > \{q_{max} + 5\} > \{q_{max} + 6\} > \cdots\cdots$$

And we can find the following.

$$\begin{aligned}
&\text{Total sum of } \{\colorbox{yellow}{\ \ }\} = \qquad \{q_{max}\} + \{q_{max} + 2\} + \{q_{max} + 4\} + \{q_{max} + 6\} + \cdots\cdots \\
&\qquad\qquad\qquad\qquad\quad \vee \qquad\quad \vee \qquad\quad \vee \qquad\quad \vee \ \leftarrow \text{Value comparison} \\
&\text{Total sum of } \{\colorbox{lightgray}{\ \ }\} = \{q_{max} + 1\} + \{q_{max} + 3\} + \{q_{max} + 5\} + \{q_{max} + 7\} + \cdots\cdots
\end{aligned}$$

Therefore [Total sum of $\{\colorbox{yellow}{\ \ }\}$ > Total sum of $\{\colorbox{lightgray}{\ \ }\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\colorbox{yellow}{\ \ }\} = A >$ Total sum of $\{\colorbox{lightgray}{\ \ }\}$].
We have the following (55).

$$f(n_{max} + 1) - \{q_{max} - 1\} < 2A \tag{55}$$

4.2.3. Condition for A ($\{q_{max}\}$ is not included within A.)

We have the following (56). $\{q_{max}\}$ is not included within A.

$$\begin{aligned}
f(n_{max} + 1) &= \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 3) - f(n_{max} + 4)\} \\
&\quad + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots
\end{aligned}$$
$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{56}$$

From the above (56) we have the following (57).

$$f(n_{max} + 1) - \{q_{max}\}$$
$$= \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots \tag{57}$$
$$\leftarrow \cdots\cdots\cdots\cdots\cdots \text{ Range 1 } \cdots\cdots\cdots\cdots\cdots \rightarrow | \leftarrow \cdots\cdots\cdots \text{ Range 2 } \cdots\cdots\cdots$$

(Range 1) and (Range 2) are determined as above. In (Range 1) value comparison is as follows.

$\{1\} < \{2\} < \{3\} < \{4\} < \cdots < \{q_{max}-4\} < \{q_{max}-3\} < \{q_{max}-2\} < \{q_{max}-1\}$

And we can find the following.

$$\begin{array}{lllllll} \text{Total sum of } \{\quad\} = & \{1\} + & \{3\} + & \{5\} + & \{7\} + \cdots + & \{q_{max}-3\} + & \{q_{max}-1\} \\ & & \vee & \vee & \vee & \vee & \vee \quad \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\quad\} = & & \{2\} + & \{4\} + & \{6\} + \cdots + & \{q_{max}-4\} + & \{q_{max}-2\} \end{array}$$

Therefore [Total sum of $\{\quad\}$ > Total sum of $\{\quad\}$] holds.
In (Range 2) value comparison is as follows.

$\{q_{max}+1\} > \{q_{max}+2\} > \{q_{max}+3\} > \{q_{max}+4\} > \{q_{max}+5\} > \{q_{max}+6\} > \{q_{max}+7\} > \cdots\cdots$

And we can find the following.

$$\begin{array}{lllll} \text{Total sum of } \{\quad\} = & \{q_{max}+1\} + & \{q_{max}+3\} + & \{q_{max}+3\} + & \{q_{max}+7\} + \cdots\cdots \\ & \vee & \vee & \vee & \vee \quad \leftarrow \text{Value comparison} \\ \text{Total sum of } \{\quad\} = & \{q_{max}+2\} + & \{q_{max}+4\} + & \{q_{max}+6\} + & \{q_{max}+8\} + \cdots\cdots \end{array}$$

Therefore [Total sum of $\{\quad\}$ > Total sum of $\{\quad\}$] holds.
In (Range 1)+(Range 2) we have [Total sum of $\{\quad\} = A >$ Total sum of $\{\quad\}$].
We have the following (58).

$$f(n_{max}+1) - \{q_{max}\} < 2A \tag{58}$$

4.2.4. Condition for $A > B$

From (55) and (58) we have the following inequality.

$$f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] < 2A$$

As shown in item 4.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n+1)|$. Then the following holds.

$$\{f(3) - f(2)\} >[\{q_{max}\} \text{ or } \{q_{max}-1\}]$$
$$\{f(3) - f(2)\} >f(n_{max}) - f(n_{max}+1)$$

We have the following inequality from the above 3 inequalities.

$$\begin{aligned} 2A >&f(n_{max}+1) - [\{q_{max}\} \text{ or } \{q_{max}-1\}] > f(n_{max}+1) - \{f(3) - f(2)\} \\ >&f(n_{max}) - \{f(3) - f(2)\} - \{f(3) - f(2)\} = f(n_{max}) - 2\{f(3) - f(2)\} \end{aligned} \tag{59}$$

We have the following (60) for $A > B$ from (52) and (59).

$$2A > f(n_{max}) - 2\{f(3) - f(2)\} > f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \tag{60}$$

From (60) we can have the final condition for $A > B$ as follows.

$$(4/3)f(2) > f(3) \tag{61}$$

(Graph 5) shows $(4/3)f(2) - f(3) = (4/3)(\frac{1}{2^{1/2-a}} - \frac{1}{2^{1/2+a}}) - (\frac{1}{3^{1/2-a}} - \frac{1}{3^{1/2+a}})$.

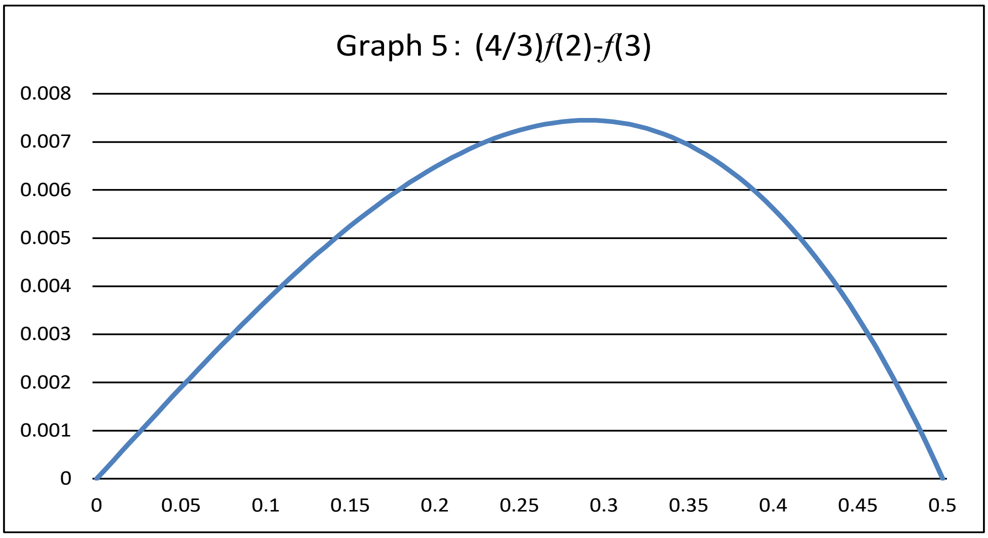


Table 1 : The values of $(4/3)f(2) - f(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

(Graph 6) shows [differentiated $\{(4/3)f(2) - f(3)\}$ regarding a] i.e. $(4/3)f'(2) - f'(3) = (4/3)\{\log 2(\frac{1}{2^{1/2-a}} + \frac{1}{2^{1/2+a}})\} - \{\log 3(\frac{1}{3^{1/2-a}} + \frac{1}{3^{1/2+a}})\}$.

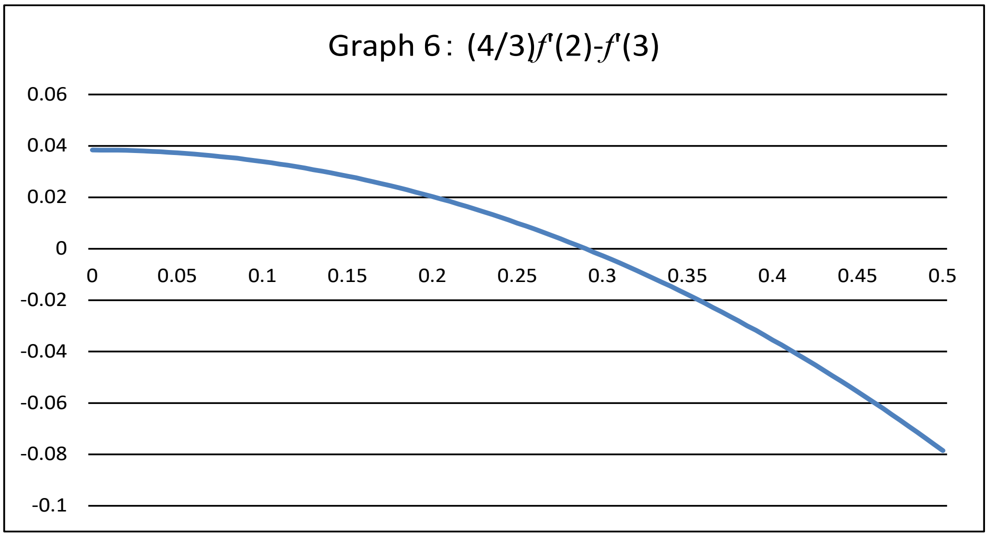


Table 2 : The values of $(4/3)f'(2) - f'(3)$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)-f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	-0.00272	-0.01785	-0.03547	-0.05567	-0.07852

From (Graph 5) and (Graph 6) we can find $[(4/3)f(2) - f(3) > 0$ in $0 < a < 1/2]$ that means $A > B$ i.e. $F(a) > 0$ in $0 < a < 1/2$.

4.3. Verification of $A > B$ (n_{max} is even number.)

n_{max} is even number as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \cdots\cdots$$
$$= \{f(2) - f(3)\} + \{f(4) - f(5)\} + \cdots + \{f(n_{max} - 4) - f(n_{max} - 3)\} + \{f(n_{max} - 2) - f(n_{max} - 1)\}$$
$$+\{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

We can have A and B as follows.

$$B = \{f(3) - f(2)\} + \{f(5) - f(4)\} + \{f(7) - f(6)\} + \cdots + \{f(n_{max} - 3) - f(n_{max} - 4)\} + \{f(n_{max} - 1) - f(n_{max} - 2)\}$$
$$A = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\} + \{f(n_{max} + 4) - f(n_{max} + 5)\} + \cdots\cdots$$

$$f(n_{max}) = \{f(n_{max}) - f(n_{max} + 1)\} + \{f(n_{max} + 1) - f(n_{max} + 2)\} + \{f(n_{max} + 2) - f(n_{max} + 3)\}$$
$$+\{f(n_{max} + 3) - f(n_{max} + 4)\} + \cdots\cdots$$
$$= \{0\} + \{1\} + \{2\} + \{3\} + \{4\} + \cdots + \{q_{max} - 3\} + \{q_{max} - 2\} + \{q_{max} - 1\} + \{q_{max}\} + \{q_{max} + 1\} + \{q_{max} + 2\} + \{q_{max} + 3\} + \cdots\cdots$$

After the same process as in item 4.2.1 we can have the following (62).

$$f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{62}$$

As shown in item 4.1.3.1 $\{f(3) - f(2)\}$ is the maximum in all $|f(n) - f(n + 1)|$. Then the following holds.

$$\{f(3) - f(2)\} > [\{q_{max}\} \text{ or } \{q_{max} - 1\}]$$
$$f(n_{max}) > f(n_{max} - 1)$$

We have the following (63) from the same process as in item 4.2.2 and item 4.2.3 and the above inequalities.

$$2A > f(n_{max}) - [\{q_{max}\} \text{ or } \{q_{max} - 1\}] > f(n_{max}) - \{f(3) - f(2)\}$$
$$> f(n_{max} - 1) - \{f(3) - f(2)\} \tag{63}$$

We have the following (64) for $A > B$ from (62) and (63).

$$2A > f(n_{max} - 1) - \{f(3) - f(2)\} > f(n_{max} - 1) - f(2) + \{f(3) - f(2)\} > 2B \tag{64}$$

From (64) we can have the final condition for $A > B$ as follows.

$$(3/2)f(2) > f(3) \tag{65}$$

In the inequality of $[(3/2)f(2) > (4/3)f(2) > f(3) > 0]$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 4.2.4 we already confirmed that the following (61) is true in $0 < a < 1/2$.

$$(4/3)f(2) > f(3) \tag{61}$$

Therefore the above (65) is true in $0 < a < 1/2$. Now we can confirm $F(a) > 0$ in $0 < a < 1/2$.

4.4. Conclusion

As shown in item 4.2 and item 4.3 $F(a) = 0$ has the only solution of $a = 0$ due to $[0 \leq a < 1/2], [F(0) = 0]$ and $[F(a) > 0$ in $0 < a < 1/2]$.

4.5. Graph of $F(a)$

We can approximate $F(a)$ with the average of $\{F(a, n-1) + F(a, n)\}/2$. But we approximate $F(a)$ for better accuracy by the following (66). (Graph 7) shows $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$.

$$\frac{\frac{F(a,n-1)+F(a,n)}{2} + \frac{F(a,n)+F(a,n+1)}{2}}{2} = F(a)_n \tag{66}$$

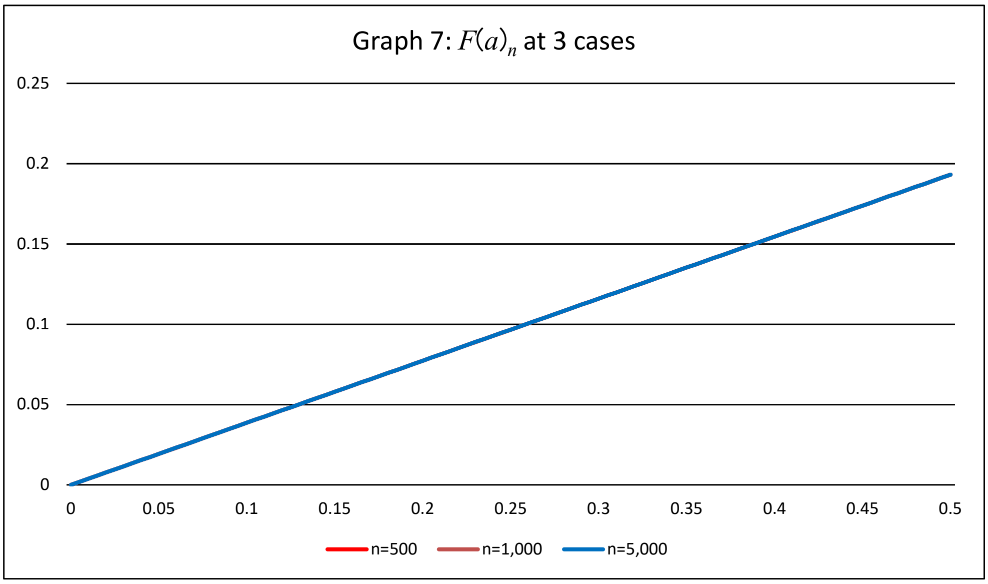


Table 3 : The values of $F(a)_n$ at 3 cases

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
n=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
n=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
n=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

3 line graphs overlapped. Because $F(a)_n$ calculated at 3 cases of $n = 500, 1000, 5000$ are equal to 4 digits after the decimal point. The range of a is $0 \leq a < 1/2$. $a = 1/2$ is not included in the range. But we added $F(1/2)_n$ to calculation due to the following reason. $[f(n)$ at $a = 1/2]$ is $(1-1/n)$ and $F(1/2)$ fluctuates due to $\lim_{n\to\infty} f(n) = 1$. But the value of the above (66) converges to the fixed value on the condition of $\lim_{n\to\infty} \{f(n+1) - f(n)\} = 0$. The condition holds due to $f(n+1) - f(n) = 1/(n + n^2)$.
$F(a)$ is a monotonically increasing function as shown in (Graph 7). So $F(a) = 0$ has the only solution and the solution must be $a = 0$ due to the following facts. Therefore Riemann hypothesis must be true.

4.5.1 In 1914 G. H.Hardy proved that there are infinite zero points on the line of $Re(s) = 1/2$.

4.5.2 All zero points found until now exist on the line of $Re(s) = 1/2$.

References

[1] Yukio Kusunoki, Introduction to infinite series, Asakura syoten, (1972), page 22, (written in Japanese)

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